Infinitely many solutions to fractional differential equations with instantaneous and non-instantaneous impulses

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Abstract: The goal of this paper is to study fractional differential equations involving instantaneous and non-instantaneous impulses with Sturm-Liouville boundary conditions. By using critical point theory and variational approach, infinitely many solutions are obtained. The interesting point is that the potential has an oscillating asymptotic behavior. Also one example is presented to illustrate the main result.

Keywords: Variational approach; Fractional differential equations; Instantaneous impulses; Non-instantaneous impulses; Sturm-Liouville boundary conditions.

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1 Introduction

In scientific research, many physical, chemical and biological phenomena can be described by differential equations. In recent years, the study of differential equations has attracted much attention, especially the differential equations with impulsive effect. The most outstanding characteristic of impulsive differential system is that it can fully consider the impact of sudden change on the state and can more profoundly reflect the changing law of things. In real life, many phenomena affected by external uncertainty will change suddenly. According to the duration of action, they can be divided into instantaneous impulses and non-instantaneous impulses [1, 3–5, 7–9, 12, 14].

V'Milman first proposed the instantaneous impulses [8] in 1960. However, in many practical applications, instantaneous impulses cannot describe all phenomena. In 2013, non-instantaneous

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impulses were first introduced by Hernandez and O'Regan [3]. They studied the existence of solutions for the following differential equations

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ u(t) = g_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, ..., N, \\ u(0) = x_0. \end{cases} \tag{1.1}$$

Since then, the study of non-instantaneous impulses attracts much attention. In [7], Bai and Nieto first studied the variational structure of the following differential equations with non-instantaneous impulses

$$\begin{cases} -u''(t) = \sigma_i(t), t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ u'(t) = \alpha_i, t \in (t_i, s_i], i = 1, 2, ..., N, \\ u'(s_i^-) = u'(s_i^+), i = 1, 2, ..., N, \\ u(0) = u(T) = 0, u'(0) = \alpha_0. \end{cases} \tag{1.2}$$

They used the variational method to get existence and uniqueness of weak solutions as critical points. In [12], Tian and Zhang studied the following second-order differential equations with instantaneous and non-instantaneous impulses

$$\begin{cases} -u''(t) = f_i(t, u(t)), t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ \Delta u'(t_i) = I_i(u(t_i)), i = 1, 2, ..., N, \\ u'(t) = u'(t_i^+), t \in (t_i, s_i], i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-), i = 1, 2, ..., N, \\ u(0) = u(T) = 0. \end{cases} \tag{1.3}$$

By using a variational method, the existence of at least one classical solution was proved.

On the other hand, fractional differential equations have been widely used in the fields of viscoelasticity, neuron and electrochemistry. We suggest that readers refer to the literatures [2, 6, 10, 11, 13, 15].

To our best knowledge, the existence of at least one or two solutions of impulsive differential equations with Dirichlet boundary conditions are obtained recently. However, there are few papers studying the existence of infinitely many solutions for fractional differential equations involving instantaneous and non-instantaneous impulses with Strum-Liouville boundary conditions. In order to fill this gap, we will consider the following problem

$$\begin{cases} -\frac{d}{dt}(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))) = f_i(t, u(t)), t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ \Delta(\frac{1}{2}{}_0D_t^{-\beta}u'(t_i) + \frac{1}{2}{}_tD_T^{-\beta}u'(t_i)) = I_i(u(t_i)), i = 1, 2, ..., N, \\ \frac{1}{2}{}_0D_t^{-\beta}u'(t) + \frac{1}{2}{}_tD_T^{-\beta}u'(t) = \frac{1}{2}{}_0D_t^{-\beta}u'(t_i^+) + \frac{1}{2}{}_tD_T^{-\beta}u'(t_i^+), t \in (t_i, s_i], i = 1, 2, ..., N, \\ \frac{1}{2}{}_0D_t^{-\beta}u'(s_i^+) + \frac{1}{2}{}_tD_T^{-\beta}u'(s_i^+) = \frac{1}{2}{}_0D_t^{-\beta}u'(s_i^-) + \frac{1}{2}{}_tD_T^{-\beta}u'(s_i^-), i = 1, 2, ..., N, \\ au(0) - b(\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(0) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(0)) = 0, \\ cu(T) + d(\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(T) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(T)) = 0, \end{cases} \tag{1.4}$$

where $\beta \in [0, 1)$, ${}_0D_t^{-\beta}$, ${}_tD_T^{-\beta}$ are the left and right Riemann-Lionville fractional integrals of β order, respectively, $a, c > 0, b, d \geq 0$ and $0 = s_0 < t_1 < s_1 < ... < s_N < t_{N+1} = T$, $I_i \in C(\mathrm{R},\mathrm{R})$, and $f_i \in C((s_i, t_{i+1}] \times \mathrm{R},\mathrm{R})(i = 1, 2, ..., N)$,

$$\Delta(\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(t_i) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(t_i)) = (\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(t_i^+) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(t_i^+)) - (\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(t_i^-) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(t_i^-)),$$

$$\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(t_i^\pm) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(t_i^\pm) = \lim_{t \to t_i^\pm}(\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(t_i) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(t_i)),$$

$$\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(s_i^\pm) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(s_i^\pm) = \lim_{t \to s_i^\pm}(\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(s_i) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(s_i)).$$

The new insights presented in the paper are as follows. Firstly, following on from problem (1.3), we generalize the second-order differential equations to fractional differential equations and establish the variational structure of problem (1.4). Secondly, compared with (1.3), problem (1.4) extends Dirichlet boundary conditions to Sturm-Liouville boundary conditions. Considering instantaneous impulses, non-instantaneous impulses and Sturm-Liouville boundary conditions at the same time, we overcome the difficulty that the weak solution of problem (1.4) is a classical solution. If $\beta = 0, b = 0$ and $d = 0$, problem (1.4) reduces to problem (1.3). Problem (1.4) expands the range of application of problem (1.3). Thirdly, we consider the case that the potential F has an oscillating asymptotic behavior and prove the existence of infinitely many solutions of problem (1.4). Our results generalize and complement the existing results in the literature [12].

This paper is organized as follows. In section 2, we present some fundamental definitions and lemmas for $bd \neq 0$. Moreover, the variational structure of problem (1.4) is revealed. In section 3, we prove that the weak solution of problem (1.4) is a classical solution. Finally we give the main results for $bd \neq 0$. In section 4, we discuss three cases and get the main results for $bd = 0$. In section 5, we give an example to illustrate the main results for $bd \neq 0$.

2 Preliminaries for $bd \neq 0$

In this section, we recall some basic definitions and lemmas for the fractional space and functional φ. For the definitions of fractional integrals and derivatives, we refer the reader to monographs [2, 6, 10, 11, 13, 15].

