Topology Uniformity Pinning Control for Multi-agent Flocking

Jintao Liu*, Ming He*+, Qiang Liu, Jie Wang, Mingguang Zou, Weishi Zhang

***Abstract*- The optimal pinning node selection for multi-agent flocking is a NP-hard problem. The current pinning node selection strategies mainly depend on complex network node characteristics which are lack of rigorous mathematical proof for flocking control. This paper studies the effect and selection strategy of pinning node based on matrix eigenvalue theory. Firstly, the effect on the eigenvalue of Laplacian matrix by pinning node is analyzed. Secondly, the synchronization index which reflects topology uniformity of multi-agent system is proposed to exert maximum influence on the system synchronizability. A practicable optimal pinning node selection method based on synchronization index is proposed and analyzed by the eigenvalue perturbation method. Finally, the simulations show the rate of system convergence by using optimal synchronizability pinning node is better than the maximum degree centrality.**

***Keywords*-multi-agent, pinning control, flocking control , synchronization index**

I. Introduction

Bees swarm need only a few individuals(about 5%) to lead the whole group to fly to a new nest [1]. Inspired, some scholars have proposed pinning control method [2], [3]. Pinning control method is of great application value for unmanned aerial vehicles swarms and other multi-agent systems. In practical engineering system, for the limitations of communication, only a few nodes may be used as pinning nodes, even one pinning node in the extreme case. There are also some surveys, such as [4] and [5] reviewed the pinning control and pinning synchronization on complex dynamical networks, while [6] focus on pinning-based consensus and flocking control of mobile multi-agent networked systems.

How to choose the best pinning node is a key research topic, and is currently an unsolved problem in recent years. Many scholars[7]–[11] have studied on how to find the minimum amount of control input node-set to ensure the system is structurally controllable. On directed graphs, YY Liu provided a criterion to determine whether a set of driving nodes can make the system controllable, namely, "minimum input theorem"[12]. However, how to choose the minimum set of pinning nodes is still an NP-hard problem[13]–[15]. There is no direct solution of the best or the minimal pinning node set[11].

The existing approximate solutions can be roughly divided into the following categories:

1) Brute force search. Permutate and combine of all nodes in a set, and then select the best-performing set. This method can provide a globally optimal solution, but it is of the worst efficiency, especially for large networks.

2) Heuristic search method. Select the pinning node according to some important features (degree, intermediate degree, compact center degree, swarming coefficient, and etc.). The advantage of this method is high computational efficiency, but the disadvantage is that the controllability and performance of the system cannot be proved directly [12]. E. N. Sanchez et al.used recurrent high order neural networks to identify system parameters[16]. F.-D. Kong and J.-P. Suninvestigated the synchronization problem of complex dynamical networks on time scales, but the problem how to determine the number of pinned nodes is still unsolved[17].

3) Evolutionary optimization algorithm. Evolutionary optimization algorithms select pinning node by calculating an objective function namely the synchronization criterion. As shown in literature [18], a Cat Swarm Optimization (CSO) method is used for spatial search. The downside is that the number of calculations also grows exponentially with the size of the network. Therefore, this method can only be applied to relatively small networks.

4) The Laplacian matrix eigenvalue method. The eigenvalues of the Laplacian matrix are taken as the measure of controllability. For example, in literature [11], a centrality measurement method based on the sensitivity analysis of the Laplacian matrix is proposed to obtain the approximate solution of the optimal set. As merely one eigenvalue decomposition calculation on the Laplacian matrix is required, the computational efficiency is very high.

In this paper, a simple flocking control strategy for second-order multi-agent systems is proposed. The Lyapunov method is used to prove that the multi-agent swarm can realize stable and collision-free flocking motion. The influence of the dynamic performance of the system by adding a pinning node

This work was supported by National Key R&D Program of China No. 2018YFC0806900; China Postdoctoral Science Foundation funded Project No. 2019M663972, 2018M633757; Primary Research & Development Plan of Jiangsu Province under Grants No. BE2016904, BE2017616, BE2018754, and BE2019762. Jiangsu Province Postdoctoral Science Foundation funded Project 2019K185.

Jintao Liu, Ming He, Qiang Liu, Mingguang Zou, Weishi Zhang are with the command and control engineering college, People's Liberation Army Engineering University, Nanjing, China. Jie Wang is with Naval Aviation University, Huludao, China.

*Authors contribute equally, +Corresponding author. Email: ming_he_2020@126.com (Ming He)

is analyzed. It provides a theoretical basis for selecting the optimal control node.

The main contributions of this paper are as follows:

An optimal selection method of pinning node for flocking synchronizability convergence is proposed in this article. The smallest eigenvalue of dynamic matrix plays the dominant rule in system convergence. From this reason, the change range of the smallest eigenvalue after pinning node added is estimated by eigenvalue perturbation theory. So, the pinning node that has the greatest impact is selected to accelerate convergence.

The remainder of this paper is organized as follows. In Section 2, some related work about flocking control and pinning nodes selection strategy are briefly introduced. In Section 3, model depiction and related mathematics preliminaries are introduced. In Section 4, the control protocol is proposed. In Section 5, the optimization index of topology uniformity and the effect of pinning node are analyzed. Some results are proposed which give us useful insight into the problem of pinning controls. In Section 6, some numerical examples of second-order multi-agent systems flocking are simulated to illustrate effectiveness of the proposed approach. Finally, we summarize the main ideas and conclusions in Section 7.

II. Related Work

A. Pinning nodes Selection Based on Network Topology Characteristics

Various centralities characterizing the structural characteristics of the network [19][20] can be used as an enlightening basis for the selection of pinning nodes, such as distribution degree, correlation degree, degree centrality, connectivity centrality, betweenness centrality, proximity centrality, eigenvector centrality, swarming coefficient, and bridgeness of edges [21]. Zhou et al. [20] proposed the use of pseudo-leaders as pinning nodes, and Gao et al. [21] proposed the use of degree central nodes of connected subnets as pinning nodes, both of which achieved good results. Some studies have even pointed out that the failure or loss of some key nodes can lead to network paralysis or the loss of control ability [22]–[24].

But degree centrality is not always the best choice, when combined with specific network functions. Due to its sparsity, the network may have a large number of nodes with the maximum degree. Literature [21] did not conduct in-depth research on how to choose the pinning node in the case of the same degree centrality. One of the main tasks of the pinning node is to transfer the information of the virtual leader to other nodes. So Maksim et al. [25] proposed to use the important index obtained by k-shell decomposition to measure the propagation performance of nodes in the network. Some important improvements based on this method include mixing degree decomposition [26], k-shell decomposition based on the gravity point of view[27] and so on.

B. Analysis Method Based on Laplacian Matrix Spectrum

Similarly, the spectrum of the Laplacian matrix contains information about the structural properties of the graph, including its connectivity (represented by inequalities) and the number of spanning trees (represented by identities). When edges are added or removed, the spectrum of the Laplacian matrix also varies with the structure of the graph. In particular, the value of λ_2 directly reflects the robustness of graph connectivity, and even the convergence of multi-agent distributed collaborative control [28].There are several related studies. Such as E. Estrada et al. proposed a measure of sub-graph centrality based on adjacency matrix spectrum[29]. A. M. Amani et al. proposed a spectral based measure of centrality to evaluate and rank the importance of nodes in pinning control[30] and introduced a measurement method to rank nodes according to the influence of the network synchronization by the removed nodes from the network[31].. T. Watanabe and N. Masuda proposed a perturbation strategy node deletion method to increase the spectral gap of the graph (i.e. the minimum eigenvalue of the second Laplacian matrix) by deleting nodes in a certain order to enhance the consistency and convergence performance of the network[32].

