NON-HOMOGENEOUS P-LAPLACIAN EQUATIONS ON THE SIERPINSKI GASKET

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Abstract. Let $\mathcal{S}$ be the Sierpiński gasket in $\mathbb{R}^2$ and $\mathcal{S}_0$ denote the boundary of $\mathcal{S}$. In this paper, we study the following non-homogeneous p-Laplacian equation

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2}u + f \text{ in } \mathcal{S}\setminus\mathcal{S}_0 \\ u &= 0 \text{ on } \mathcal{S}_0, \end{aligned}$$

where p, q, λ are real numbers such that $\lambda > 0$, $1 < p < q$ and the function $f : \mathcal{S} \to \mathbb{R}$ is suitably chosen. The existence of at least two nontrivial weak solutions to the above non-homogeneous equation on the Sierpiński gasket will be established.

1. Introduction

Let p, q and λ be real numbers such that $1 < p < q$ and $\lambda > 0$. Let $\mathcal{S}$ be the Sierpiński gasket in $\mathbb{R}^2$ and $\mathcal{S}_0$ denote the boundary of $\mathcal{S}$. Consider the following non-homogeneous p-Laplacian equation on the Sierpinski gasket.

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2}u + f \text{ in } \mathcal{S}\setminus\mathcal{S}_0; \\ u &= 0 \text{ on } \mathcal{S}_0, \end{aligned} \tag{1.1}$$

where Δ_p denotes the p-Laplacian and $f : \mathcal{S} \to \mathbb{R}$ is a continuous function.
Let us assume the following hypothesis.

(H1) If $\int_{\mathcal{S}} |u|^q \mathrm{d}\mu = 1$ then

$$\int_{\mathcal{S}} fu\mathrm{d}\mu < K_{p,q}(\|u\|^p_{\mathcal{E}_p})^{\frac{q-1}{q-p}} ,$$

where

$$K_{p,q} = \frac{(q-p)(p-1)^{\frac{p-1}{q-p}}}{\lambda^{\frac{p-1}{q-p}}(q-1)^{\frac{q-1}{q-p}}} .$$

If $u \in \mathrm{dom}_0(\mathcal{E}_p)$ and satisfies

$$\lambda \int_{\mathcal{S}} |u|^{q-2}uv\mathrm{d}\mu + \int_{\mathcal{S}} fv \in \mathcal{E}_p(u,v)$$

for all $v \in \mathrm{dom}_0(\mathcal{E}_p)$, then u is called a weak solution of (1.1).

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The motivation to study non-homogeneous p-Laplacian equation on the Sierpinski gasket came from the study of these equations on regular domains and a brief details is presented below.
Tarentello [11] studied the following problem

$$-\Delta u = |u|^{p-2}u + f \text{ on } \Omega \qquad (1.2)$$
$$u = 0 \text{ on } \partial\Omega$$

and proved that if f satisfies some suitable conditions then (1.2) admits two distinct solutions. In 2007, Hirano studied the following problem

$$-\Delta u + u = |u|^{p-2}u + f$$

where $u \in H^1(\mathbb{R}^N), f \in L^2(\mathbb{R}^N), f \geq 0$ and $f \not\equiv 0$. With these conditions, he was able to show the existence of multiple solutions. Marcos do Ó et al. [6] studied the following quasi linear non-homogeneous elliptic equation.

$$-\Delta u + V(x)|u|^{N-2}u = f(x,u) + \epsilon h(x) \text{ in } \mathbb{R}^N, N \geq 2\ ,$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and $f : \mathbb{R}^N \to \mathbb{R}$ behaves suitably. They used Ekeland variational principle and mountain pass theorem to show the existence of solutions. Hirano and Kim [4] studied the problem:

$$-\Delta u + \alpha u = |u|^{2^*-2}u + f \text{ in } \mathbb{R}^N, N \geq 3\ ,$$

where $2^* = 2N/(N-2), \alpha > 0, f \geq 0, f \not\equiv 0$ and $f \in L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For $3 \leq N \leq 5$ with some suitable conditions, they showed that above problem possesses at least three solutions. A vast literature is available for homogeneous Laplacian problem on the Sierpiński gasket, readers can see [2, 5, 9]. Also, there are some literature on homogeneous p-Laplacian equation on the Sierpiński gasket, readers are encouraged to see [7, 8, 10]. But in case of non-homogeneous problem on the Sierpiński gasket, to our knowledge there is no literature available. This motivated us to study the non-homogeneous problem on the Sierpński gasket.

The outline of our paper is as follows. In Section 2, we discuss about the weak p-Laplacian on Sierpiński gasket and state the main theorem. In Section 3, we define the Euler functional (I) associated to the problem (1.1). We define fibering map (ϕ_u) and do its analysis in Section 4. Finally, in Section 5, we give the detailed proof of main theorem stated in Section 2.

2. Preliminaries and Main Results

Let $\mathcal{S}_0 = \{q_1, q_2, q_3\}$ be three points on $\mathbb{R}^2$ equidistant from each other. Let $F_i(x) = \frac{1}{2}(x - q_i) + q_i$ for $i = 1, 2, 3$ and $F(A) = \cup_{i=1}^3 F_i(A)$. It is well known that F has a unique fixed point $\mathcal{S}$, that is, $\mathcal{S} = F(\mathcal{S})$ (see, for instance, [1, Theorem 9.1]), which is called the Sierpiński gasket. Another way to view this Sierpiński gasket is $\mathcal{S} = \overline{\cup_{j\geq 0} F^j(\mathcal{S}_0)}$, where F^j denotes F composed with itself j times. We know that $\mathcal{S}$ is a compact set in $\mathbb{R}^2$. It is well known that the Hausdorff dimension of $\mathcal{S}$ is $\frac{\ln 3}{\ln 2}$ and the $\frac{\ln 3}{\ln 2}$-dimensional Hausdorff measure is finite ($0 < \mathcal{H}^{\frac{\ln 3}{\ln 2}}(\mathcal{S}) < \infty$) (see, [1, Theorem 9.3]). Throughout this paper, we will use this measure.

We define the p-energy with the help of a three variable real valued function A_p which is convex, homogeneous of degree p, invariant under addition of constant, permutation of indices and the markov property. The m^{th} level Sierpiński gasket is $\mathcal{S}^{(m)} = \cup_{j=0}^{m} F^j(\mathcal{S}_0)$. We construct the m^{th} level crude energy as

$$E_p^{(m)}(u) = \sum_{|\omega|=m} A_p\left(u(F_\omega q_1), u(F_\omega q_2), u(F_\omega q_3)\right)$$

and m^{th} level renormalized p-energy is given by

$$\mathcal{E}_p^{(m)}(u) = (r_p)^{-m} E_p^{(m)}(u)\ ,$$

where r_p is the unique (with respect to p, independent of A_p) renormalizing factor and $0 < r_p < 1$. Now we can observe that $\mathcal{E}_p^{(m)}(u)$ is a monotonically increasing function of m because of renormalization. So we define the p-energy function as

$$\mathcal{E}_p(u) = \lim_{m\to\infty} \mathcal{E}_p^{(m)}(u)$$

which exists for all u as an extended real number. Now we define $\text{dom}(\mathcal{E}_p)$ as the space of continuous functions u satisfying $\mathcal{E}_p(u) < \infty$. The space $\text{dom}(\mathcal{E}_p)$ modulo constant functions forms a Banach space endowed with the norm $\|\cdot\|_{\mathcal{E}_p}$ defined as

$$\|u\|_{\mathcal{E}_p} = \mathcal{E}_p(u)^{1/p}.$$

Now we will proceed to define energy form from energy function as follows:

$$\mathcal{E}_p(u,v) := \frac{1}{p}\ \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_p(u+tv)\Big|_{t=0}. \tag{2.1}$$

Note that we do not know whether $\mathcal{E}_p(u+tv)$ is differentiable or not but we know by the convexity of A_p that $\mathcal{E}_p(u)$ is a convex function. So, we interpret the equation (2.1) as an interval valued equation. That is,

$$\mathcal{E}_p(u,v) = [\mathcal{E}_p^-(u,v), \mathcal{E}_p^+(u,v)]$$

is a nonempty compact interval and the end points are the one-sided derivatives. For more details, see [3].