Definition 2.1. Let $\alpha \in (\frac{1}{2}, 1], p \in [1, +\infty)$. The fractional derivative space

$$E^{\alpha,p} = \{u : [0,T] \to \text{R}: u \text{ is absolutely continuous and } {}_0^cD_t^\alpha u \in L^p([0,T],\text{R})\}$$

is defined by the closure of $C^\infty([0,T],\text{R})$ with the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p + |{}_0^cD_t^\alpha u(t)|^p dt\right)^{\frac{1}{p}}.$$

Remark 2.1. When $p = 2$, we write $E^{\alpha,2} = E^\alpha$ and let $X = E^\alpha$. It is obvious that the fractional derivative space $E^{\alpha,p}$ is the space of function $u \in L^p([0,T],\text{R})$ having an $\alpha - order$ Caputo factional derivative ${}_0^cD_t^\alpha u \in L^p([0,T],\text{R})$.

Lemma 2.1. *Let $\alpha \in (0,1], p \in (1,+\infty)$. The space $E^{\alpha,p}$ is a reflexive and separable Banach space. Now set*

$$\tilde{X} := \left\{u \in X : \int_0^T u(s)ds = 0\right\},$$

we can split X into $X = R\bigoplus \tilde{X}$, and each $u \in X$ can be uniquely written as $u = \bar{u} + \tilde{u}$, where $\bar{u} \in R$ and $\tilde{u} \in \tilde{X}$.

Proof. From Lemma 4.2 of [11], we get X is a reflexive and separable Banach space.

Firstly, we shall prove $X = \text{R} + \tilde{X}$. By Lemma 2.22 from [6], one obtains

$$u(t) = u(\xi) + (I_{\xi+}^\alpha \, {}^cD_{\xi+}^\alpha)u(t),$$

where $u(\xi) = \frac{\int_0^T u(s)ds}{T} \in \text{R}$. Since

$$\int_0^T (I_{\xi+}^\alpha \, {}^cD_{\xi+}^\alpha)u(t)dt = \int_0^T (u(t) - u(\xi))\, dt = \int_0^T \left(u(t) - \frac{1}{T}\int_0^T u(s)ds\right) dt = 0,$$

we have $(I_{\xi+}^\alpha \, {}^cD_{\xi+}^\alpha)u(t) \in \tilde{X}$.

Secondly, we will show $\text{R}\bigcap \tilde{X} = \{0\}$. For $u \in \text{R}\bigcap \tilde{X}$, we have $u \equiv c$ and

$$\int_0^T u(s)ds = cT = 0,$$

which means $c = 0$. The proof is completed. □

Lemma 2.2. *[11] Let $\alpha \in (\frac{1}{2}, 1]$, $b, d > 0$ and $u \in E^\alpha$. The norm $\|u\|_{\alpha,2}$ is equivalent to*

$$\|u\| = \Big(-\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)dt + \frac{c}{d}(u(T))^2 + \frac{a}{b}(u(0))^2\Big)^{\frac{1}{2}}.$$

Moreover, one has $\|u\|_{\alpha,2} \le M_1\|u\|$, where

$$M_1 = \Big(\max\Big\{2T\frac{b}{a}, -\frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2\cos\pi\alpha}\Big\} - \frac{1}{\cos\pi\alpha}\Big)^{\frac{1}{2}}.$$

Lemma 2.3. *[11] For $u \in E^\alpha$, there exists $M_3 > 0$ such that $\|u\|_\infty \le M_3\|u\|$, where*

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|,$$

$$M_3 = \sqrt{2}M_1 \max\Big\{T^{-\frac{1}{2}}, \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha+1)}\Big\} + \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}\sqrt{|\cos\pi\alpha|}}$$

and M_1 is defined in lemma 2.2.

Let $\alpha = 1 - \frac{\beta}{2}$, $b, d > 0$, then $\alpha \in (\frac{1}{2}, 1]$. We define energy functional $\varphi : X \to$R by

$$\begin{aligned}\varphi(u) = &-\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt + \frac{c}{2d}(u(T))^2 + \frac{a}{2b}(u(0))^2\\ &+\sum_{i=1}^N \int_0^{u(t_i)} I_i(s)ds - \sum_{i=0}^N \int_{s_i}^{t_{i+1}} F_i(t, u(t))dt,\end{aligned} \tag{2.1}$$

where $F_i(t, u) = \int_0^u f_i(t, s)ds$. Since f_i and I_i are continuous, we can have $\varphi \in C^1(X,$R$)$ and

$$\begin{aligned}\langle\varphi'(u), v\rangle = &\frac{1}{2}\int_0^T ({}_0D_t^{-\beta} u'(t) + {}_tD_T^{-\beta} u'(t))v'(t)dt + \frac{c}{d}u(T)v(T) + \frac{a}{b}u(0)v(0)\\ &+\sum_{i=1}^N I_i(u(t_i))v(t_i) - \sum_{i=0}^N \int_{s_i}^{t_{i+1}} f_i(t, u(t))v(t)dt.\end{aligned} \tag{2.2}$$

Definition 2.2. A function $u \in X$ is said to be a weak solution of problem (1.4) , if u satisfied $\langle\varphi'(u), v)\rangle = 0$ for all $v \in X$.

Definition 2.3. A funtion $u \in X$ such that ${}_0D_t^{-\beta} u'(t), {}_tD_t^{-\beta} u'(t) \in C^1(s_i, t_{i+1}]$ is said to be a classical solution of problem (1.4), If u satisfies equations, instantaneous impulses, non-instantaneous impulses and Sturm-Liouville boundary conditions in problem (1.4).

Lemma 2.4. *The function $\varphi : X \to R$ is weakly lower semi-continuous.*

Proof. Let $\varphi(u) = \varphi_1(u) + \varphi_2(u)$, where

$$\varphi_1(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt + \frac{c}{2d}(u(T))^2 + \frac{a}{2b}(u(0))^2,$$

$$\varphi_2(u)=\sum_{i=1}^{N}\int_0^{u(t_i)}I_i(s)ds+\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,u(t))dt.$$

Firstly, by $\varphi_1(u) = \frac{1}{2}\|u\|^2$, we have that φ_1 is continuous, which means φ_1 is lower semi-continuous. From Lemma 5.3 of [11], since φ_1 is convex, we have φ_1 is weakly lower semi-continuous. Then, let $(u_k)_{k=1}^{\infty}$ be weakly convergent to u in E^{α}. From proposition 5.5 of [10], we get $u_k \to u$ in $C[0,T]$. Together with $I_i \in C(\mathrm{R},\mathrm{R}), f_i \in C((s_i,t_{i+1}]\times\mathrm{R},\mathrm{R})$, we have

$$\begin{aligned}\lim_{k\to\infty}\varphi_2(u_k)&=\lim_{k\to\infty}\left(\sum_{i=1}^{N}\int_o^{u_k(t_i)}I_i(s)ds-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,u_k(t))dt\right)\\&=\sum_{i=1}^{N}\int_0^{u(t_i)}I_i(s)ds-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,u(t))dt\\&=\varphi_2(u).\end{aligned}$$

Obviously, φ_2 is weakly continous . Therefore φ_2 is weakly lower semi-continuous. The proof is completed. □

3 Main results for $bd \neq 0$

In this section, we study the case that the potential F has an oscillating asymptotic behavior. In order to prove the existence of infinitely many solutions for problem (1.4), we assume

(H_1) there exist constant $\alpha > 0$ such that $|I_i(x)| \le \alpha|x|$ for all $x \in\mathrm{R}$.