Deleting a node from a complex network can be modeled as deleting the relevant rows and columns of the Laplacian matrix, as well as reducing the number of diagonal entries that represent other nodes connected to this node. For large scale networks, the removal of a single node is negligible. Therefore, the effect of removing node k is approximated by removing row k and column k of Laplacian matrix.

III. Dynamic Model and Mathematics Preliminary

A. Multi-agent dynamic model

Considering that N agents work in n -dimensional Euclidean spaces, their dynamics equations can be written as:

$$\begin{cases} \dot{\mathbf{q}}_i = \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \mathbf{u}_i \end{cases} \quad i = 1,...,N \tag{1}$$

Among them, $\mathbf{q}_i, \mathbf{p}_i, \mathbf{u}_i \in \mathbb{R}^n$ are the position vector, velocity vector and acceleration vector of the agent respectively[33].

An undirected graph $\mathbf{G}(t)$ composed of multiple agents at time t consists of nodes and edges, where the set of nodes is represented by $\mathbf{V} = \{1, \cdots, m\}$, and the set of edges is represented by $\mathbf{E}(t) = \{(i, j) \in \mathbf{V} \times \mathbf{V}, j \in \mathcal{N}_i(t)\}$. Let r be the perceived radius of the multi agents, then the neighborhood of agent i is defined as:

$$\mathcal{N}_i(t) = \left\{ j \middle| \| \mathbf{q}_i - \mathbf{q}_j \| < r, j \in \mathbf{V}, j \neq i \right\} \tag{2}$$

Multi-agent networks can be represented by Laplacian matrix. The Laplacian matrix $\mathbf{L}$ is defined as

$$\mathbf{L} = \left[L_{ij} \right]_{m \times m}, \text{ where } L_{ii} = \sum_{j=1, i \neq j}^{m} a_{ij}, L_{ij} = -a_{ij} \tag{3}$$

Where only if $j \in \mathcal{N}_i$, $a_{ij} = 1$, otherwise $a_{ij} = 0$; $\mathbf{L}$ satisfies the following sum of squares:

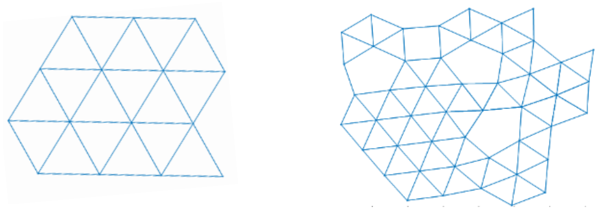
$$|\, \mathbf{z}^T (\mathbf{L} \otimes \mathbf{I}_m) \mathbf{z} = \frac{1}{2} \sum_{j \in \mathcal{N}_i} a_{ij} (\mathbf{z}_j - \mathbf{z}_i)^2 \ . \tag{4}$$

In order to avoid collision between agents, the agents need to be no less than a safe distance r_s while no more than the perceived radius r, the expected distance of multi-agent system is defined as:

$$\| \mathbf{q}_i - \mathbf{q}_j \|_2 = d, (i,j) \in \mathbf{E}, r_s < d < r \,. \tag{5}$$

However, due to the interaction of attraction and repulsion between agents, it will be difficult to reach the ideal α-lattices system[34] and eventually evolve into α-lattices -like system as

$$-\delta + d \le \| \mathbf{q}_i - \mathbf{q}_j \|_2 \le \delta + d, (i,j) \in \mathbf{E} \,. \tag{6}$$



(a) (b)

Fig. 1 (a) α-lattices system, (b) α-lattices-like system

Olfati-saber [35] firstly proposed the basic framework of the flocking control algorithm, and constructed the flocking control algorithm with leader, without leader and in the obstacle environment respectively by using three agents. Olfati-saber method assumed that the location and speed information of the virtual leader was available to all members, which was obviously not practical in the actual practice.

Subsequent work in this field has proved that when only a small number of nodes are informed of the status of the virtual leader, flocking control can also be achieved. The flocking control of the multi-agent can be realized by means of pinning control [36]. In this way, uninformed members can be driven by the informed members, so the multi-agent requires less communication costs. Pinning control is a kind of flocking control with virtual leader, which can achieve multi-agent synchronization as well as allows the community to evolve towards the given goal. Although [36] has shown that a subset of the nodes is enough to make the status of the virtual leader informed, but did not provide the method to select the pinning node nor the rules of which nodes should be informed.

B. *Matrix properties*

Lemma 1: $\mathbf{L}$ is the Laplacian matrix of the connected graph, $\mathbf{B}$ is the non-all-zero diagonal matrix $\mathbf{B} = diag(b_1, b_2, \cdots, b_n)$, $\mathbf{C} = \mathbf{L} + \mathbf{B}$, and $\mathbf{C}$ will be the positively definite matrix.

Proof: because $\mathbf{L}, \mathbf{B}$ are semidefinite matrices

for any non-zero column vector $\mathbf{x}, \mathbf{y}$

$$\mathbf{x}\mathbf{L}\mathbf{x}^T \ge 0, \mathbf{y}\mathbf{B}\mathbf{y}^T \ge 0 \,, \tag{7}$$

then for any non-zero column vector $\mathbf{u}$

$$\mathbf{u}(\mathbf{L} + \mathbf{B})\mathbf{u}^T \ge 0 \,, \tag{8}$$

then $\mathbf{C} = \mathbf{L} + \mathbf{B}$ is a semidefinite matrix

where $k \ne 0, \mathbf{H} \ne \mathbf{0}$.

Given that $\mathbf{C}$ is a semi-positive matrix, it is proved by reduction to proof that $\mathbf{C}$ is a positive matrix.

If $\mathbf{C}$ is not a positive matrix, then there exists some $\mathbf{z} \ne \mathbf{0}$ so that $\mathbf{z}\mathbf{C}\mathbf{z}^T = 0$, also because $\mathbf{L}$ is Laplacian matrix of a connected graph.

If and only if $\mathbf{z} = k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $\mathbf{z}\mathbf{L}\mathbf{z}^T = 0$,

while $\mathbf{B} = diag(b_1, b_2, \cdots, b_n)$, we have

$$\mathbf{z}\mathbf{B}\mathbf{z}^T = k\sum_{i=1}^{n} b_i \ne 0 \,, \tag{9}$$

$$\mathbf{z}\mathbf{C}\mathbf{z}^T = \mathbf{z}(\mathbf{L} + \mathbf{B})\mathbf{z}^T \ne 0 \,, \tag{10}$$

which contradicts the assumption.

Therefore, the conclusion has been proved that $\mathbf{C}$ is a positive definite matrix.

Corollary 1: $\mathbf{C}, \mathbf{C}^{-1}$ are M-matries.

First define the following matrices:

$$\mathbf{Z}_{N\times N} = \{ \mathbf{A} = (a_{ij}) \in \mathbb{R}_{N\times N} : a_{ij} \le 0 \;\; if \;\; i \ne j (i, j = 1, ..., N) \}$$

denote the set of real square $N \times N$ matrices whose off-diagonal elements are all nonpositive.