We recall some results which will be required to prove our results.

Lemma 2.1. [10, Lemma 3.2] *There exists a constant $K_p > 0$ such that for all $u \in dom(\mathcal{E}_p)$ we have*

$$|u(x) - u(y)| \le K_p \mathcal{E}_p(u)^{1/p} (r_p^{1/p})^m$$

whenever x and y belong to the same or adjacent cells of order m.

Lemma 2.2. [8, Lemma 2.2] *If $u \in dom_0(\mathcal{E}_p)$ then there exists a real positive constant K such that $\|u\|_\infty \le K\|u\|_{\mathcal{E}_p}$.*

Lemma 2.3. *Let $\{u_n\}$ be a bounded sequence in $dom_0(\mathcal{E}_p)$. Then $\{u_n\}$ is an equicontinuous family of functions. Moreover, it has a subsequence which converges to a continuous function u_0. Also, $u_0 \in dom_0(\mathcal{E}_p)$.*

Proof. As $\{u_n\}$ is a bounded sequence in $\mathrm{dom}_0(\mathcal{E}_p)$. By Lemma 2.1, there exists a constant $K_p > 0$ such that for all $u \in \mathrm{dom}(\mathcal{E}_p)$ we have $|u(x) - u(y)| \leq K_p(\mathcal{E}_p(u))^{1/p}(r^{1/p})^m$ whenever x and y belongs to the same cell or adjacent cells of order m. Let $B = \sup\{\|u_n\|_{\mathcal{E}_p} : n \in \mathbb{N}\}$ and $\epsilon > 0$ be given. As $0 < r_p < 1$, we can choose $m \in \mathbb{N}$ such that $K_p B(r_p^m)^{1/p} < \epsilon$ and choose $\delta = 2^{-m}$. Then $\|x - y\|_\infty < \delta$ implies that $|u_n(x) - u_n(y)| < \epsilon$ for all $n \in \mathbb{N}$. Hence $\{u_n\}$ is an equicontinuous family of functions. As $\{u_n\} \subset \mathrm{dom}_0(\mathcal{E}_p)$, by Lemma 2.2, we have $\|u_n\|_\infty \leq K\|u_n\|_{\mathcal{E}_p}$ for all $n \in \mathbb{N}$ and hence $\|u_n\|_\infty \leq KB$ for all $n \in \mathbb{N}$. Therefore, $\{u_n\}$ is a uniformly bounded family of functions. By the Arzela-Ascoli theorem, there exists a subsequence of $\{u_n\}$, call it $\{u_{n_k}\}$ converging to a continuous function u_0, that is, $\|u_{n_k} - u_0\|_\infty \to 0$ as $n \to \infty$.
Now we claim that $u_0 \in \mathrm{dom}_0(\mathcal{E}_p)$. To see this, we consider

$$\mathcal{E}_p(u_0) = \sup_m \mathcal{E}_p^{(m)}(u_0) = \sup_m \lim_{n\to\infty} \mathcal{E}_p^{(m)}(u_n) \leq \sup_m \limsup_{n\to\infty} \mathcal{E}_p(u_n) = \limsup_{n\to\infty} \mathcal{E}_p(u_n).$$

This completes the proof. □

The following theorem is our main result.

Theorem 2.4. *Let f satisfies hypothesis (*H1*). Then Problem* (1.1) *has at least two nontrivial weak solutions.*

3. Euler Functional Analysis

The Euler functional conjoin with the problem (1.1) is defined as

$$I(u) = \frac{1}{p}\|u\|_{\mathcal{E}_p}^p - \frac{\lambda}{q}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu - \int_{\mathcal{S}} fu\mathrm{d}\mu. \tag{3.1}$$

Now, we will define a subset of $\mathrm{dom}_0(\mathcal{E}_p)$ in such a way that the Euler functional is bounded below over it. Consider the set

$$\begin{aligned}\mathcal{N}(\mathcal{S}) &= \left\{u \in \mathrm{dom}_0(\mathcal{E}_p) \setminus \{0\} : \lambda\int_{\mathcal{S}} |u|^q \mathrm{d}\mu + \int_{\mathcal{S}} fu\mathrm{d}\mu \in \mathcal{E}_p(u,u)\right\}\\ &= \left\{u \in \mathrm{dom}_0(\mathcal{E}_p) \setminus \{0\} : \lambda\int_{\mathcal{S}} |u|^q \mathrm{d}\mu + \int_{\mathcal{S}} fu\mathrm{d}\mu = \|u\|_{\mathcal{E}_p}^p\right\}\end{aligned}$$

It is easy to verify that $u \in \mathcal{N}(\mathcal{S})$ if and only if

$$\|u\|_{\mathcal{E}_p}^p - \lambda\int_{\mathcal{S}} |u|^q \mathrm{d}\mu - \int_{\mathcal{S}} fu\mathrm{d}\mu = 0. \tag{3.2}$$

Theorem 3.1. *The Euler functional I is coercive and bounded below on $\mathcal{N}(\mathcal{S})$.*

Proof. As $u \in \mathcal{N}(\mathcal{S})$, we have $\|u\|_{\mathcal{E}_p}^p - \lambda \int_{\mathcal{S}} |u|^q \mathrm{d}\mu - \int_{\mathcal{S}} fu = 0$. So,

$$\begin{aligned}
I(u) &= \frac{1}{p}\|u\|_{\mathcal{E}_p}^p - \frac{\lambda}{q}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu - \int_{\mathcal{S}} fu\mathrm{d}\mu \\
&= \frac{1}{p}\|u\|_{\mathcal{E}_p}^p - \frac{1}{q}\left(\|u\|_{\mathcal{E}_p}^p - \int_{\mathcal{S}} fu\right) - \int_{\mathcal{S}} fu\mathrm{d}\mu \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_{\mathcal{E}_p}^p + \left(\frac{1}{q} - 1\right)\int_{\mathcal{S}} fu\mathrm{d}\mu \\
&\geq \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_{\mathcal{E}_p}^p + \left(\frac{1}{q} - 1\right)\int_{\mathcal{S}} |f||u|\mathrm{d}\mu \\
&\geq \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_{\mathcal{E}_p}^p + \left(\frac{1}{q} - 1\right)\|u\|_\infty\int_{\mathcal{S}} |f|\mathrm{d}\mu \\
&\geq \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_{\mathcal{E}_p}^p + \left(\frac{1}{q} - 1\right)K\|u\|_{\mathcal{E}_p}\int_{\mathcal{S}} |f|\mathrm{d}\mu
\end{aligned}$$