(H_2) there exist $\beta(t) \in L^1([0,T],\mathrm{R})$ such that $|f_i(t,x)| \le \beta(t)|x|$ for a.e $t \in [0,T]$ and all $x \in\mathrm{R}$. In addition, $\|\beta(t)\|_{L_1} < \frac{1}{M_3^2} - N\alpha$, where $\alpha \le \frac{1}{NM_3^2}$ and M_3 is defined in Lemma 2.3.

$$(H_3)(i)\liminf_{s\to+\infty}\sup_{e\in R,|e|=s}\left\{\frac{c}{2d}e^2+\frac{a}{2b}e^2+\sum_{i=1}^{N}\int_0^{e}I_i(s)ds-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,e)dt\right\}=-\infty,$$

$$(ii)\liminf_{s\to+\infty}\sup_{l\in R,|l|=s}\left\{\frac{(\|l\|+M_3|l|\|\beta\|_{L_1}+N\alpha M_3)^2}{2-2M_3^2\|\beta\|_{L_1}-2N\alpha M_3^2}-\sum_{i=1}^{N}\int_0^{l}I_i(s)ds+\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,l)dt\right\}$$
$$<+\infty.$$

Lemma 3.1. *Suppose that (H_1) and (H_2) are satisfied. If $\varphi(u_n)$ and $(\bar{u}_n)$ are bounded for any sequence $(u_n) \subset X$, (u_n) is bounded in X.*

Proof. By using conditions $(H_1), (H_2)$ and Lemma 2.3, we have:

$$\begin{aligned}
\varphi(u) &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt + \frac{c}{2d}(u(T))^2 + \frac{a}{2b}(u(0))^2 + \sum_{i=1}^{N}\int_0^{u(t_i)} I_i(s)ds \\
&\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,u(t))dt \\
&= \frac{1}{2}\|u(t)\|^2 - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,u(t))dt + \sum_{i=1}^{N}\int_0^{u(t_i)} I_i(s)ds \\
&= \frac{1}{2}\|\bar{u}+\tilde{u}(t)\|^2 - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u}+\tilde{u}(t))dt + \sum_{i=1}^{N}\int_0^{\bar{u}+\tilde{u}(t_i)} I_i(s)ds \\
&\geq \frac{1}{2}\|\tilde{u}(t)\|^2 - \|\bar{u}\|\|\tilde{u}(t)\| - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\int_0^1 \frac{d}{ds}F_i(t,\bar{u}+s\tilde{u}(t))dsdt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt \\
&\quad + \sum_{i=1}^{N}\int_0^{\bar{u}} I_i(s)ds + \sum_{i=1}^{N}\int_{\bar{u}}^{\bar{u}+\tilde{u}(t_i)} I_i(s)ds \\
&\geq \frac{1}{2}\|\tilde{u}(t)\|^2 - \|\bar{u}\|\|\tilde{u}(t)\| - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\int_0^1 \beta(t)|\bar{u}+s\tilde{u}(t)||\tilde{u}(t)|dsdt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt \\
&\quad + \sum_{i=1}^{N}\int_0^{\bar{u}} I_i(s)ds - \sum_{i=1}^{N}\int_{\bar{u}}^{\bar{u}+\tilde{u}(t_i)} \alpha|s|ds \\
&\geq \frac{1}{2}\|\tilde{u}(t)\|^2 - \|\bar{u}\|\|\tilde{u}(t)\| - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\int_0^1 \beta(t)|\bar{u}||\tilde{u}(t)|dsdt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\int_0^1 \beta(t)s|\tilde{u}(t)|^2dsdt \\
&\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt + \sum_{i=1}^{N}\int_0^{\bar{u}} I_i(s)ds - \sum_{i=1}^{N}\int_{\bar{u}}^{\bar{u}+\tilde{u}(t_i)} \alpha|s|ds \\
&\geq \frac{1}{2}\|\tilde{u}(t)\|^2 - \|\bar{u}\|\|\tilde{u}(t)\| - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \beta(t)|\bar{u}||\tilde{u}(t)|dt - \frac{1}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \beta(t)|\tilde{u}(t)|^2dt \\
&\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt + \sum_{i=1}^{N}\int_0^{\bar{u}} I_i(s)ds - \sum_{i=1}^{N}\alpha(|\bar{u}||\tilde{u}(t_i)| + \frac{1}{2}|\tilde{u}(t_i)|^2) \\
&\geq \frac{1}{2}\|\tilde{u}(t)\|^2 - \|\bar{u}\|\|\tilde{u}(t)\| - |\bar{u}|\|\beta\|_{L_1}\|\tilde{u}(t)\|_\infty - \frac{1}{2}\|\beta\|_{L_1}\|\tilde{u}(t)\|_\infty^2 - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt \\
&\quad + \sum_{i=1}^{N}\int_0^{\bar{u}} I_i(s)ds - N\alpha(|\bar{u}|\|\tilde{u}(t)\|_\infty + \frac{1}{2}\|\tilde{u}(t)\|_\infty^2) \\
&\geq \left(\frac{1}{2} - \frac{M_3^2}{2}\|\beta\|_{L_1} - \frac{N\alpha M_3^2}{2}\right)\|\tilde{u}(t)\|^2 - (\|\bar{u}\| + M_3|\bar{u}|\|\beta\|_{L_1} + N\alpha M_3)\|\tilde{u}(t)\| \\
&\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt + \sum_{i=1}^{N}\int_0^{\bar{u}} I_i(s)ds.
\end{aligned}$$

Therefore,

$$\varphi(u) \geq -\frac{(\|\bar{u}\| + M_3|\bar{u}|\|\beta\|_{L_1} + N\alpha M_3)^2}{2 - 2M_3^2\|\beta\|_{L_1} - 2N\alpha M_3^2} - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt + \sum_{i=1}^{N}\int_0^{\bar{u}} I_i(s)ds. \tag{3.1}$$

Now, let (u_n) be a sequence in X such that $(\bar{u}_n)$ is bounded , which means $\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt$ and $\sum_{i=1}^{n}\int_0^{\bar{u}} I_i(s)ds$ are bounded. Since $\varphi(u_n)$ is bounded, one has $\|\tilde{u}_n\| \leq M'(M' > 0)$. Therefore (u_n) is bounded in X. □

Lemma 3.2. *If $u \in X$ is a weak solution of problem (1.4) , $u \in X$ is a classical solution of problem (1.4).*

Proof. If u is a weak solution of problem (1.4) , then $\langle \varphi'(u), v\rangle = 0$ for all $v \in X$. We will divide three steps to complete the proof.