A nonsingular matrix A It is called M matrix[37] if A $\in \mathbf{Z}_{N\times N}$ and all the eigenvalues of A have positive real parts. By Lemma 1, we can get that $\mathbf{C}$, $\mathbf{C}^{-1}$ are positive definite matrices, then $\mathbf{C}, \mathbf{C}^{-1} \in \mathbf{M}_n$

IV. CONTROL PROTOCOL DESIGN AND ANALYSIS

A. *Controller design*

Assume in time t control input is:

$$\mathbf{u}_i(t) = \mathbf{f}_i^{\alpha}(t) + h_i(t)\mathbf{f}_i^{\gamma}(t) \tag{11}$$

$\mathbf{f}_i^{\alpha}$ is used to do the separation, alignment and aggregation rules of flocking motion. Different from literature [35], in order to facilitate analysis and calculation, the complex potential energy function is rewritten, and the specific mathematical expression is:

$$\begin{aligned} \mathbf{f}_i^{\alpha}(t) = c_1^{\alpha} \sum_{j \in \mathcal{N}_i(t)} a_{ij} \left((\mathbf{q}_j(t) - \mathbf{q}_i(t)) - \mathbf{r} \right) \\ + c_2^{\alpha} \sum_{j \in \mathcal{N}_i(t)} a_{ij} \left(\mathbf{p}_j(t) - \mathbf{p}_i(t) \right) \end{aligned} \tag{12}$$

Where $\mathbf{r} = sign(\mathbf{q}_j - \mathbf{q}_i)d$, we call it the desired distance control term.

Remark 1:When the expected distance is $d = 0$, the equation (11) degenerates from the flocking problem to the second order consensus problem. Let's decompose the *sign* function according to coordinates.

$\mathbf{f}_i^{\gamma}$ receives the information of virtual leader, and the mathematical expression is:

Where $\mathbf{p}_{\gamma} \in \mathbb{R}^n$ is the velocity vector of the virtual leader at time t ; $h_i(t)$ is the parameter related to the pinning node selection, and the mathematical expression of $h_i(t)$ at time t is:

$$h_i(t)=\begin{cases}1, \text{ node } i \text{ is pinning node}\\ 0, \text{ node } i \text{ is NOT pinning node}\end{cases}$$

$$\mathbf{H}=diag(h_1,h_2,\cdots,h_n)$$

Define an augmented Laplacian matrix

$$\mathbf{A}=\mathbf{L}+k\mathbf{H} \tag{13}$$

The set of all agent nodes is $\mathbf{V}$. Take the non-empty subset of $\mathbf{V}$ as $\mathbf{V}_b\subset\mathbf{V}$, $\mathbf{V}_b:=\{i\in\mathbf{V}|h_i=1\}$. In different literature, $\mathbf{V}_b$ is called control node, driving node[12], [38], pinning node[39], [40], leader[34]. The remaining nodes $\mathbf{V}\setminus\mathbf{V}_b$ are called follower.

Assumption 1:The initial status of the undirected graph $\mathbf{G}(0)$ is connected.

Remark 2: If the initial state is not connected and divided into n subnets, then each subnet needs at least one pinning point, the same conclusion can be obtained.

Assumption 2: The initial state has no collisions.

Assumption 3: The speed of virtual leader is fixed.

Theorem 1: Considering the equation of motion as shown in formula (1), and the control input as shown in formula (6), then take any one or more nodes as the pinning node and there are following conclusions:

1) The relative positions of all agents eventually approach to lattices;

2) The speeds of all agents tend to virtual pilot speed $\mathbf{p}_\gamma(t)$ (an improvement on the proof of speed consistency in literature [41]);

3) There will be no collisions between agents;

4) The pinning node's position tends to the virtual leader position $\mathbf{q}_\gamma(t)$ (An extension of the conclusion on the location of literature [42]).

Proof: Assumption that the tracking error between agent and virtual leader is $\tilde{\mathbf{q}}_i(t)=\mathbf{q}_i(t)-\mathbf{q}_\gamma(t)$, $\tilde{\mathbf{p}}_i(t)=\mathbf{p}_i(t)-\mathbf{p}_\gamma(t)$.

The distance between agents is . $\mathbf{q}_{ij}(t)=\mathbf{q}_i(t)-\mathbf{q}_j(t)$.

The total energy of the multi-agent $Q(t)$ is composed of the potential energy and the kinetic energy

$$Q(t)=\frac{1}{2}\sum_{i=1}^{m}\left(U_i(t)+\tilde{\mathbf{p}}_i(t)^T\tilde{\mathbf{p}}_i(t)\right). \tag{14}$$

The potential energy function is

$$U_i(t)=\sum_{j=1,j\neq i}^{n}\Phi_i(t)+h_ic_1\tilde{\mathbf{q}}_i^T\tilde{\mathbf{q}}_i, \tag{15}$$

$\Phi_i(t)$ is expressed as

$$\Phi_i(t)=\frac{1}{2}\sum_{j\in\mathcal{N}_i(t)}\left(\mathbf{q}_{ij}(t)-\mathbf{r}\right)^T\left(\mathbf{q}_{ij}(t)-\mathbf{r}\right). \tag{16}$$

When there is no overlap between agents, $\mathbf{q}_i(0)\neq\mathbf{q}_j(0)$ and there is

$$\frac{\partial\Phi_i}{\partial\mathbf{q}_i}=\frac{\partial\Phi_i}{\partial\mathbf{q}_{ij}}=\sum_{j\in\mathcal{N}_i(t)}\left(\mathbf{q}_{ij}(t)-\mathbf{r}\right)\mathbf{p}_i(t), \tag{17}$$

$$\nabla\Phi_i=\sum_{j\in\mathcal{N}_i(t)}\left(\mathbf{q}_{ij}(t)-\mathbf{r}\right)\mathbf{p}_i(t). \tag{18}$$

The energy equation can be rewritten as

$$Q(t)=\frac{1}{2}\sum_{i=1}^{n}\left(U_i(t)+\tilde{\mathbf{p}}_i(t)^T\tilde{\mathbf{p}}_i(t)\right). \tag{19}$$

The derivative of $Q(t)$ is

$$\begin{aligned}\dot{Q}(t)&=-c_2^\alpha\tilde{\mathbf{p}}(t)^T\left(\mathbf{L}(t)\otimes\mathbf{I}_m\right)\tilde{\mathbf{p}}(t)\\&\quad-c_2^\gamma\sum_{i=1}^{m}h_i(t)\tilde{\mathbf{p}}_i(t)^T\tilde{\mathbf{p}}_i(t)\\&=-\tilde{\mathbf{p}}(t)^T\left(\left(c_2^\alpha\mathbf{L}(t)+c_2^\gamma\mathbf{H}(t)\right)\otimes\mathbf{I}_m\right)\tilde{\mathbf{p}}(t).\end{aligned} \tag{20}$$

$\mathbf{L}$ is a Laplacian matrix and also is a semidefinite matrix. $\mathbf{H}=diag[h_1\cdots h_n]$ is a definite matrix. From lemma 1 $c_2^\alpha\mathbf{L}(t)+c_2^\gamma\mathbf{H}(t)$ is a positive matrix. Then $\left(\left(c_2^\alpha\mathbf{L}(t)+c_2^\gamma\mathbf{H}(t)\right)\otimes\mathbf{I}_m\right)$ is a positive matrix. So at time t $\dot{Q}<0$,and the total energy of the system continues to decay.

$$Q(t)<Q_0<Q_{\max} \tag{21}$$

From the definition of the potential function $U(q)$, $\sum_{i=1}^{N}U_i(\|r\|_\alpha)\leq Q(0)\leq Q_{\max}$, at time t, the distance between two adjacent nodes that are neighbors should be less than r, that is, the existing edge in the network will not break. Assuming that after $\vartriangle t$ time, m new edges are added into the network, then

$$Q(t+\vartriangle t)=Q_0+mU(\|r-\varepsilon\|_\alpha)<Q_{\max} \tag{22}$$

This indicates that over time, the existing edges in the system will have no possibility of disconnection, while the initial network $G(0)$ is connected, so the system will always remain connected.

Next we need to answer the following two questions[38]:

Question 1: Will the controllability be achieved by choosing a set of nodes? This is a problem of controllability.