This implies $I(u) \to +\infty$ as $\|u\|_{\mathcal{E}_p} \to +\infty$ since, $p > 1$. Hence, the functional I is coercive and bounded below. $\square$

4. Fibering map Analysis

We will define fibering maps for each $u \in \mathrm{dom}_0(\mathcal{E}_p)$ as follows: $\phi_u : (0, +\infty) \to \mathbb{R}$ is defined as $\phi_u(t) = I(tu)$, that is,

$$\phi_u(t) = \frac{t^p}{p}\|u\|_{\mathcal{E}_p}^p - \frac{\lambda t^q}{q}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu - t\int_{\mathcal{S}} fu\mathrm{d}\mu. \tag{4.1}$$

As ϕ_u is a smooth functions we can find its derivatives as follows:

$$\phi_u'(t) = t^{p-1}\|u\|_{\mathcal{E}_p}^p - \lambda t^{q-1}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu - \int_{\mathcal{S}} fu\mathrm{d}\mu. \tag{4.2}$$

$$\phi_u''(t) = (p-1)t^{p-2}\|u\|_{\mathcal{E}_p}^p - (q-1)\lambda t^{q-2}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu. \tag{4.3}$$

Observe that $u \in \mathcal{N}(\mathcal{S})$ if and only if $\phi_u'(1) = 0$ and more generally, $tu \in \mathcal{N}(\mathcal{S})$ if and only if $\phi_{tu}'(1) = 0$, equivalently, $\phi_u'(t) = 0$. For further study, we will subdivide $\mathcal{N}(\mathcal{S})$ into sets corresponding to local minima, local maxima and point of inflection at 1. Define the sets as follows :

$$\begin{aligned}
\mathcal{N}^+(\mathcal{S}) &= \{u \in \mathcal{N}(\mathcal{S}) : \phi_u''(1) > 0\} \\
\mathcal{N}^0(\mathcal{S}) &= \{u \in \mathcal{N}(\mathcal{S}) : \phi_u''(1) = 0\} \\
\text{and } \mathcal{N}^-(\mathcal{S}) &= \{u \in \mathcal{N}(\mathcal{S}) : \phi_u''(1) < 0\}
\end{aligned}$$

Now define the map

$$M_u(t) \coloneqq t^{p-1}\|u\|_{\mathcal{E}_p}^p - \lambda t^{q-1}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu$$

and observe that $\phi_u'(t) = 0$ if and only if $M_u(t) = \int_{\mathcal{S}} fu \mathrm{d}\mu$.

Our aim is to study the nature of graph of M_u for which we compute

$$M_u'(t) = (p-1)t^{p-2}\|u\|_{\mathcal{E}_p}^p - \lambda(q-1)t^{q-2}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu$$

Then $M_u'(t) = 0$. This implies,

$$\begin{aligned} t^{p-2}\left((p-1)\|u\|_{\mathcal{E}_p}^p - \lambda(q-1)t^{q-p}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu\right) &= 0 \\ \Longrightarrow \lambda(q-1)t^{q-p}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu &= (p-1)\|u\|_{\mathcal{E}_p}^p \\ \Longrightarrow t &= \left(\frac{(p-1)\|u\|_{\mathcal{E}_p}^p}{\lambda(q-1)\int_{\mathcal{S}} |u|^q \mathrm{d}\mu}\right)^{\frac{1}{q-p}}. \end{aligned}$$

Let us denote

$$t_0 = \left(\frac{(p-1)\|u\|_{\mathcal{E}_p}^p}{\lambda(q-1)\int_{\mathcal{S}} |u|^q \mathrm{d}\mu}\right)^{\frac{1}{q-p}}. \tag{4.4}$$

It can be observed that $M_u(t) \to -\infty$ as $t \to \infty$. Hence, $M_u(t)$ is increasing on $(0, t_0)$ and decreasing on $(t_0, +\infty)$.

If $\int_{\mathcal{S}} fu \mathrm{d}\mu > 0$ and sufficiently small $\big($*i.e.* $0 < \int_{\mathcal{S}} fu \mathrm{d}\mu < M_u(t_0)\big)$ then $M_u(t) = \int_{\mathcal{S}} fu \mathrm{d}\mu$ has two solutions. Hence, $\phi_u'(t) = 0$ has two solutions, let us say $0 < t_1 < t_0 < t_2$. Also, we can compute to see that $\phi_u''(t_1) > 0$ and $\phi_u''(t_2) < 0$. Therefore, $t_1 u \in \mathcal{N}^+(\mathcal{S})$ and $t_2 u \in \mathcal{N}^-(\mathcal{S})$. If $\int_{\mathcal{S}} fu \mathrm{d}\mu \leq 0$ then $M_u(t) = \int_{\mathcal{S}} fu \mathrm{d}\mu$ has only one solution. This implies, $\phi_u'(t) = 0$ has only one solution, let us say $t_3 > t_0$. Again we can compute to see that $\phi_u''(t_3) < 0$ implying that $t_3 u \in \mathcal{N}^-(\mathcal{S})$.

Lemma 4.1. *Let* $\mathcal{B} = \left\{u \in dom_0(\mathcal{E}_p) : \int_{\mathcal{S}} |u|^q = 1\right\}$. *Let f satisfies hypothesis* (H1)*. Then* $\inf\{K_{p,q}\|u\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} fu \mathrm{d}\mu : u \in \mathcal{B}\} = \delta > 0$ *is achieved on* $\mathcal{B}$.

Proof. Let us define a map $P : \mathrm{dom}_0(\mathcal{E}_p) \to \mathbb{R}$ by

$$P(u) = K_{p,q}\|u\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} fu \mathrm{d}\mu .$$

Clearly,

$$\begin{aligned} P(u) &= K_{p,q}\|u\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} fu \mathrm{d}\mu \\ &\geq K_{p,q}\|u\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \|u\|_\infty \int_{\mathcal{S}} |f| \mathrm{d}\mu \\ &\geq K_{p,q}\|u\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - K\|u\|_{\mathcal{E}_p} \int_{\mathcal{S}} |f| \mathrm{d}\mu \end{aligned}$$

As $\frac{p(q-1)}{q-p} > 1$, we get P is coercive and bounded below. Hence, δ is a finite quantity. Let $\{u_m\} \subset \mathcal{B}$ be a sequence such that $\lim_{m\to\infty}\left(K_{p,q}\|u_m\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} fu_m\right) = \delta$. If $\|u_m\|_{\mathcal{E}_p} \to \infty$ then $P(u_m) \to \infty$ as