Step 1. We will prove that u satisfies the equations in (1.4).

Without loss of generality, we assume that $v \in C_0^\infty(s_i, t_{i+1}], v' \in C_0^\infty(s_i, t_{i+1}]$ such that $v \equiv 0$ for $t \in [0, s_i] \bigcup (t_{i+1}, T]$. Substituting $v(t)$ into (2.2), we have

$$\frac{1}{2}\int_{s_i}^{t_{i+1}} ({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u')v'dt = \int_{s_i}^{t_{i+1}} f_i(t,u(t))v(t)dt, \tag{3.2}$$

i.e.,

$$\begin{aligned}
0 &= \int_{s_i}^{t_{i+1}} \left(\frac{1}{2}({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u')v' - f_i(t,u(t))v(t)\right) dt \\
&= \int_{s_i}^{t_{i+1}} \left(\frac{1}{2}({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u')v' - v(t)\frac{d}{dt}\int_0^t f_i(s,u(s))ds\right) dt \\
&= \int_{s_i}^{t_{i+1}} \left(\frac{1}{2}({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u')v' - v(t)\int_0^t f_i(s,u(s))ds\Big|_{s_i}^{t_{i+1}} + v'(t)\int_0^t f_i(s,u(s))ds\right) dt \\
&= \int_{s_i}^{t_{i+1}} \left(\frac{1}{2}({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u') + \int_0^t f_i(s,u(s))ds\right) v'(t)dt.
\end{aligned}$$

By Dubois-Reymond Lemma, for all $v'(t) \in C_0^\infty$, we have

$$\frac{1}{2}({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u') + \int_0^t f_i(t,u(t))ds = constant.$$

Since $f_i \in C((s_i, t_{i+1}] \times \mathrm{R}, \mathrm{R})$, we have

$$\frac{d}{dt}(\frac{1}{2}\,{}_0D_t^{-\beta}\, u' + \frac{1}{2}\,{}_tD_T^{-\beta}\, u') + f_i(t,u(t)) = 0,$$

which implies

$$-\frac{d}{dt}(\frac{1}{2}\,{}_0D_t^{-\beta}\, u'(t) + \frac{1}{2}\,{}_tD_T^{-\beta}\, u'(t)) = f_i(t,u(t)). \tag{3.3}$$

Because $f_i \in C((s_i, t_{i+1}] \times \mathrm{R}, \mathrm{R})$, we get $\frac{1}{2}({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u') \in C^1(s_i, t_{i+1}]$. Therefore u satisfies the equations in problem (1.4).

Step 2. We show u satisfies instantaneous and non-instantaneous impulses.

Substituting (3.3) into (2.2), we obtain

$$\frac{1}{2}\int_0^T ({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u')v'dt + \sum_{i=1}^{N} I_i(u(t_i))v(t_i) + \frac{c}{d}u(T)v(T) + \frac{a}{b}u(0)v(0)$$
$$+ \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \frac{d}{dt}(\frac{1}{2}\,{}_0D_t^{-\beta}\, u'(t) + \frac{1}{2}\,{}_tD_T^{-\beta}\, u'(t))v(t)dt = 0,$$

i.e.,

$$\begin{aligned}&\sum_{i=0}^{N}\frac{1}{2}({}_0D_t^{-\beta}\, u'(t_{i+1}^-) + {}_tD_T^{-\beta}\, u'(t_{i+1}^-))v(t_{i+1}^-) - \sum_{i=0}^{N}\frac{1}{2}({}_0D_t^{-\beta}\, u'(s_i^+) + {}_tD_T^{-\beta}\, u'(s_i^+))v(s_i^+)\\ &+\sum_{i=1}^{N}\int_{t_i}^{s_i}\frac{1}{2}({}_0D_t^{-\beta}\, u' + {}_tD_T^{-\beta}\, u')v'dt + \sum_{i=1}^{N} I_i(u(t_i))v(t_i) + \frac{c}{d}u(T)v(T) + \frac{a}{b}u(0)v(0) = 0.\end{aligned} \tag{3.4}$$

Without loss of generality, we take the test function $v \in C_0^\infty(t_i, s_i], v' \in C_0^\infty(t_i, s_i]$ such that $v(t) \equiv 0$ for $t \in [0, t_i] \bigcup (s_i, T]$. Substituting $v(t)$ into (3.4), from Dubois-Reymond Lemma, we have

$$\frac{1}{2}({}_0D_t^{-\beta}\, u'(t) + {}_tD_T^{-\beta}\, u'(t) = constant, t \in (t_i, s_i],$$

i.e., for $t \in (t_i, s_i]$,

$$\frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+)) = \frac{1}{2}({}_0D_t^{-\beta}\, u'(s_i^-) + {}_tD_T^{-\beta}\, u'(s_i^-)) = \frac{1}{2}({}_0D_t^{-\beta}\, u'(t) + {}_tD_T^{-\beta}\, u'(t)). \tag{3.5}$$