Question 2: How to select a minimum number of nodes collection, and make the system controllable? That is the problem of minimum set.

The first question could be solved by analyzing the spectrum of the Laplacian matrix with structural characteristics of the graph, in particular λ_2. Its size directly reflects the convergence of graph connectivity, robustness and even multi-agent distributed collaborative control [28]. With pinning control, the Laplacian matrix L is further transformed into the augmented Laplacian matrix A, as shown in (13). By estimating the approximate range of the eigenvalues of A and the variation trend of the eigenvalues caused by the pinning control, we can preliminarily obtain the convergence effect of pinning control on the system.

Theorem 2[43]: The controllability of a strongly-connected network can be increased by either adding a pinning node or increasing one of the pinning gains.

Remark 3: Z. Cheng et al. [43] presents a network control strategy based on left Perron vector, but it requires iterative calculation of eigenvalues and eigenvectors of $\mathbf{L}+k\mathbf{H}$. It is difficult to solve the eigenvalues in practical engineering due to the large amount of computation. The method of eigenvalue perturbation can be used to approximate the optimal pinning point.

Remark 4: With regard to the first-order consistency problem, when no pinning control is applied, since the minimum eigenvalue of Laplician is 0, the final result of convergence in multi-agent equilibrium is not necessarily 0,depending on the average value of the initial state of multi-agent [44], and the second minimum eigenvalue determines the convergence rate of consistency. When pinning control is applied, as the minimum eigenvalue of the system is less than zero, the equilibrium state finally converges to the state of the pinning point.

V. SELECTION OF PINING NODE

A. Synchronization index

A Synchronizability Index $\bar{R}$ is defined in [45]

$$\bar{R}=\frac{\lambda_N}{\lambda_2}, \tag{23}$$

$\bar{R}$ is the ratio of the eigenvalues of the Laplacian matrix, the maximum eigenvalue λ_N divided by the second smallest eigenvalue λ_2. The smaller the eigenvalue $\bar{R}$ is, the better the network synchronization performance will be.

Remark 5: The synchronization index of Laplacian matrix is an important indicator to represent the synchronization of the network. The smaller the synchronization index is, the better the synchronization performance of the network will be. The smaller the synchronization index is, the closer the ratio of the maximum eigenvalue of Laplacian matrix to the subminimum eigenvalue is to 1, which indicates that all the eigenvalues of Laplacian matrix are almost equal.

However, when the connection between multi-agent systems is not completely connected, it leads to $\lambda_2=0$ and $\bar{R}$ nonsense.

So we define a new Synchronizability Index R

$$R=\frac{\lambda_m}{\lambda_N}, \tag{24}$$

Where λ_m is the minimum non-zero eigenvalue. It only requires the solution of two eigenvalues of Laplacian matrix, which is easy to calculate[11].

Remark 6: The larger the synchronization index is, the closer the ratio of the maximum eigenvalue of Laplacian matrix to the subminimum eigenvalue is to 1, which indicates that all the eigenvalues of Laplacian matrix are almost equal.

Remark 7: The synchronization index reflects the uniformity of network topology connection, and for the α-lattices system, the synchronization index also reflects the uniformity of agent in spatial distribution.

Remark 8: The synchronization index of Laplacian matrix is an important index to represent the synchronization of the network. The smaller the synchronization index is, the better the synchronization performance of the network will be.

Remark 9: Conclusion from Master Stability Function analysis[46]. With the stability functions for synchronized coupled systems, the spread of Laplacian eigenvalues can be used as a synchronizability index. So a simple measure for the spread of eigenvalues is the ratio of the largest to the smallest [47][48].

B. Perturbation analysis

The node with largest value of R is of stronger effect on network controllability[49]. Further, the change of R caused by the pinning node is expressed as:

$$\Delta R=\frac{\lambda_N\Delta\lambda_m-\lambda_m\Delta\lambda_N}{\lambda_N^2}. \tag{25}$$

It's worth noting that since λ_N is much larger than λ_m in the initial phase, ΔR is much more affected by $\Delta\lambda_m$ than $\Delta\lambda_n$. Therefore, when adding a pinning node, if the effect on $\Delta\lambda_m$ is greatest, the effect on ΔR will generally be greatest.

From the eigenvalue perturbation theory[50] , changes of the eigenvalue λ_n caused by perturbation of the parameter p is[30]:

$$\frac{d\lambda_n}{dp}=\mathbf{y}_n^T\frac{d\mathbf{L}(p)}{dp}\mathbf{x}_n,\quad n=1,2,\ldots,N, \tag{26}$$

where $\mathbf{y}_n^T,\mathbf{x}_n$ are the normalized left and right eigenvectors of $\mathbf{L}$ respectively, and $\mathbf{y}_n^T\mathbf{x}_n=1$, and node i is selected as the pinning node.

When the pinning control is applied, $\bar{\mathbf{L}}=\mathbf{L}-k\mathbf{H}$. So the minimum eigenvalue of the new matrix $\bar{\mathbf{L}}$ is $\bar{\lambda}_1>0$ according to the previous analysis.

The perturbation of R by pinning node is approximate to:

$$\frac{dR}{dl_{ii}}=\frac{\left(y_1^ix_1^i\right)\lambda_N-\left(y_N^ix_N^i\right)\lambda_1}{\left(\lambda_N\right)^2}, \tag{27}$$

where i represents the i th element of the vector. Since the minimum eigenvalue of $\mathbf{L}$ is $\lambda_1=0$, the corresponding eigenvector is $\mathbf{x}_1=\frac{1}{\sqrt{N}}\mathbf{1}_N$.

For an undirected graph with $\mathbf{y}_n=\mathbf{x}_n$, it can be written as:

$$\frac{dR}{dl_{ii}}=\frac{1}{\lambda_N}\left(\frac{1}{\sqrt{N}}-R\left(x_N^i\right)^2\right). \tag{28}$$

Define Controllability Centrality[51] $\Psi(i)$ as

$$\Psi(i)=\left(x_N^i\right)^2,\quad i=1,2,\ldots,N. \tag{29}$$

Because $\frac{d\lambda_k}{dl_{ii}}=-\left(\mathbf{x}_N^i\right)^2$ and the minimum eigenvalue of $\mathbf{L}$ is $\lambda_1=0$, the corresponding eigenvector is $\mathbf{x}_1=\frac{1}{\sqrt{N}}\mathbf{1}_N$.

Remark 10: We assume that the disturbance k_1 caused by the control gain is consistent $k_1>0$ on all diagonal elements of matrix $\mathbf{L}$. So we will get the theoretical optimal result of the theory, when the perturbations are much smaller than the diagonal terms of the original Laplacian matrix $\mathbf{L}$, namely the perturbation is far more less than the minimum degree of the network. For larger disturbance or small degree, the precision of the perturbation method may decrease.

VI. Simulation

In this section, the motion of multi-agent based on constant speed virtual leader in two-dimensional space plane is studied. Assuming the number of agents is n=35, and the initial state of all agents is shown in Figure 1, where the red arrow represents the speed of the agent, the black circle beside it indicates that the agent is a pinning node, and the virtual leader is represented by a five-pointed star. Let r=10 be the perception radius of the agent, and d=2 be the radius of the target α-lattices structure. The virtual leader's initial position is $(25, 25)$, and the speed is constant $(0.5, 0.5)$.