P is coercive. This implies $\delta = \infty$, which is a contradiction. Thus, $\{u_m\}$ is bounded in $\mathrm{dom}_0(\mathcal{E}_p)$. Using Lemma 2.3, we can obtain a function $u_0 \in \mathrm{dom}_0(\mathcal{E}_p)$ such that up to a subsequence $u_m \to u_0$ uniformly. By Lebesgue dominated convergence theorem we obtain $\lim_{m\to\infty}\int_{\mathcal{S}}|u_m|^q \mathrm{d}\mu = \int_{\mathcal{S}}|u_0|^q\mathrm{d}\mu = 1$ which gives $u_0 \in \mathcal{B}$. We know $\mathcal{E}_p(u_0) \le \limsup_{m\to\infty}\mathcal{E}_p(u_k)$ then $K_{p,q}\|u_0\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} \le \limsup_{m\to\infty} K_{p,q}\|u_m\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}}$ and $\lim_{m\to\infty}\int_{\mathcal{S}} fu_m\mathrm{d}\mu = \int_{\mathcal{S}} fu_0\mathrm{d}\mu$. This implies,

$$K_{p,q}\|u_0\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} fu_0\mathrm{d}\mu \le \limsup_{m\to\infty}\left(K_{p,q}\|u_m\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} fu_m\mathrm{d}\mu\right) = \delta \le K_{p,q}\|u_0\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} fu_0\mathrm{d}\mu.$$

So, $\delta = K_{p,q}\|u_0\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} fu_0\mathrm{d}\mu$ which proves that, δ is achieved. $\delta > 0$ holds true by hypothesis (H1). □

Corollary 4.2. *For any $\rho > 0$, we have*

$$\inf_{\int_{\mathcal{S}}|u|^q=\rho} \frac{K_{p,q}(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}}|u|^q\mathrm{d}\mu)^{\frac{p-1}{q-p}}} - \int_{\mathcal{S}} fu\mathrm{d}\mu \ge \delta\rho^{1/q}.$$

Proof. Let $\int_{\mathcal{S}}|u|^q\mathrm{d}\mu = \rho$ then $\int_{\mathcal{S}}|\frac{u}{\rho^{1/q}}|^q = 1$. From previous lemma we obtain that

$$\begin{aligned}
&K_{p,q}\left\|\frac{u}{\rho^{1/q}}\right\|_{\mathcal{E}_p}^{\frac{p(q-1)}{q-p}} - \int_{\mathcal{S}} f\left(\frac{u}{\rho^{1/q}}\right)\mathrm{d}\mu \ge \delta \\
i.e.\ &\frac{K_{p,q}\|u\|^{\frac{p(q-1)}{q-p}}}{\rho^{\frac{p(q-1)}{q(q-p)}}} - \frac{1}{\rho^{1/q}}\int_{\mathcal{S}} fu\mathrm{d}\mu \ge \delta \\
i.e.\ &\frac{K_{p,q}\|u\|^{\frac{p(q-1)}{q-p}}}{\rho^{\frac{p(q-1)}{(q-p)}}} - \int_{\mathcal{S}} fu\mathrm{d}\mu \ge \delta\rho^{1/q} \\
i.e.\ &\frac{K_{p,q}\|u\|^{\frac{p(q-1)}{q-p}}}{(\int_{\mathcal{S}}|u|^q)^{\frac{p(q-1)}{(q-p)}}} - \int_{\mathcal{S}} fu\mathrm{d}\mu \ge \delta\rho^{1/q}.
\end{aligned}$$

This holds true for all $u \in \mathrm{dom}_0(\mathcal{E}_p)$ such that $\int_{\mathcal{S}}|u|^q = \rho$. Hence,

$$\inf_{\int_{\mathcal{S}}|u|^q=\rho} \frac{K_{p,q}(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}}|u|^q\mathrm{d}\mu)^{\frac{p-1}{q-p}}} - \int_{\mathcal{S}} fu\mathrm{d}\mu \ge \delta\rho^{1/q}.$$

□

Lemma 4.3. *If f is nonzero and satisfies hypothesis (H1) then $\mathcal{N}^0(\mathcal{S})$ is an empty set.*

Proof. To prove $\mathcal{N}^0(\mathcal{S})$ is empty, we need to show that for any $u \in \mathcal{N}(\mathcal{S})$, $\phi_u(t)$ has no critical point which is a saddle point. From above, $M_u(t)$ has a unique global maximum of at t_0.

Now, consider

$$
\begin{aligned}
M_u(t_0) &= t_0^{p-1}\|u\|_{\mathcal{E}_p}^p - \lambda t_0^{q-1}\int_{\mathcal{S}} |u|^q \mathrm{d}\mu \\
&= \left(\frac{(p-1)\|u\|_{\mathcal{E}_p}^p}{\lambda(q-1)\int_{\mathcal{S}}|u|^q\mathrm{d}\mu}\right)^{\frac{p-1}{q-p}} \|u\|_{\mathcal{E}_p}^p - \lambda\left(\frac{(p-1)\|u\|_{\mathcal{E}_p}^p}{\lambda(q-1)\int_{\mathcal{S}}|u|^q\mathrm{d}\mu}\right)^{\frac{q-1}{q-p}} \int_{\mathcal{S}}|u|^q\mathrm{d}\mu \\
&= \frac{(p-1)^{\frac{p-1}{q-p}}(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}}}{(\lambda(q-1)\int_{\mathcal{S}}|u|^q\mathrm{d}\mu)^{\frac{p-1}{q-p}}} - \frac{\lambda(p-1)^{\frac{q-1}{q-p}}(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}}}{(\lambda(q-1))^{\frac{q-1}{q-p}}(\int_{\mathcal{S}}|u|^q\mathrm{d}\mu)^{\frac{p-1}{q-p}}} \\
&= \frac{(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}}|u|^q\mathrm{d}\mu)^{\frac{p-1}{q-p}}}\left(\frac{(p-1)^{\frac{p-1}{q-p}}}{(\lambda(q-1))^{\frac{p-1}{q-p}}} - \frac{\lambda(p-1)^{\frac{q-1}{q-p}}}{(\lambda(q-1))^{\frac{q-1}{q-p}}}\right) \\
&= \frac{(p-1)^{\frac{p-1}{q-p}}}{(\lambda(q-1))^{\frac{p-1}{q-p}}}\frac{(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}}|u|^q\mathrm{d}\mu)^{\frac{p-1}{q-p}}}\left(1-\frac{p-1}{q-1}\right) \\
&= \left(\frac{q-p}{q-1}\right)\left(\frac{p-1}{\lambda(q-1)}\right)^{\frac{p-1}{q-p}}\frac{(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}}|u|^q\mathrm{d}\mu)^{\frac{p-1}{q-p}}} \\
&= \frac{(q-p)(p-1)^{\frac{p-1}{q-p}}}{\lambda^{\frac{p-1}{q-p}}(q-1)^{\frac{q-1}{q-p}}}\frac{(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}}|u|^q\mathrm{d}\mu)^{\frac{p-1}{q-p}}}.
\end{aligned}
$$

From Hypothesis (H1) we have $\int_{\mathcal{S}} fu\mathrm{d}\mu < \frac{(q-p)(p-1)^{\frac{p-1}{q-p}}}{\lambda^{\frac{p-1}{q-p}}(q-1)^{\frac{q-1}{q-p}}}(\|u\|_{\mathcal{E}_p}^p)^{\frac{q-1}{q-p}} = M_u(t_0)$, so it must have critical points which is either a local minima or local maxima but not a saddle point. This completes the proof. ∎