Substituting (3.5) into (3.4), one gets

$$
\begin{aligned}
0 =& \sum_{i=0}^{N} \frac{1}{2}({}_0D_t^{-\beta}\, u'(t_{i+1}^-) + {}_tD_T^{-\beta}\, u'(t_{i+1}^-))v(t_{i+1}^-) - \sum_{i=0}^{N} \frac{1}{2}({}_0D_t^{-\beta}\, u'(s_i^+) + {}_tD_T^{-\beta}\, u'(s_i^+))v(s_i^+) \\
&+ \sum_{i=1}^{N} \frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+))v(s_i) - \sum_{i=1}^{N} \frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+))v(t_i) \\
&+ \sum_{i=1}^{N} I_i(u(t_i))v(t_i) + \frac{c}{d}u(T)v(T) + \frac{a}{b}u(0)v(0) \\
=& \sum_{i=1}^{N} \frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^-) + {}_tD_T^{-\beta}\, u'(t_i^-))v(t_i^-) - \sum_{i=1}^{N} \frac{1}{2}({}_0D_t^{-\beta}\, u'(s_i^+) + {}_tD_T^{-\beta}\, u'(s_i^+))v(s_i^+) \\
&+ \sum_{i=1}^{N} \frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+))v(s_i) - \sum_{i=1}^{N} \frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+))v(t_i) \\
&+ \sum_{i=1}^{N} I_i(u(t_i))v(t_i) + \frac{1}{2}({}_0D_t^{-\beta}\, u'(T) + {}_tD_T^{-\beta}\, u'(T))v(T) - \frac{1}{2}({}_0D_t^{-\beta}\, u'(0) + {}_tD_T^{-\beta}\, u'(0))v(0) \\
&+ \frac{c}{d}u(T)v(T) + \frac{a}{b}u(0)v(0) \\
=& \sum_{i=1}^{N} \left[\frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^-) + {}_tD_T^{-\beta}\, u'(t_i^-)) - \frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+)) + I_i(u(t_i))\right] v(t_i) \\
&+ \sum_{i=1}^{N} \left[\frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+)) - \frac{1}{2}({}_0D_t^{-\beta}\, u'(s_i^+) + {}_tD_T^{-\beta}\, u'(s_i^+))\right] v(s_i) \\
&+ \left[\frac{c}{d}u(T) + \frac{1}{2}({}_0D_t^{-\beta}\, u'(T) + {}_tD_T^{-\beta}\, u'(T))\right] v(T) + \left[\frac{a}{b}u(0) - \frac{1}{2}({}_0D_t^{-\beta}\, u'(0) + {}_tD_T^{-\beta}\, u'(0))\right] v(0).
\end{aligned}
$$

Without loss of generally, we assume $v(s_i) = v(0) = v(T) = 0$, which means the instantaneous impulses

$$
\frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+)) - \frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^-) + {}_tD_T^{-\beta}\, u'(t_i^-)) = I_i(u(t_i))
$$

are hold. Then, without loss of genenerally, we assume $v(t_i) = v(0) = v(T) = 0$, one has

$$
\frac{1}{2}({}_0D_t^{-\beta}\, u'(t_i^+) + {}_tD_T^{-\beta}\, u'(t_i^+)) = \frac{1}{2}({}_0D_t^{-\beta}\, u'(s_i^+) + {}_tD_T^{-\beta}\, u'(s_i^+)).
$$

Together with (3.5), we have

$$
\frac{1}{2}({}_0D_t^{-\beta}\, u'(s_i^-) + {}_tD_T^{-\beta}\, u'(s_i^-)) = \frac{1}{2}({}_0D_t^{-\beta}\, u'(s_i^+) + {}_tD_T^{-\beta}\, u'(s_i^+)).
$$

Therefore the non-instantaneous impulses are hold.

Step 3. We show u satisfies Sturm-Liouville boundary conditions of problem (1.4). Without loss of generally, let $v(t_i) = v(s_i) = v(0) = 0$ or $v(t_i) = v(s_i) = v(T) = 0$. Form Step 2, we have

$$
\begin{cases}
au(0) - b(\frac{1}{2}\,{}_0D_t^{-\beta}\, u'(0) + \frac{1}{2}\,{}_tD_T^{-\beta}\, u'(0)) = 0, \\
cu(T) + d(\frac{1}{2}\,{}_0D_t^{-\beta}\, u'(T) + \frac{1}{2}\,{}_tD_T^{-\beta}\, u'(T)) = 0.
\end{cases}
$$

So u satisfies Sturm-Lionville boundary conditions of problem (1.4). The proof is completed. $\square$

Theorem 3.3. *If hypothesis* $(H_1),(H_2)$ *and* (H_3) *are satisfied, the problem (1.4) has infinitely many solutions.*

Proof. We will divide five steps to complete the proof.

Step 1. We will proof that there exists a sequence (s_n) such that $\lim_{n\to\infty} s_n = +\infty$ and

$$\lim_{n\to\infty}\left(\sup_{e\in R,|e|=s_n}\varphi(e)\right) = -\infty.$$

Substituting $e = u$ into (2.1), we have

$$\varphi(e) = \frac{c}{2d}e^2 + \frac{a}{2b}e^2 + \sum_{i=1}^{N}\int_0^e I_i(s)ds - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,e)dt.$$

The result is hold by $(H_3)(i)$.

Step 2. We will show there exists a sequence (s_n) such that $\lim_{n\to\infty} s_n = +\infty$ and

$$\lim_{n\to\infty}(\inf_{l\in R,|l|=s_n,\tilde{u}\in\tilde{X}}\varphi(l+\tilde{u})) =: L > -\infty.$$

Let $s > 0, l\in$R$, |l| = s$ and $\tilde{u}\in\tilde{X}$. From (3.1), with l instead of $\bar{u}$, we get

$$\inf_{l\in R,|l|=s,\tilde{u}\in\tilde{X}}\varphi(l+\tilde{u}) \geq \inf_{l\in R,|l|=s}\left\{-\frac{(\|l\| + M_3|l|\|\beta\|_{L_1} + N\alpha M_3)^2}{2-2M_3^2\|\beta\|_{L_1} - 2N\alpha M_3^2} - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,l)dt + \sum_{i=1}^{N}\int_0^l I_i(s)ds\right\}.$$

The result follows from $(H_3)(ii)$.

Step 3. We will prove there exists a sequence $(u^*_{n_m})$ such that $u^*_{n_m}$ is a critical point of I in X.

Let

$$\Pi_n = \{u\in X : |\bar{u}| \leq s_n\}(n\in N).$$

By proposition 3.1, we have

$$\varphi(u) \geq -\frac{(\|\bar{u}\| + M_3|\bar{u}|\|\beta\|_{L_1} + N\alpha M_3)^2}{2-2M_3^2\|\beta\|_{L_1} - 2N\alpha M_3^2} - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,\bar{u})dt + \sum_{i=1}^{N}\int_0^{\bar{u}} I_i(s)ds.$$

Since $u\in\Pi_n$, we have φ is bounded on Π_n. Let

$$\mu_n := \inf_{u\in\Pi_n}\varphi(u_n)(n\in N),$$

and let (u_k) be a minimizing sequence in Π_n such that

$$\varphi(u_k) \to \mu_n (k \to \infty).$$

Then we have (u_k) is bounded in X by lemma 3.1. X is a reflexive Banach space, hence (u_k) has a weakly convergent subsequence (u_{k_m}) and $u_{k_m} \rightharpoonup u_n^* \in X$. From Mazur's lemma, since Π_n is a convex closed subset of X, one gets $u_n^* \in \Pi_n$. In addition, we get

$$\mu_n = \lim_{k\to\infty} \varphi(u_k) = \lim_{k\to\infty} \varphi(u_{k_m}) \geq \varphi(u_n^*) \geq \inf_{u\in\Pi_n} \varphi(u) = \mu_n.$$