Analyze the multi-agent system as a whole, define the location and velocity of the multi-agent center（CoM，Center of Mass） as the average of the locations and velocities of all agents:

$$\begin{cases} \bar{\mathbf{q}} = \dfrac{1}{m}\sum_{i=1}^{m}\mathbf{q}_i \\ \bar{\mathbf{p}} = \dfrac{1}{m}\sum_{i=1}^{m}\mathbf{p}_i \end{cases}. \tag{30}$$

The definition of virtual leader and CoM tracking error is:

$$\begin{cases} \tilde{\mathbf{q}} = \mathbf{q}_r - \bar{\mathbf{q}} \\ \tilde{\mathbf{p}} = \mathbf{p}_r - \bar{\mathbf{p}} \end{cases}. \tag{31}$$

The total velocity tracking error of the multi-agent system is defined as:

$$Err = \sum_{i=1}^{N}\left\|\mathbf{q}_i - \mathbf{q}_r\right\|_2 . \tag{32}$$

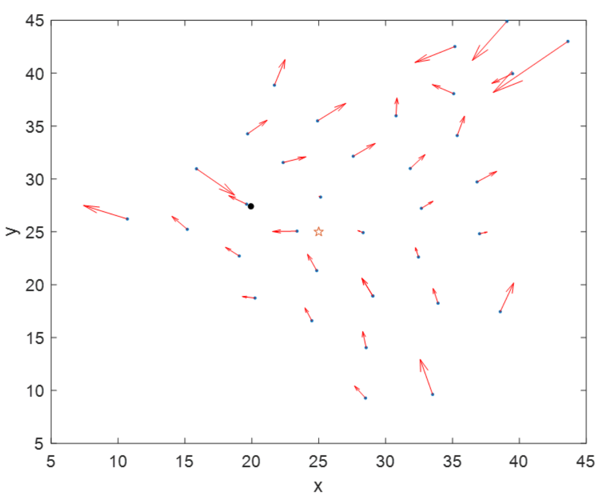


Figure 1 Agent initial state

In the Figure 1, each point represents an agent, and the angle of the arrow represents the direction of the agent, the length of the arrow represents velocity magnitude. The virtual leader is represented by a pentagram.

The following is a comparison of flocking control based on optimal, degree centrality maximization and degree centrality minimization.

A. Simulation 1

Selection based on optimal pinning point.

The Figure 2 shows the trajectories of all the agents. The four graphs in Figure 3 respectively show the positions and motions of the agents at different times. Figure 4 and Figure 5 show velocity convergence error curves of agents on x and y axis. Figure 6 shows the total velocity error convergence of multi-agent system according to the calculation formula (32).

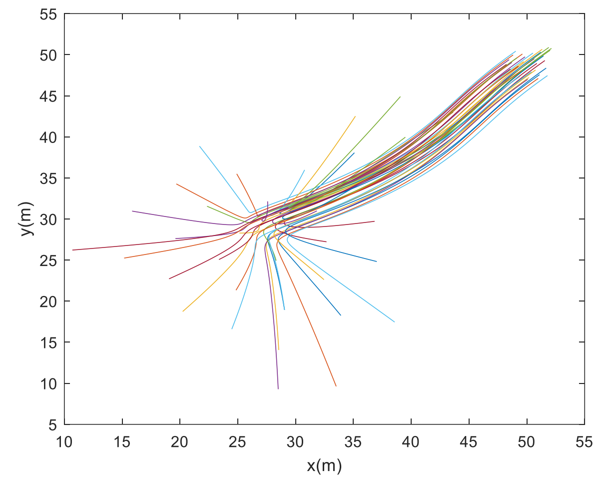


Figure 2 Trajectories of agents

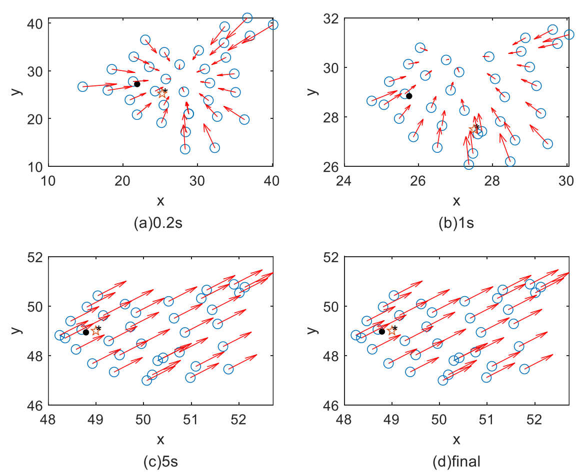


Figure 3 The status of agents at different times

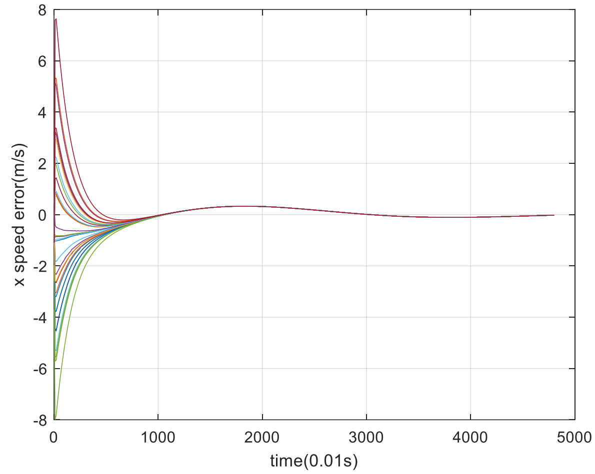


Figure 4 Velocity convergence error curves of agents on x axis

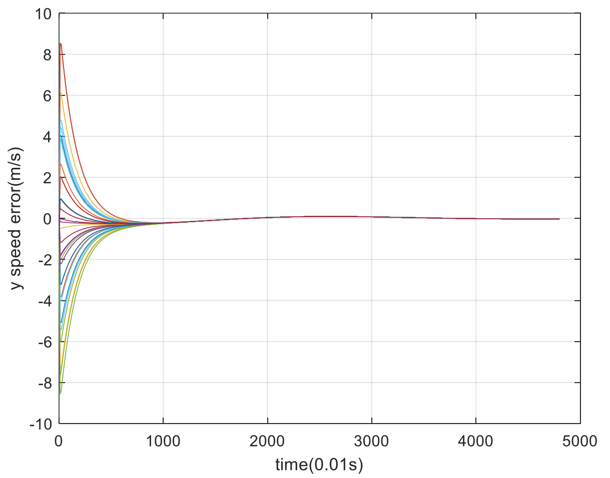


Figure 5 Velocity convergence error curves of multi-agent on y axis

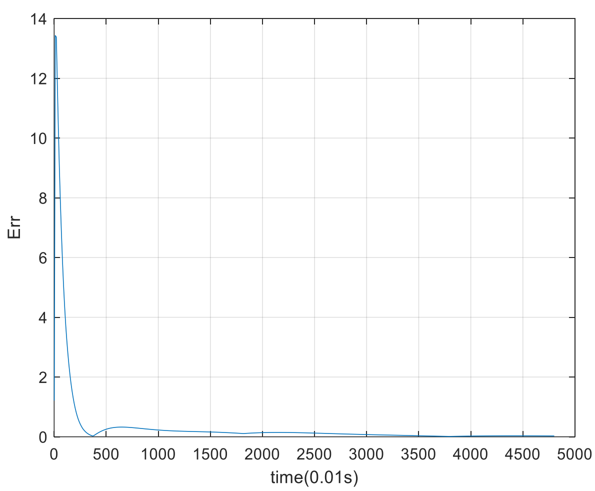


Figure 6 The total velocity error convergence of multi-agent system

B. Simulation 2

Based on degree centrality of the maximum point of pinning selection.

The Figure 7 shows the trajectories of all the agents. The four graphs in Figure 8 respectively show the positions and motions of the agents at different times. Figure 9 and Figure 10 show velocity convergence error curves of agents on x and y axis. Figure 11 shows the total velocity error convergence of multi-agent system according to the calculation formula (32).