5. Main results

Lemma 5.1. *Let $\gamma^+ = \inf_{u\in\mathcal{N}^+(\mathcal{S})} I(u)$. Then there exists a constant $C_1 > 0$ such that $\gamma^+ \leq -\frac{(p-1)(q-p)}{pq}C_1$.*

Proof. Let $\hat{u} \in \mathrm{dom}_0(\mathcal{E}_p)$ be a solution of the problem

$$
\begin{cases} -\Delta_p u = f \text{ in } \mathcal{S}\setminus\mathcal{S}_0 \\ u = 0 \text{ in } \mathcal{S}_0 \end{cases} \tag{5.1}
$$

Strichartz and Wong [10] proved existence of a solution $\hat{u}$ to the above problem. So,

$$\int_{\mathcal{S}} f\hat{u}\mathrm{d}\mu = \int_{\mathcal{S}} -\Delta_p(\hat{u})\hat{u} = \|\hat{u}\|_{\mathcal{E}_p}^p > 0.$$

Since hypothesis (H1) holds, we know that there exists $t_1 > 0$ such that $t_1\hat{u} \in \mathcal{N}^+(\mathcal{S})$. We have,

$$I(t_1\hat{u}) = \frac{1}{p}\|t_1\hat{u}\|_{\mathcal{E}_p}^p - \frac{\lambda}{q}\int_{\mathcal{S}}|t_1\hat{u}|^q\mathrm{d}\mu - \int_{\mathcal{S}} t_1 f\hat{u}\mathrm{d}\mu.$$

As $t_1\hat{u} \in \mathcal{N}^+(\mathcal{S})$, it implies that $t_1\hat{u} \in \mathcal{N}(\mathcal{S})$. Hence, $t_1^{p-1}\|\hat{u}\|^p_{\mathcal{E}_p} - \lambda t_1^{q-1}\int_{\mathcal{S}} |\hat{u}|^q \mathrm{d}\mu - t_1\int_{\mathcal{S}} f\hat{u}\mathrm{d}\mu = 0$. Since $\phi''_{t_1\hat{u}}(1) > 0$, we get

$$\begin{aligned}
&(p-1)\|t_1\hat{u}\|^p_{\mathcal{E}_p} - (q-1)\lambda\int_{\mathcal{S}} |t_1\hat{u}|^q \mathrm{d}\mu > 0\\
&i.e.\ (q-1)\lambda\int_{\mathcal{S}} |t_1\hat{u}|^q \mathrm{d}\mu < (p-1)\|t_1\hat{u}\|^p_{\mathcal{E}_p}\\
&i.e.\ \lambda\int_{\mathcal{S}} |t_1\hat{u}|^q \mathrm{d}\mu < \left(\frac{p-1}{q-1}\right)\|t_1\hat{u}\|^p_{\mathcal{E}_p}
\end{aligned} \tag{5.2}$$

Therefore,

$$\begin{aligned}
I(t_1\hat{u}) &= \left(\frac{1}{p}-1\right)\|t_1\hat{u}\|^p_{\mathcal{E}_p} + \left(1-\frac{1}{q}\right)\lambda\int_{\mathcal{S}} |t_1\hat{u}|^q \mathrm{d}\mu\\
&= -\left(\frac{p-1}{p}\right)\|t_1\hat{u}\|^p_{\mathcal{E}_p} + \left(\frac{q-1}{q}\right)\lambda\int_{\mathcal{S}} |t_1\hat{u}|^q \mathrm{d}\mu\\
&< -\left(\frac{p-1}{p}\right)\|t_1\hat{u}\|^p_{\mathcal{E}_p} + \left(\frac{q-1}{q}\right)\left(\frac{p-1}{q-1}\right)\|t_1\hat{u}\|^p_{\mathcal{E}_p}\\
&< -\left(\frac{p-1}{p}\right)\|t_1\hat{u}\|^p_{\mathcal{E}_p} + \left(\frac{p-1}{q}\right)\|t_1\hat{u}\|^p_{\mathcal{E}_p}\\
&= (p-1)\left(\frac{1}{q}-\frac{1}{p}\right)\|t_1\hat{u}\|^p_{\mathcal{E}_p}\\
&< 0.
\end{aligned} \tag{5.3}$$

It gives that $\gamma^+ = \inf_{u\in\mathcal{N}^+(\mathcal{S})} I(u) \le I(t_1\hat{u}) < \frac{(p-1)(p-q)}{pq}C_1$ where $C_1 = \|t_1\hat{u}\|^p_{\mathcal{E}_p} > 0$. Hence proved. □

Theorem 5.2. *Let f satisfies Hypothesis (H1). Then $\inf_{u\in\mathcal{N}^+(\mathcal{S})} I(u)$ is achieved.*

Proof. Since, I is bounded below $\mathcal{N}(\mathcal{S})$, it is also bounded below on $\mathcal{N}^+(\mathcal{S})$. Hence, there exist a sequence $\{u_n\} \in \mathcal{N}^+(\mathcal{S})$ such that

$$\lim_{n\to\infty} I(u_n) = \inf_{u\in\mathcal{N}^+(\mathcal{S})} I(u).$$

Claim(1) : $\{u_n\}$ is bounded on $\mathrm{dom}_0(\mathcal{E}_p)$.
If it is unbounded on $\mathrm{dom}_0(\mathcal{E}_p)$ then there exist a sub sequence $\{u_{n_k}\}$ such that $\|u_{n_k}\|_{\mathcal{E}_p} \to +\infty$ as $k \to +\infty$. So, $\lim_{k\to\infty} I(u_{n_k}) = \lim_{n\to\infty} I(u_n) = +\infty = \inf_{u\in\mathcal{N}^+(\mathcal{S})} I(u)$. This implies, $\mathcal{N}^+(\mathcal{S})$ must be an empty set, which is a contradiction. Hence $\{u_n\}$ is bounded in $\mathrm{dom}_0(\mathcal{E}_p)$. Using Lemma 2.3, we get a subsequence of $\{u_n\}$, still call it $\{u_n\}$, converging uniformly to u_0 and $u_0 \in \mathrm{dom}_0(\mathcal{E}_p)$.
Claim(2) : $\int_{\mathcal{S}} fu_0\mathrm{d}\mu > 0$.

We consider

$$\begin{aligned}I(u_n) &= \frac{1}{p}\|u_n\|^p_{\mathcal{E}_p} - \frac{\lambda}{q}\int_{\mathcal{S}} |u_n|^q \mathrm{d}\mu - \int_{\mathcal{S}} f u_n \mathrm{d}\mu. \\ &= \frac{1}{p}\|u_n\|^p_{\mathcal{E}_p} - \frac{1}{q}\left(\|u_n\|^p_{\mathcal{E}_p} - \int_{\mathcal{S}} f u_n \mathrm{d}\mu\right) - \int_{\mathcal{S}} f u_n \mathrm{d}\mu. \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p_{\mathcal{E}_p} - \left(1 - \frac{1}{q}\right)\int_{\mathcal{S}} f u_n \mathrm{d}\mu.\end{aligned}$$

This implies $\left(1-\frac{1}{q}\right)\int_{\mathcal{S}} f u_n \mathrm{d}\mu = \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p_{\mathcal{E}_p} - I(u_n) \geq -I(u_n)$. Taking limit $n \to \infty$ we get $\int_{\mathcal{S}} f u_0 \mathrm{d}\mu \geq -\left(\frac{q}{q-1}\right)\gamma^+ > 0$ as we know that $\gamma^+ < 0$ from Lemma 5.1.