One has $\varphi(u_n^*) = \inf_{u\in\Pi_n} \varphi(u) = \mu_n$, which means φ has a local minimum point u_n^*. Then, we denote a subsequence of (s_n) as (s_{n_m}) such that $0 < s_n < s_{n_m}$ for all $m \in$N. Let some $c \in (-\infty, L)$. As

$$\partial\Pi_{n_m} = \{b \in R : |b| = s_{n_m}\} + \tilde{X},$$

we have $m_0 \in N$ such that

$$\varphi(u_{n_m}^*) = \inf_{u\in\Pi_{n_m}} \varphi(u) \leq \inf_{|e|=s_n} \varphi(e) \leq \sup_{|e|=s_n} \varphi(e)(m > m_0).$$

By step 1, we can find a c such that

$$\varphi(u_{n_m}^*) \leq \sup_{|e|=s_n} \varphi(e) < c.$$

From step 2, we have $\lim_{n\to\infty}(\inf_{l\in R,|l|=s_n,\tilde{u}\in\tilde{X}} \varphi(l+\tilde{u})) =: L > c$. Therefore

$$\inf_{u\in\partial\Pi_{n_m}} \varphi(u) > c(m > m_0),$$

which means $u_{n_m}^* \notin \partial\Pi_{n_m}$, i.e., $\varphi'(u_{n_m}^*) = 0$. Since $u_{n_m}^* \in int\Pi_n$, Π_n contains an open X-neighborhood of $u_{n_m}^*$. Thus $u_{n_m}^*$ is a free critical point of φ in Π_n, i.e., $u_{n_m}^*$ is a critical point of φ in X.

Step 4. We will show $u_{n_m}^*$ is a solution of problem (1.4).

Since $u_{n_m}^*$ is a critical point of I in X, by lemma 3.2, we have $u_{n_m}^*$ is a solution of problem (1.4).

Step 5. We will prove the existence of infinitely many solutions for the problem (1.4).

From step 1 and step 3, we have

$$\varphi(u_{n_m}^*) = \inf_{|\bar{u}|\leq s_{n_m}} \varphi(u) \leq \inf_{|\bar{u}|\leq s_n} \varphi(u) \leq \sup_{|e|=s_n} \varphi(e),$$

which means $\lim_{n\to\infty} \varphi(u_{n_m}^*) = -\infty$, hence the result is hold. The proof is completed. □

4 Main results for $bd=0$

Case 1. If $b=0, d\neq 0$, then problem (1.4) is reduced to

$$\begin{cases} -\frac{d}{dt}(\frac{1}{2}{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}{}_tD_T^{-\beta}(u'(t)))=f_i(t,u(t)), t\in(s_i,t_{i+1}], i=0,1,2,...,N,\\ \Delta(\frac{1}{2}{}_0D_t^{-\beta}u'(t_i)+\frac{1}{2}{}_tD_T^{-\beta}u'(t_i))=I_i(u(t_i)), i=1,2,...,N,\\ \frac{1}{2}{}_0D_t^{-\beta}u'(t)+\frac{1}{2}{}_tD_T^{-\beta}u'(t)=\frac{1}{2}{}_0D_t^{-\beta}u'(t_i^+)+\frac{1}{2}{}_tD_T^{-\beta}u'(t_i^+), t\in(t_i,s_i], i=1,2,...,N,\\ \frac{1}{2}{}_0D_t^{-\beta}u'(s_i^+)+\frac{1}{2}{}_tD_T^{-\beta}u'(s_i^+)=\frac{1}{2}{}_0D_t^{-\beta}u'(s_i^-)+\frac{1}{2}{}_tD_T^{-\beta}u'(s_i^-), i=1,2,...,N,\\ u(0)=0, cu(T)+d(\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(T)+\frac{1}{2}\,{}_tD_T^{-\beta}\,u'(T))=0. \end{cases} \tag{4.1}$$

We define the fractional derivative space $X_1=\{u\in X: u(0)=0\}$ with the norm

$$\|u\|_{X_1}=\left(-\int_0^T({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)dt+\frac{c}{d}(u(T))^2\right)^{\frac{1}{2}}$$

and functional $\varphi_1: X_1\to$R by

$$\varphi_1(u)=-\frac{1}{2}\int_0^T({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt+\frac{c}{2d}(u(T))^2+\sum_{i=1}^N\int_0^{u(t_i)}I_i(s)ds-\sum_{i=0}^N\int_{s_i}^{t_{i+1}}F_i(t,u(t))dt.$$

Similar to Lemma 2.2 and Lemma 2.3, we obtain that $\|\cdot\|_{X_1}$ is equivalent to $\|\cdot\|_{\alpha,2}$ and $\|u\|_\infty\leq M_6\|u\|_{X_1}$, where $M_6=\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}\sqrt{|\cos\pi\alpha|}}$. We assume

$(H_4)(i)\ \liminf\limits_{s\to+\infty}\sup\limits_{e_1\in R,|e_1|=s}\left\{\frac{c}{2d}{e_1}^2+\sum\limits_{i=1}^N\int_0^{e_1}I_i(s)ds-\sum\limits_{i=0}^N\int_{s_i}^{t_{i+1}}F_i(t,e_1)dt\right\}=-\infty,$

$(ii)\ \liminf\limits_{s\to+\infty}\sup\limits_{l_1\in R,|l_1|=s}\left\{\frac{(\|l_1\|_{X_1}+M_6|l_1|\|\beta\|_{L_1}+N\alpha M_6)^2}{2-2M_6^2\|\beta\|_{L_1}-2N\alpha M_6^2}-\sum\limits_{i=1}^N\int_0^{l_1}I_i(s)ds+\sum\limits_{i=0}^N\int_{s_i}^{t_{i+1}}F_i(t,l_1)dt\right\}$

$<+\infty.$

Remark 4.1. We have the following claims. Firstly, The space X_1 is a reflexive and separable Banach space and X_1 can be split into $X_1=$R$\bigoplus\tilde{X}_1$, where $\tilde{X}_1:=\left\{u\in X_1:\int_0^T u(s)ds=0\right\}$. Secondly, the functional $\varphi_1: X_1\to$R is weakly lower semi-continuous. Thirdly, suppose that (H_1) and (H_2) are satisfied. If $\varphi_1(u_n)$ and $(\bar{u}_n)$ are bounded for any sequence $(u_n)\subset X_1$, (u_n) is bounded in X_1. Finally, If $u\in X_1$ is a weak solution of problem (4.1) , $u\in X_1$ is a classical solution of problem (4.1). The claims can be proved analogously to the proof of Lemma 2.1, Lemma 2.4, Lemma 3.1 and Lemma 3.2.