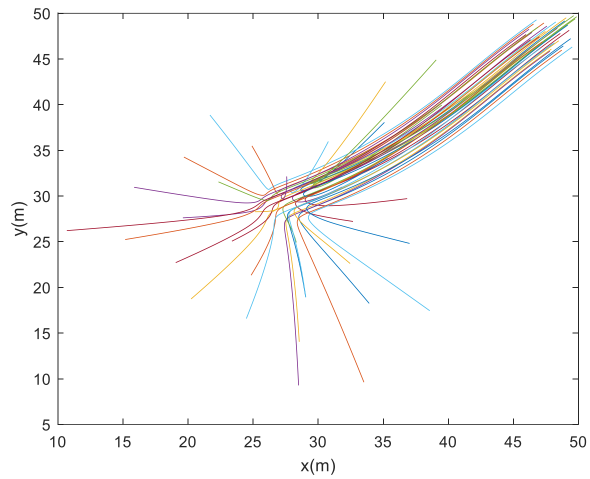


Figure 7 Trajectories of agents

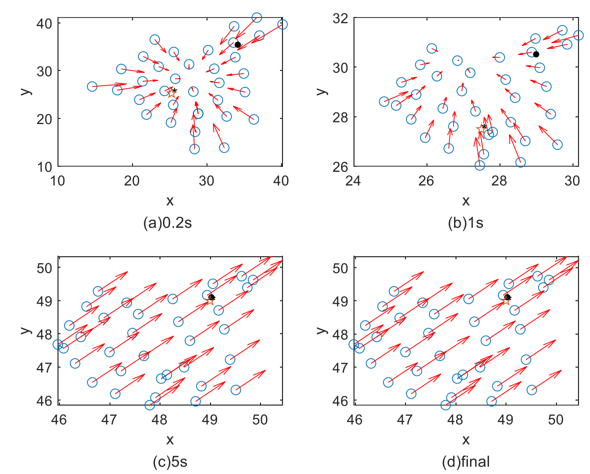


Figure 8 The status of agents at different times

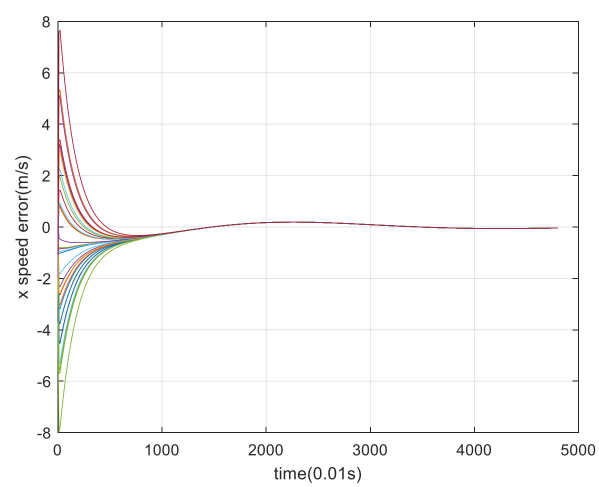


Figure 9 Velocity convergence error curves of agents on x axis

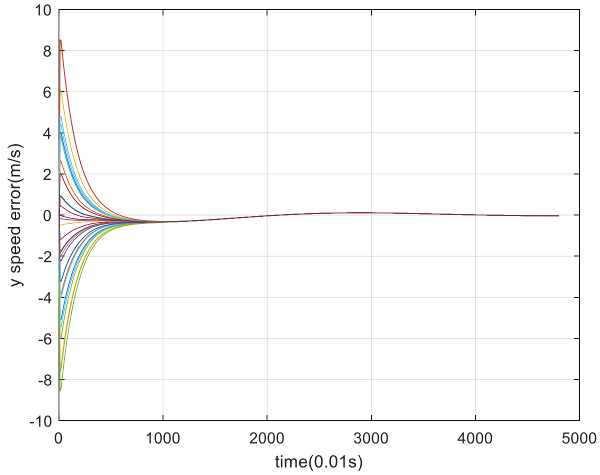


Figure 10 Velocity convergence error curves of agents on y axis

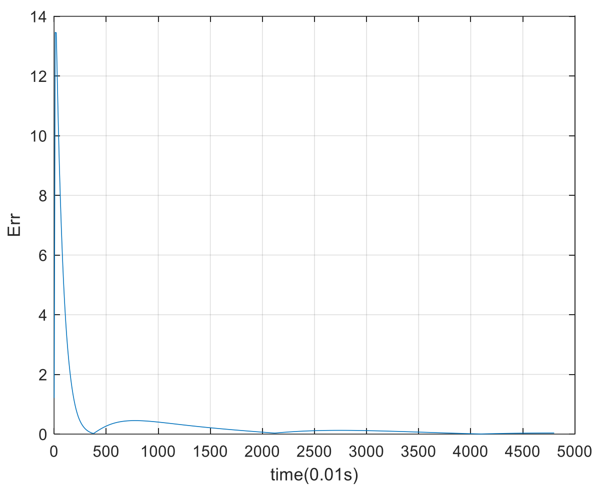


Figure 11 The total velocity error convergence of multi-agent system

C. *Simulation 3*

Based on degree centrality of the minimum point of pinning selection.

The Figure 12 shows the trajectories of all the agents. The four graphs in Figure 13 respectively show the positions and motions of the agents at different times. Figure 14 and Figure 15 show velocity convergence error curves of agents on x and y axis. Figure 16 shows the total velocity error convergence of multi-agent system according to the calculation formula (32).