Claim(3) : $\int_{\mathcal{S}} f u_0 \mathrm{d}\mu < K_{p,q}\frac{(\|u_0\|^p_{\mathcal{E}_p})^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}} |u_0|^q \mathrm{d}\mu)^{\frac{p-1}{q-p}}}$.

Since, $\int_{\mathcal{S}} f u_0 \mathrm{d}\mu > 0$, we get $u_0 \not\equiv 0$. So, $\int_{\mathcal{S}} |u_0|^q \mathrm{d}\mu = a > 0$. From Corollary 4.2 we infer that $\frac{K_{p,q}(\|u_0\|^p_{\mathcal{E}_p})^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}} |u_0|^q \mathrm{d}\mu)^{\frac{p-1}{q-p}}} - \int_{\mathcal{S}} f u_0 \mathrm{d}\mu \geq \delta a^{1/q} > 0$. This implies $0 < \int_{\mathcal{S}} f u_0 \mathrm{d}\mu < \frac{K_{p,q}(\|u_0\|^p_{\mathcal{E}_p})^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}} |u_0|^q \mathrm{d}\mu)^{\frac{p-1}{q-p}}}$.
Hence, there exists $t_{u_0} > 0$ such that $\phi'_{u_0}(t_{u_0}) = 0$ and $t_{u_0}u_0 \in \mathcal{N}^+(\mathcal{S})$ from Lemma 4.3. By Lebesgue dominated convergence theorem, we have $\lim_{n\to\infty}\int_{\mathcal{S}} |u_n|^q = \int_{\mathcal{S}} |u_0|^q$ and $\lim_{n\to\infty}\int_{\mathcal{S}} f u_n \mathrm{d}\mu = \int_{\mathcal{S}} f u_0 \mathrm{d}\mu$. If $\mathcal{E}_p(u_0) < \limsup_{n\to\infty}\mathcal{E}_p(u_n)$ then $\phi'_{u_0}(t) < \limsup_{n\to\infty}\phi'_{u_n}(t)$. Since $\{u_n\} \subset \mathcal{N}^+(\mathcal{S})$, $\phi'_{u_n}(1) = 0$ for all $n \in \mathbb{N}$. Also, $0 = \phi'_{u_0}(t_{u_0}) < \limsup_{n\to\infty}\phi'_{u_n}(t_{u_0})$. This implies $\phi'_{u_n}(t_{u_0}) > 0$ for some large n. Hence, $t_{u_0} \geq 1$. Because of $t_{u_0}u_0 \in \mathcal{N}^+(\mathcal{S})$, if $t_{u_0} > 1$, then

$$\inf_{u\in\mathcal{N}^+(\mathcal{S})} I(u) \leq I(t_{u_0}u_0) = \phi_{u_0}(t_{u_0}) < \phi_{u_0}(1) < \limsup_{n\to\infty}\phi_{u_n}(1) = \limsup_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) = \inf_{u\in\mathcal{N}^+(\mathcal{S})} I(u)$$

which is a contradiction. Thus, $t_{u_0} = 1$ and $\mathcal{E}_p(u_0) = \limsup_{n\to\infty}\mathcal{E}_p(u_n)$ Hence,

$$I(u_0) = \limsup_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) = \inf_{u\in\mathcal{N}^+(\mathcal{S})} I(u)$$

This proves u_0 is a minimizer of I on $\mathcal{N}^+(\mathcal{S})$. □

Theorem 5.3. *Let $\gamma^- = \inf_{v\in\mathcal{N}^-(\mathcal{S})} I(v)$. Then γ^- is achieved.*

Proof. Since I is bounded below on $\mathcal{N}(\mathcal{S})$, there exists a sequence $\{v_n\} \subset \mathcal{N}^-(\mathcal{S})$ such that $\lim_{n\to\infty} I(v_n) = \gamma^-$. As I is coercive and bounded below, it can be inferred that v_n is bounded in $\mathrm{dom}_0(\mathcal{E}_p)$. Applying Lemma 2.3, obtain a subsequence of $\{v_n\}$, still call it $\{v_n\}$ such that v_n converges to a function v_0 uniformly and $v_0 \in \mathrm{dom}_0(\mathcal{E}_p)$. By Lebesgue dominated convergence theorem, it follows that

$$\lim_{n\to\infty}\int_{\mathcal{S}} |v_n|^q \mathrm{d}\mu = \int_{\mathcal{S}} |v_0|^q \mathrm{d}\mu \text{ and } \lim_{n\to\infty}\int_{\mathcal{S}} f v_n \mathrm{d}\mu = \int_{\mathcal{S}} f v_0 \mathrm{d}\mu.$$

Using Corollary 4.2, we obtain

$$\int_{\mathcal{S}} f v_0 \mathrm{d}\mu < \frac{K_{p,q}(\|v_0\|^p_{\mathcal{E}_p})^{\frac{q-1}{q-p}}}{(\int_{\mathcal{S}} |v_0|^q \mathrm{d}\mu)^{\frac{p-1}{q-p}}}.$$

Hence, there exists t_{v_0} such that $t_{v_0}v_0 \in \mathcal{N}^-(\mathcal{S})$. If we presume $\mathcal{E}_p(v_0) < \limsup_{n\to\infty} \mathcal{E}_p(v_n)$ then we get $\phi'_{v_0}(t) < \limsup_{n\to\infty} \phi'_{v_n}(t)$. Since $\{v_n\} \subset \mathcal{N}^-(\mathcal{S})$, $\phi'_{v_n}(1) = 0$ for all $n \in \mathbb{N}$. Also, $0 = \phi'_{v_0}(t_{v_0}) < \limsup_{n\to\infty} \phi'_{v_n}(t_{v_0})$ which implies $\phi'_{v_n}(t_{v_0}) > 0$ for some large n. Hence, $t_{v_0} \leq 1$. As $t_{v_0}v_0 \in \mathcal{N}^-(\mathcal{S})$, we get

$$I(t_{v_0}v_0) = \phi_{v_0}(t_{v_0}) < \limsup_{n\to\infty} \phi_{v_n}(t_{v_0}) \leq \limsup_{n\to\infty} \phi_{v_n}(1) \leq \limsup_{n\to\infty} I(v_n) = \lim_{n\to\infty} I(v_n) = \inf_{v\in\mathcal{N}^-(\mathcal{S})} I(v)$$

which is a contradiction. Hence, $\mathcal{E}_p(v_0) = \limsup_{n\to\infty} \mathcal{E}_p(v_n)$. This implies $\phi'_{v_0}(1) = \limsup_{n\to\infty} \phi'_{v_n}(1) = 0$ and $\phi''_{v_0}(1) \leq 0$. From Lemma 4.3, $\mathcal{N}^0(\mathcal{S})$ is an empty set. Hence $\phi''_{v_0}(1) < 0$ and $v_0 \in \mathcal{N}^-(\mathcal{S})$. So,

$$I(v_0) = \limsup_{n\to\infty} I(v_n) = \lim_{n\to\infty} I(v_n) = \inf_{v\in\mathcal{N}^-(\mathcal{S})} I(v)$$

Therefore v_0 is a minimizer of I on $\mathcal{N}^-(\mathcal{S})$. $\square$

Lemma 5.4. *Let $u_0 \in \mathcal{N}^+(\mathcal{S})$ be such that $I(u_0) = \inf_{u\in\mathcal{N}^+(\mathcal{S})} I_\lambda(u)$ and $v_0 \in \mathcal{N}^-(\mathcal{S})$ be such that $I_\lambda(v_0) = \inf_{v\in\mathcal{N}^-(\mathcal{S})} I(v)$. Then for each $w \in dom_0(\mathcal{E}_p)$, the following hold true :*

(i) there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ there exists a unique $t_\epsilon > 0$ such that $t_\epsilon(u_0 + \epsilon w) \in \mathcal{N}^+(\mathcal{S})$. Also, $t_\epsilon \to 1$ as $\epsilon \to 0$.