Theorem 4.1. *If hypothesis $(H_1),(H_2)$ and $(H_4)(i)(ii)$ are satisfied, the problem (4.1) has infinitely many solutions.*

Case 2. If $b \neq 0, d = 0$, then problem (1.4) is reduced to

$$\begin{cases} -\frac{d}{dt}(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))) = f_i(t, u(t)), t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ \Delta(\frac{1}{2}{}_0D_t^{-\beta}u'(t_i) + \frac{1}{2}{}_tD_T^{-\beta}u'(t_i)) = I_i(u(t_i)), i = 1, 2, ..., N, \\ \frac{1}{2}{}_0D_t^{-\beta}u'(t) + \frac{1}{2}{}_tD_T^{-\beta}u'(t) = \frac{1}{2}{}_0D_t^{-\beta}u'(t_i^+) + \frac{1}{2}{}_tD_T^{-\beta}u'(t_i^+), t \in (t_i, s_i], i = 1, 2, ..., N, \\ \frac{1}{2}{}_0D_t^{-\beta}u'(s_i^+) + \frac{1}{2}{}_tD_T^{-\beta}u'(s_i^+) = \frac{1}{2}{}_0D_t^{-\beta}u'(s_i^-) + \frac{1}{2}{}_tD_T^{-\beta}u'(s_i^-), i = 1, 2, ..., N, \\ au(0) - b(\frac{1}{2}\,{}_0D_t^{-\beta}\,u'(0) + \frac{1}{2}\,{}_tD_T^{-\beta}\,u'(0)) = 0, u(T) = 0. \end{cases} \tag{4.2}$$

We define the fractional derivative space $X_2 = \{u \in X : u(T) = 0\}$ with the norm

$$\|u\|_{X_2} = \left(-\int_0^T ({}_0^cD_t^\alpha\, u, {}_t^cD_T^\alpha\, u)dt + \frac{a}{b}(u(0))^2\right)^{\frac{1}{2}}$$

and functional $\varphi_2 : X_2 \to$R by

$$\varphi_2(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha\, u(t), {}_t^cD_T^\alpha\, u(t))dt + \frac{a}{2b}(u(0))^2 + \sum_{i=1}^N \int_0^{u(t_i)} I_i(s)ds - \sum_{i=0}^N \int_{s_i}^{t_{i+1}} F_i(t, u(t))dt.$$

Let

$$w_n(t) = \begin{cases} d_n, & t \in [0, \frac{1}{2}T], \\ -\frac{2d_nt}{T} + 2d_n, & t \in [\frac{1}{2}T, T]. \end{cases}$$

Similar to Lemma 2.2 and Lemma 2.3, we obtain that $\|\cdot\|_{X_2}$ is equivalent to $\|\cdot\|_{\alpha,2}$ and $\|u\|_\infty \leq M_7\|u\|_{X_2}$, where $M_7 = (\frac{d_n}{T(1-\alpha)\Gamma(1-\alpha)}t)^2\frac{T^{3-2\alpha}}{2^{3-4\alpha}} + \frac{a}{b}$. We assume

$(H_5)(i)$ $\liminf\limits_{s\to+\infty} \sup\limits_{e_2 \in R, |e_2|=s} \left\{\frac{a}{2b}{e_2}^2 + \sum_{i=1}^N \int_0^{e_2} I_i(s)ds - \sum_{i=0}^N \int_{s_i}^{t_{i+1}} F_i(t, e_2)dt\right\} = -\infty,$

(ii) $\liminf\limits_{s\to+\infty} \sup\limits_{l_2 \in R, |l_2|=s} \left\{\frac{(\|l_2\|_{X_2} + M_7|l_2|\|\beta\|_{L_1} + N\alpha M_7)^2}{2 - 2M_7^2\|\beta\|_{L_1} - 2N\alpha M_7^2} - \sum_{i=1}^N \int_0^{l_2} I_i(s)ds + \sum_{i=0}^N \int_{s_i}^{t_{i+1}} F_i(t, l_2)dt\right\}$

$< +\infty.$

Remark 4.2. With M_1 replaced by $\frac{1}{\sqrt{|\cos\pi\alpha|}}$, we have the following claims. Firstly, The space X_2 is a reflexive and separable Banach space and X_2 can be split into $X_2 =$R$\bigoplus \tilde{X}_2$, where $\tilde{X}_2 := \left\{u \in X_2 : \int_0^T u(s)ds = 0\right\}$. Secondly, the functional $\varphi_2 : X_2 \to$R is weakly lower semi-continuous. Thirdly, suppose that (H_1) and (H_2) are satisfied. If $\varphi_2(u_n)$ and $(\bar{u}_n)$ are bounded for any sequence $(u_n) \subset X_2$, (u_n) is bounded in X_2. Finally, If $u \in X_2$ is a weak solution of problem (4.2) , $u \in X_2$ is a classical solution of problem (4.2). The claims can be proved analogously to the proof of Lemma 2.1, Lemma 2.4, Lemma 3.1 and Lemma 3.2.

Theorem 4.2. *If hypothesis $(H_1),(H_2)$ and $(H_5)(i)(ii)$ are satisfied, the problem (4.2) has infinitely many solutions.*

Case 3. If $b=0, d=0$, then problem (1.4) is reduced to

$$\begin{cases} -\frac{d}{dt}(\frac{1}{2}{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}{}_tD_T^{-\beta}(u'(t)))=f_i(t,u(t)), t\in(s_i,t_{i+1}], i=0,1,2,...,N,\\ \Delta(\frac{1}{2}{}_0D_t^{-\beta}u'(t_i)+\frac{1}{2}{}_tD_T^{-\beta}u'(t_i))=I_i(u(t_i)), i=1,2,...,N,\\ \frac{1}{2}{}_0D_t^{-\beta}u'(t)+\frac{1}{2}{}_tD_T^{-\beta}u'(t)=\frac{1}{2}{}_0D_t^{-\beta}u'(t_i^+)+\frac{1}{2}{}_tD_T^{-\beta}u'(t_i^+), t\in(t_i,s_i], i=1,2,...,N,\\ \frac{1}{2}{}_0D_t^{-\beta}u'(s_i^+)+\frac{1}{2}{}_tD_T^{-\beta}u'(s_i^+)=\frac{1}{2}{}_0D_t^{-\beta}u'(s_i^-)+\frac{1}{2}{}_tD_T^{-\beta}u'(s_i^-), i=1,2,...,N,\\ u(0)=0, u(T)=0. \end{cases} \tag{4.3}$$

We define the fractional derivative space $X_3=\{u\in X: u(0)=0, u(T)=0\}$ with the norm

$$\|u\|_{X_3}=\left(-\int_0^T({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)dt\right)^{\frac{1}{2}}$$

and functional $\varphi_3: X_3\to$R by

$$\varphi_3(u)=-\frac{1}{2}\int_0^T({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt+\sum_{i=1}^N\int_0^{u(t_i)}I_i(s)ds-\sum_{i=0}^N\int_{s_i}^{t_{i+1}}F_i(t,u(t))dt.$$