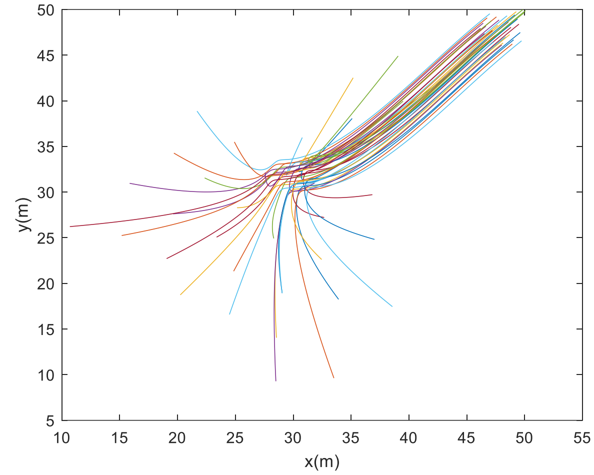


Figure 12 Trajectories of agents

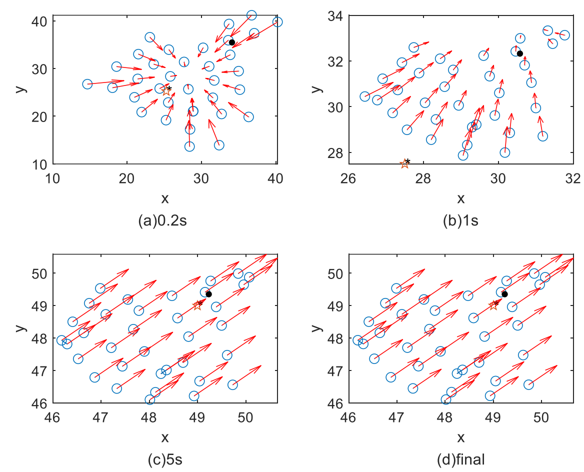


Figure 13 Multi-agent status at different time

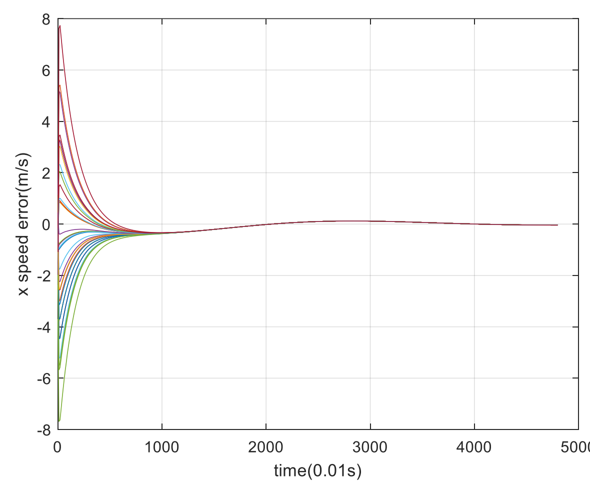


Figure 14 Velocity convergence error curves of agents on x axis

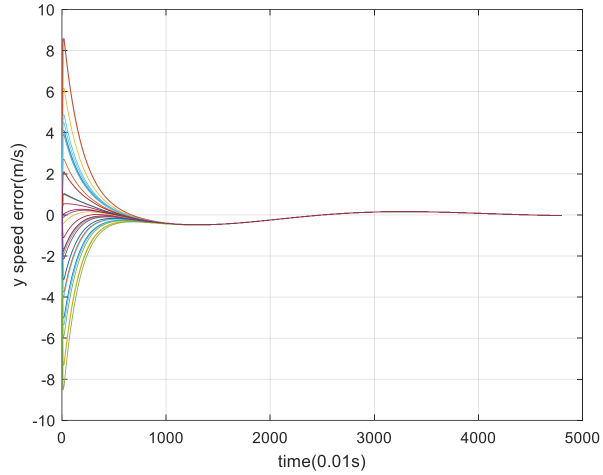


Figure 15 Velocity convergence error curves of agents on y axis

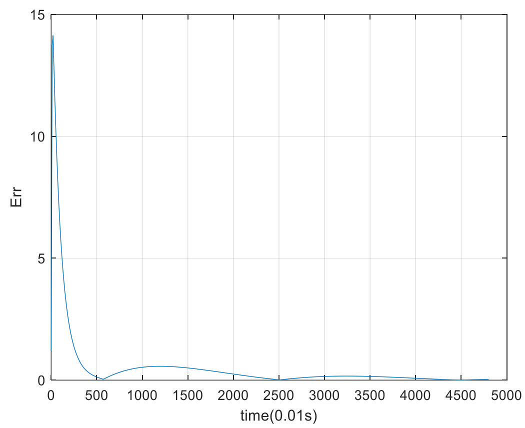


Figure 16 The total velocity error convergence of multi-agent system

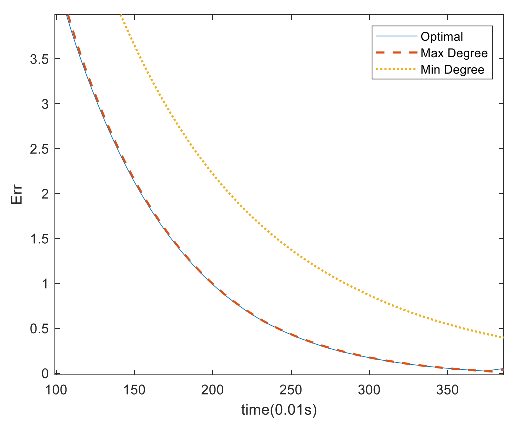


Figure 17 Influence of different pinning points on velocity consistency error

D. *Experiment analysis*

From three groups of simulation, the proposed algorithm can achieve effective flocking control. The speed of all agents is basically the same in a short time, and gradually approaches the speed of the virtual leader. The distance between agents gradually approaches the ideal distance over time, and is always greater than 0, which verifies the correctness of the controller design.

It can be seen from Figure 17 that the selection of different pinning points has a great influence on the formation speed of multi-agent flocking. The convergence rate of velocity consistency error based on the optimal pinning point is the fastest, the pinning point of maximum degree centrality is the second, and the pinning point of minimum degree centrality is the slowest. It is indicated that the formation speed of agent flocking is directly related to degree centrality.

After realizing the flocking movement, the velocity tracking error of the agents and the virtual leader have no direct relation with the choice of the pinning point.

VII. Conclusion

We have proposed a simple flocking control strategy for second-order multi-agent systems by which the control system can realize stable and collision-free flocking motion. The influence of the dynamic performance of the system by adding a pinning node was analyzed. The controllability of a strongly-connected network can be increased by either adding a pinning node or increasing one of the pinning gains. All the eigenvalues of the new system are greater than zero. We have also analyzed the synchronizability index pinning point selection strategies. The synchronization index of Laplacian matrix is an important index to represent the synchronization of the network. The smaller the synchronization index is, the better the synchronization performance of the network will be. The optimal pinning node has also been analyzed by the eigenvalue perturbation method. The simulations show the pinning point selection based on the synchronization index is the best, and the convergence rate of velocity consistency error is faster than the maximum degree centrality.

References

[1] I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin, “Effective leadership and decision-making in animal groups on the move,” *Nature*, vol. 433, no. 7025, pp. 513–516, 2005.

[2] X. F. Wang and G. Chen, “Pinning control of scale-free dynamical networks,” *Phys. A Stat. Mech. Its Appl.*, vol. 310, no. 3, pp. 521–531, 2002.

[3] X. Li, X. Wang, and G. Chen, “Pinning a complex dynamical network to its equilibrium,” *IEEE Trans. Circuits Syst. I Regul. Pap.*, vol. 51, no. 10, pp. 2074–2087, 2004.

[4] G. Chen, “Pinning control and synchronization on complex dynamical networks,” *Int. J. Control. Autom. Syst.*, vol. 12, no. 2, pp. 221–230, 2014.

[5] W. Xing, P. Shi, R. K. Agarwal, and Y. Zhao, “A survey on global pinning synchronization of complex networks,” *J. Franklin Inst.*, vol. 356, no. 6, pp. 3590–3611, 2019.

[6] X. Wang and H. Su, “Pinning control of complex networked systems:A decade after and beyond,” *Annu. Rev. Control*, vol. 38, no. 1, pp. 103–111, 2014.

[7] A. Olshevsky, “Minimal Controllability Problems,” *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 3, pp. 249–258, Sep. 2014.

[8] S. Pequito, S. Kar, and A. P. Aguiar, “A Framework

for Structural Input/Output and Control Configuration Selection in Large-Scale Systems," *IEEE Trans. Automat. Contr.*, vol. 61, no. 2, pp. 303–318, Feb. 2016.

[9] A. Chapman and M. Mesbahi, "On strong structural controllability of networked systems: A constrained matching approach," in *2013 American Control Conference*, Jun. 2013, pp. 6126–6131.

[10] S. S. Mousavi, M. Haeri, and M. Mesbahi, "On the Structural and Strong Structural Controllability of Undirected Networks," *IEEE Trans. Automat. Contr.*, vol. 63, no. 7, pp. 2234–2241, Jul. 2018.

[11] A. M. Amani, M. Jalili, X. Yu, and L. Stone, "Controllability of complex networks: Choosing the best driver set," *Phys. Rev. E*, vol. 98, no. 3, pp. 1–6, 2018.

[12] Y. Y. Liu, J. J. Slotine, and A. L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 7346, p. 167, 2011.

[13] A. Olshevsky, "Minimal Controllability Problems," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 3, pp. 249–258, Sep. 2014.

[14] A. Chapman and M. Mesbahi, "On strong structural controllability of networked systems: A constrained matching approach," in *2013 American Control Conference*, Jun. 2013, pp. 6126–6131.