(ii) there exists $\epsilon_1 > 0$ such that for each $\epsilon \in (-\epsilon_1, \epsilon_1)$ there exists a unique $\tilde{t}_\epsilon > 0$ such that $\tilde{t}_\epsilon(v_0 + \epsilon w) \in \mathcal{N}^-(\mathcal{S})$. Also, $\tilde{t}_\epsilon \to 1$ as $\epsilon \to 0$.

Proof. (i) Let us define a function $\mathbb{F} : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}$ by

$$\mathbb{F}(x, y, z, t) = xt^{p-1} - \lambda y t^{q-1} - z.$$

Then

$$\frac{\partial \mathbb{F}}{\partial t}(x, y, z, t) = (p-1)xt^{p-2} - (q-1)\lambda y t^{q-2}.$$

Since $u_0 \in \mathcal{N}^+_\lambda(\mathcal{S})$, $\phi'_{u_0}(1) = 0$ and $\phi''_{u_0}(1) > 0$. Therefore,

$$\mathbb{F}\left(\|u_0\|^p_{\mathcal{E}_p}, \int_{\mathcal{S}} |u_0|^q \mathrm{d}\mu, \int_{\mathcal{S}} f u_0 \mathrm{d}\mu, 1\right) = \phi'_{u_0}(1) = 0$$

and

$$\frac{\partial \mathbb{F}}{\partial t}\left(\|u_0\|^p_{\mathcal{E}_p}, \int_{\mathcal{S}} |u_0|^q \mathrm{d}\mu, \int_{\mathcal{S}} f u_0 \mathrm{d}\mu, 1\right) = \phi''_{u_0}(1) > 0.$$

The function $\mathtt{f}_1(\epsilon) = \int_{\mathcal{S}} f(u_0 + \epsilon w)\mathrm{d}\mu$ is a continuous function and $\mathtt{f}_1(0) > 0$ by Theorem 5.2. By the continuity of $\mathtt{f}_1$, there exists $\epsilon_0 > 0$ such that $\mathtt{f}_1(\epsilon) > 0$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. So, for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ there exists t_ϵ such that $t_\epsilon(u_0 + \epsilon w) \in \mathcal{N}^+(\mathcal{S})$. This implies that

$$\mathbb{F}\left(\|u_0 + \epsilon w\|^p_{\mathcal{E}_p}, \int_{\mathcal{S}} |u_0 + \epsilon w|^q \mathrm{d}\mu, \int_{\mathcal{S}} f(u_0 + \epsilon w)\mathrm{d}\mu, t_\epsilon\right) = \phi'_{u_0+\epsilon w}(t_\epsilon) = 0.$$

By the implicit function theorem, there exists an open set $X \subset (0,\infty)$ containing 1, an open set $Y \subset \mathbb{R}^3$ containing $(\|u_0\|^p_{\mathcal{E}_p}, \int\limits_{\mathcal{S}} |u_0|^q \mathrm{d}\mu, \int\limits_{\mathcal{S}} f u_0 \mathrm{d}\mu)$ and a continuous function $g : Y \to X$ such that for all $\mathfrak{y} \in Y$, $\mathbb{F}(\mathfrak{y}, g(\mathfrak{y})) = 0$. So there exists a unique solution to the equation $t = g(\mathfrak{y}) \in X$. Hence,

$$t_\epsilon = g\left(\|u_0 + \epsilon w\|^p_{\mathcal{E}_p}, \int\limits_{\mathcal{S}} |u_0 + \epsilon w|^q \mathrm{d}\mu, \int\limits_{\mathcal{S}} f(u_0 + \epsilon w)\mathrm{d}\mu\right)$$

Letting $\epsilon \to 0$ and using the continuity of g, we get

$$1 = g\left(\|u_0\|^p_{\mathcal{E}_p}, \int\limits_{\mathcal{S}} |u_0|^q \mathrm{d}\mu, \int\limits_{\mathcal{S}} f u_0 \mathrm{d}\mu\right)$$

Therefore, $t_\epsilon \to 1$ as $\epsilon \to 0$.

(ii) This can be proved by taking the function

$$\mathtt{f}_2(\epsilon) = \int_{\mathcal{S}} f(v_0 + \epsilon w)\mathrm{d}\mu$$

in place of $\mathtt{f}_1(\epsilon)$ and proceeding in a similar fashion as in the proof of (i). $\square$

Theorem 5.5. *If u_0 is a minimizer of I on $\mathcal{N}^+(\mathcal{S})$ and v_0 is a minimizer of I_λ on $\mathcal{N}^-(\mathcal{S})$, then u_0 and v_0 are weak solutions to the problem* (1.1).

Proof. Let $\psi \in \mathrm{dom}_0(\mathcal{E}_p)$. Using Lemma 5.4(i), there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ there exists $\bar{t_\epsilon}$ such that $I_\lambda(\bar{t_\epsilon}(u_0 + \epsilon\psi)) \geq I_\lambda(u_0)$ and $t_\epsilon \to 1$ as $\epsilon \to 0$. Then

$$\begin{aligned}
0 &\leq \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(u_0+\epsilon\psi)) - I_\lambda(u_0)\right)\\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(u_0+\epsilon\psi)) - I_\lambda(t_\epsilon u_0) + I_\lambda(t_\epsilon u_0) - I_\lambda(u_0)\right)\\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(u_0+\epsilon\psi)) - I_\lambda(t_\epsilon u_0)\right)\\
&= \lim_{\epsilon\to 0^+}\left(\frac{1}{p}\frac{1}{\epsilon}\left(\|t_\epsilon(u_0+\epsilon\psi)\|^p_{\mathcal{E}_p} - \|t_\epsilon u_0\|^p_{\mathcal{E}_p}\right) - \frac{\lambda}{q}\frac{1}{\epsilon}\left(\int\limits_{\mathcal{S}} |t_\epsilon(u_0+\epsilon\psi)|^q \mathrm{d}\mu - \int\limits_{\mathcal{S}} |t_\epsilon u_0|^q \mathrm{d}\mu\right)\right)\\
&\quad - \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\left(\int\limits_{\mathcal{S}} f(t_\epsilon(u_0+\epsilon\psi))\mathrm{d}\mu - \int\limits_{\mathcal{S}} f t_\epsilon u_0 \mathrm{d}\mu\right)\\
&= \mathcal{E}^+_p(u_0,\psi) - \lambda\int_{\mathcal{S}} |u_0|^{q-2}u_0\psi\mathrm{d}\mu - \int_{\mathcal{S}} f\psi\mathrm{d}\mu.
\end{aligned}$$