Let

$$w_n(t)=\begin{cases}\frac{2d_nt}{T}, & t\in[0,\frac{1}{2}T],\\ -\frac{2d_nt}{T}+2d_n, & t\in[\frac{1}{2}T,T].\end{cases}$$

Similar to Lemma 2.2 and Lemma 2.3, we obtain that $\|\cdot\|_{X_3}$ is equivalent to $\|\cdot\|_{\alpha,2}$ and $\|u\|_\infty\le M_8\|u\|_{X_3}$, where $M_8=\left(\frac{2}{T(1-\alpha)\Gamma(1-\alpha)}\right)^2\left(\frac{T^{3-2\alpha}}{2^{2-2\alpha}}+\frac{T^{3-2\alpha}}{2^{1-\alpha}}\right)$. We assume

$$(H_6)(i)\liminf_{s\to+\infty}\sup_{e_3\in R,|e_3|=s}\left\{\sum_{i=1}^N\int_0^{e_3}I_i(s)ds-\sum_{i=0}^N\int_{s_i}^{t_{i+1}}F_i(t,e_3)dt\right\}=-\infty,$$

$$(ii)\liminf_{s\to+\infty}\sup_{l_3\in R,|l_3|=s}\left\{\frac{(\|l_3\|_{X_3}+M_8|l_3|\|\beta\|_{L_1}+N\alpha M_8)^2}{2-2M_8^2\|\beta\|_{L_1}-2N\alpha M_8^2}-\sum_{i=1}^N\int_0^{l_3}I_i(s)ds+\sum_{i=0}^N\int_{s_i}^{t_{i+1}}F_i(t,l_3)dt\right\}$$

$$<+\infty.$$

Remark 4.3. With M_1 replaced by $\frac{1}{\sqrt{|\cos\pi\alpha|}}$, we have the following claims. Firstly, The space X_3 is a reflexive and separable Banach space and X_3 can be split into $X_3=$R$\bigoplus\tilde{X}_3$, where $\tilde{X}_3:=\left\{u\in X_3:\int_0^T u(s)ds=0\right\}$. Secondly, the functional $\varphi_3: X_3\to$R is weakly lower semi-continuous. Thirdly, suppose that (H_1) and (H_2) are satisfied. If $\varphi_3(u_n)$ and $(\bar{u}_n)$ are bounded for any sequence $(u_n)\subset X_3$, (u_n) is bounded in X_3. Finally, If $u\in X_3$ is a weak solution of problem (4.3) , $u \in X_3$ is a classical solution of problem (4.3). The claims can be proved analogously to the proof of Lemma 2.1, Lemma 2.4, Lemma 3.1 and Lemma 3.2.

Theorem 4.3. *If hypothesis* $(H_1), (H_2)$ *and* $(H_6)(i)(ii)$ *are satisfied, the problem (4.3) has infinitely many solutions.*

5 An example

Example 5.1. Consider the following problem

$$\begin{cases} -\frac{d}{dt}(\frac{1}{2}{}_0D_t^{-0.5}(u'(t)) + \frac{1}{2}{}_tD_T^{-0.5}(u'(t))) = f_i(t, u(t)), t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ \Delta(\frac{1}{2}{}_0D_t^{-0.5}u'(t_i) + \frac{1}{2}{}_tD_T^{-0.5}u'(t_i)) = I_i(u(t_i)), i = 1, 2, ..., N, \\ \frac{1}{2}{}_0D_t^{-0.5}u'(t) + \frac{1}{2}{}_tD_T^{-0.5}u'(t) = \frac{1}{2}{}_0D_t^{-0.5}u'(t_i^+) + \frac{1}{2}{}_tD_T^{-0.5}u'(t_i^+), t \in (t_i, s_i], i = 1, 2, ..., N, \\ \frac{1}{2}{}_0D_t^{-0.5}u'(s_i^+) + \frac{1}{2}{}_tD_T^{-0.5}u'(s_i^+) = \frac{1}{2}{}_0D_t^{-0.5}u'(s_i^-) + \frac{1}{2}{}_tD_T^{-0.5}u'(s_i^-), i = 1, 2, ..., N, \\ au(0) - b(\frac{1}{2}\,{}_0D_t^{-0.5}\,u'(0) + \frac{1}{2}\,{}_tD_T^{-0.5}\,u'(0)) = 0, \\ cu(T) + d(\frac{1}{2}\,{}_0D_t^{-0.5}\,u'(T) + \frac{1}{2}\,{}_tD_T^{-0.5}\,u'(T)) = 0. \end{cases} \tag{5.1}$$

Let $T = 1$, $t_i \in (0, 1)$, $a = c = 0$, $bd \neq 0$, $N \geq 3$, $\alpha(t) = \frac{\pi^2}{N}$, $\beta(t) = \pi^2$, $I(u) = \frac{\pi^2}{N}u$, $f_i(t, u) = \frac{\pi^2}{2}u \sin[\ln(1 + u^2)] + \frac{\frac{\pi^2}{2}u^3}{1+u^2} \cos[\ln(1 + u^2)]$ and $\sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} F(t, u)dt = \frac{\pi^2}{5}|x|^2 \sin[\ln(1 + x^2)](i = 1, 2, ..., N)$. Therefore, $|I_i(u)| \leq \frac{\pi^2}{N}|u|$, $|f(t, u)| \leq \pi^2|u|$. Then, setting $e = \sqrt{e^{2k\pi+\frac{\pi}{2}} - 1}$ and $l = \sqrt{e^{2k\pi+\frac{3\pi}{2}} - 1}$ ($k \in$N), one has

$$\liminf_{k\to+\infty} \sup_{e_k \in R} \left\{ \frac{c}{2d}e_k^2 + \frac{a}{2b}e_k^2 + \sum_{i=1}^{N} \int_0^{e_k} I_i(s)ds - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} F_i(t, e_k)dt \right\} = -\infty,$$

$$\liminf_{k\to+\infty} \sup_{l_k \in R} \left\{ \frac{(\|l_k\| + M_3|l_k|\|\beta\|_{L_1} + N\alpha M_3)^2}{2 - 2M_3^2\|\beta\|_{L_1} - 2N\alpha M_3^2} - \sum_{i=1}^{N} \int_0^{l_k} I_i(s)ds + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} F_i(t, l_k)dt \right\} < +\infty.$$

Therefore conditions $(H_3)(i)$ and $(H_3)(ii)$ are satisfied. Applying Theorem (3.3), we obtain that problem (5.1) has infinitely many solutions.

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