[15] S. S. Mousavi, M. Haeri, and M. Mesbahi, "On the Structural and Strong Structural Controllability of Undirected Networks," *IEEE Trans. Automat. Contr.*, vol. 63, no. 7, pp. 2234–2241, Jul. 2018.

[16] E. N. Sanchez, D. I. Rodriguez-Castellanos, G. Chen, and R. Ruiz-Cruz, "Pinning control of complex network synchronization: A recurrent neural network approach," *Int. J. Control. Autom. Syst.*, vol. 15, no. 3, pp. 1405–1414, 2017.

[17] F.-D. Kong and J.-P. Sun, "Pinning Synchronization of Complex Dynamical Networks on Time Scales," *Int. J. Control. Autom. Syst.*, 2020.

[18] Y. Orouskhani, M. Jalili, and X. Yu, "Optimizing Dynamical Network Structure for Pinning Control," *Sci. Rep.*, no. 6, p. 24252, 2016.

[19] Mark Newman, *Networks Second Edition*. Oxford: Oxford University Press, 2018.

[20] J. Zhou, X. Wu, W. Yu, M. Small, and J. A. Lu, "Flocking of multi-agent dynamical systems based on pseudo-leader mechanism," *Syst. Control Lett.*, vol. 61, no. 1, pp. 195–202, 2012.

[21] J. Gao, X. Xu, N. Ding, and E. Li, "Flocking motion of multi-agent system by dynamic pinning control," 2017.

[22] C. L. Pu, W. J. Pei, and A. Michaelson, "Robustness analysis of network controllability," *Phys. A Stat. Mech. Its Appl.*, vol. 391, no. 18, pp. 4420–4425, 2012.

[23] B. Sergey V, P. Roni, P. Gerald, S. H Eugene, and H. Shlomo, "Catastrophic cascade of failures in interdependent networks," *Nature*, vol. 464, no. 7291, p. 1025, 2010.

[24] T. Jia, Y. Y. Liu, E. Csóka, M. Pósfai, J. J. Slotine, and A. L. Barabási, "Emergence of bimodality in controlling complex networks.," *Nat. Commun.*, vol. 4, no. 3, p. 2002, 2013.

[25] S. Ahajjam and H. Badir, "Identification of influential spreaders in complex networks using HybridRank algorithm," *Sci. Rep.*, vol. 8, no. 1, p. 888, 2018.

[26] J.-G. Liu, Z.-M. Ren, and Q. Guo, "Ranking the spreading influence in complex networks," *Phys. A Stat. Mech. its Appl.*, vol. 392, no. 18, pp. 4154–4159, 2013.

[27] J. Wang, X. Hou, K. Li, and Y. Ding, "A novel weight neighborhood centrality algorithm for identifying influential spreaders in complex networks," *Phys. A Stat. Mech. its Appl.*, vol. 475, pp. 88–105, 2017.

[28] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks:* Princeton University Press, 2010.

[29] E. Estrada and J. A. Rodríguez-Velázquez, "Subgraph centrality in complex networks," *Phys. Rev. E*, vol. 71, no. 5, p. 056103, May 2005.

[30] A. M. Amani, M. Jalili, X. Yu, and L. Stone, "Finding the most influential nodes in pinning controllability of complex networks," *IEEE Trans. Circuits Syst. II Express Briefs*, vol. 64, no. 6, pp. 685–689, Jun. 2017.

[31] A. M. Amani, M. Jalili, X. Yu, and L. Stone, "A New Metric to Find the Most Vulnerable Node in Complex Networks," in *Proceedings - IEEE International Symposium on Circuits and Systems*, May 2018, vol. 2018-May, pp. 1–5.

[32] T. Watanabe and N. Masuda, "Enhancing the spectral gap of networks by node removal," *Phys. Rev. E*, vol. 82, no. 4, p. 046102, Oct. 2010.

[33] Q. Liu, M. He, D. Xu, N. Ding, and Y. Wang, "A Mechanism for Recognizing and Suppressing the Emergent Behavior of UAV Swarm," *Math. Probl. Eng.*, vol. 2018, pp. 1–14, 2018.

[34] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: algorithms and theory," *IEEE Trans. Automat. Contr.*, vol. 51, no. 3, pp. 401–420, 2006.

[35] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Trans. Automat. Contr.*, 2006.

[36] H. Su, X. Wang, and Z. Lin, "Flocking of multi-agents with a virtual leader," *IEEE Trans. Automat. Contr.*, 2009.

[37] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*. Philadephia: SIAM Press, 1994.

[38] Y. Y. Liu and A. L. Barabási, "Control Principles of Complex Networks," *Rev. Mod. Phys.*, vol. 88, no. 3, p. 035006, 2016.

[39] G. Chen, "Pinning control and controllability of complex dynamical networks," *Int. J. Autom. Comput.*, vol. 14, no. 1, pp. 1–9, 2017.

[40] F. W. Xiao and G. Chen, "Pinning control of scale-free dynamical networks," *Phys. A-statistical Mech. Its Appl.*, vol. 310, no. 3, pp. 521–531, 2002.

[41] H. Su, X. Wang, and Z. Lin, "Flocking of Multi-Agents With a Virtual Leader," *IEEE Trans. Automat. Contr.*, vol. 54, no. 2, pp. 293–307, 2009.

[42] J. Gao, X. Xu, D. Nan, and Q. E. Li, "Flocking motion

of multi-agent system by dynamic pinning control," *Iet Control Theory Appl.*, vol. 11, no. 5, pp. 714–722, 2017.
[43] Z. Cheng, Y. Xin, J. Cao, X. Yu, and G. Lu, "Selecting pinning nodes to control complex networked systems," *Sci. China(Technological Sci.*, vol. 61, no. 10, pp. 111–119, 2018.
[44] S. S. Kia, B. Van Scoy, J. Cortes, R. A. Freeman, K. M. Lynch, and S. Martinez, "Tutorial on Dynamic Average Consensus: The Problem, Its Applications, and the Algorithms," *IEEE Control Syst. Mag.*, vol. 39, no. 3, pp. 40–72, 2019.
[45] M. Jalili, O. Askari Sichani, and X. Yu, "Optimal pinning controllability of complex networks: Dependence on network structure," *Phys. Rev. E*, vol. 91, no. 1, p. 012803, Jan. 2015.
[46] L. M. PECORA and T. L. CARROLL, "Master Stability Functions for Synchronized Coupled Systems," *Int. J. Bifurc. Chaos*, vol. 09, no. 12, pp. 2315–2320, Dec. 1999.
[47] T. Nishikawa, A. E. Motter, and G. Parisi, "Network synchronization landscape reveals compensatory structures, quantization, and the positive effect of negative interactions," *Proc. Natl. Acad. Sci. U. S. A.*, vol. 107, no. 23, pp. 10342–10347, 2010.
[48] Nykamp DQ, "The master stability function approach to determine the synchronizability of a network," *Math Insight*. .
[49] Y. Orouskhani, M. Jalili, and X. Yu, "Optimizing Dynamical Network Structure for Pinning Control," *Sci. Rep.*, vol. 6, no. 1, p. 24252, Jul. 2016.
[50] R. B. Nelson, "Simplified calculation of eigenvector derivatives," *Aiaa J.*, vol. 14, no. 9, pp. 1201–1205, 1976.
[51] A. Moradi Amani, M. Jalili, X. Yu, and L. Stone, "Finding the Most Influential Nodes in Pinning Controllability of Complex Networks," *IEEE Trans. Circuits Syst. II Express Briefs*, vol. 64, no. 6, pp. 685–689, 2017.