Note that the second equality follows by using $\lim\limits_{\epsilon\to 0^+} \frac{1}{\epsilon}(I_\lambda(\bar{t_\epsilon} u_0) - I_\lambda(u_0)) = 0$ because the limit is the same as $\phi'_{u_0}(1)$, which is zero. This implies that

$$\lambda\int_{\mathcal{S}} |u_0|^{q-2}u_0\psi\mathrm{d}\mu + \int_{\mathcal{S}} f\psi\mathrm{d}\mu \leq \mathcal{E}^+_p(u_0,\psi).$$

Similarly,

$$
\begin{aligned}
0 &\geq \lim_{\epsilon\to 0^-} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(u_0+\epsilon\psi)) - I_\lambda(u_0)\right)\\
&= \lim_{\epsilon\to 0^-} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(u_0+\epsilon\psi)) - I_\lambda(t_\epsilon u_0) + I_\lambda(t_\epsilon u_0) - I_\lambda(u_0)\right)\\
&= \lim_{\epsilon\to 0^-} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(u_0+\epsilon\psi)) - I_\lambda(t_\epsilon u_0)\right)\\
&= \mathcal{E}_p^-(u_0,\psi) - \lambda\int_{\mathcal{S}} |u_0|^{q-2}u_0\psi \mathrm{d}\mu - \int_{\mathcal{S}} f\psi \mathrm{d}\mu
\end{aligned}
$$

which implies that

$$
\lambda\int_{\mathcal{S}} |u_0|^{q-2}u_0\psi \mathrm{d}\mu + \int_{\mathcal{S}} f\psi \mathrm{d}\mu \geq \mathcal{E}_p^-(u_0,\psi).
$$

So,

$$
\mathcal{E}_p^-(u_0,\psi) \leq \lambda\int_{\mathcal{S}} |u_0|^{q-2}u_0\psi \mathrm{d}\mu + \int_{\mathcal{S}} f\psi \mathrm{d}\mu \leq \mathcal{E}_p^+(u_0,\psi).
$$

Hence

$$
\lambda\int_{\mathcal{S}} |u_0|^{q-2}u_0\psi \mathrm{d}\mu + \int_{\mathcal{S}} f\psi \mathrm{d}\mu \in \mathcal{E}_p(u_0,\psi)
$$

for all $\psi \in \mathrm{dom}_0(\mathcal{E}_p)$. Therefore, u_0 is a weak solution to the problem (1.1).

Using similar arguments as in Lemma 5.4(ii), there exists $\epsilon_1 > 0$ such that for each $\epsilon \in (-\epsilon_1, \epsilon_1)$ there exists t_ϵ such that $I_\lambda(t_\epsilon(v_0+\epsilon\psi)) \geq I_\lambda(v_0)$ and $t_\epsilon \to 1$ as $\epsilon \to 0$. Then we have

$$
\begin{aligned}
0 &\leq \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(v_0+\epsilon\psi)) - I_\lambda(v_0)\right)\\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(v_0+\epsilon\psi)) - I_\lambda(t_\epsilon v_0) + I_\lambda(t_\epsilon v_0) - I_\lambda(v_0)\right)\\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(I_\lambda(t_\epsilon(v_0+\epsilon\psi)) - I_\lambda(t_\epsilon v_0)\right)\\
&= \lim_{\epsilon\to 0^+} \left(\frac{1}{p}\frac{1}{\epsilon}\left(\|t_\epsilon(v_0+\epsilon\psi)\|_{\mathcal{E}_p}^p - \|t_\epsilon v_0\|_{\mathcal{E}_p}^p\right) - \frac{\lambda}{q}\frac{1}{\epsilon}\left(\int\limits_{\mathcal{S}} |t_\epsilon(v_0+\epsilon\psi)|^q \mathrm{d}\mu - \int\limits_{\mathcal{S}} |t_\epsilon v_0|^q \mathrm{d}\mu\right)\right)\\
&\quad - \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(\int\limits_{\mathcal{S}} f(t_\epsilon(v_0+\epsilon\psi))\mathrm{d}\mu - \int\limits_{\mathcal{S}} f t_\epsilon v_0 \mathrm{d}\mu\right)\\
&= \mathcal{E}_p^+(v_0,\psi) - \lambda\int_{\mathcal{S}} |v_0|^{q-2}v_0\psi \mathrm{d}\mu - \int_{\mathcal{S}} f\psi \mathrm{d}\mu.
\end{aligned}
$$

It implies that

$$
\lambda\int_{\mathcal{S}} |v_0|^{q-2}v_0\psi \mathrm{d}\mu + \int_{\mathcal{S}} f\psi \mathrm{d}\mu \leq \mathcal{E}_p^+(v_0,\psi).
$$

Similarly,

$$
\begin{aligned}
0 &\geq \lim_{\epsilon \to 0^-} \frac{1}{\epsilon} \left(I_\lambda(t_\epsilon(v_0 + \epsilon\psi)) - I_\lambda(v_0) \right) \\
&= \lim_{\epsilon \to 0^-} \frac{1}{\epsilon} \left(I_\lambda(t_\epsilon(v_0 + \epsilon\psi)) - I_\lambda(t_\epsilon v_0) + I_\lambda(t_\epsilon v_0) - I_\lambda(v_0) \right) \\
&= \lim_{\epsilon \to 0^-} \frac{1}{\epsilon} \left(I_\lambda(t_\epsilon(v_0 + \epsilon\psi)) - I_\lambda(t_\epsilon v_0) \right) \\
&= \mathcal{E}_p^-(v_0, \psi) - \lambda \int_{\mathcal{S}} |v_0|^{q-2} v_0 \psi \mathrm{d}\mu - \int_{\mathcal{S}} f\psi \mathrm{d}\mu
\end{aligned}
$$

which implies that

$$\lambda \int_{\mathcal{S}} |v_0|^{q-2} v_0 \psi \mathrm{d}\mu + \int_{\mathcal{S}} f\psi \mathrm{d}\mu \geq \mathcal{E}_p^-(v_0, \psi).$$

So,

$$\mathcal{E}_p^-(v_0, \psi) \leq \lambda \int_{\mathcal{S}} |v_0|^{q-2} v_0 \psi \mathrm{d}\mu + \int_{\mathcal{S}} f\psi \mathrm{d}\mu \leq \mathcal{E}_p^+(v_0, \psi)$$

Hence

$$\lambda \int_{\mathcal{S}} |v_0|^{q-2} v_0 \psi \mathrm{d}\mu + \int_{\mathcal{S}} f\psi \mathrm{d}\mu \in \mathcal{E}_p(v_0, \psi)$$

for all $\psi \in \mathrm{dom}_0(\mathcal{E}_p)$. Therefore, v_0 is a weak solution of the problem (1.1). □

Now we give the proof of Theorem 2.4 below.

Proof. (proof of Theorem 2.4) Combining all the above results, we get two distinct non-trivial solutions of the problem (1.1). □

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