On the spatially inhomogeneous particle coagulation-condensation model with singularity

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Abstract

The spatially inhomogeneous coagulation-condensation process is an interesting topic of study as the phenomenon's mathematical aspects mostly undiscovered and has multitudinous empirical applications. In this present exposition, we exhibit the existence of a continuous solution for the corresponding model with the following *singular* type coagulation kernel:

$$K(x, y) \leq \frac{(x+y)^{\theta}}{(xy)^{\mu}}, \text{ for } x, y \in (0, \infty), \text{where } \mu \in \left[0, \tfrac{1}{2}\right] \text{ and } \theta \in [0, 1].$$

The above-mentioned form of the coagulation kernel includes several practical-oriented kernels. Finally, uniqueness of the solution is also investigated.

Keywords: Coagulation-condensation equation, Space-inhomogeneous, Singular kernels, Existence, Uniqueness.

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1. Introduction

Many interesting mathematical problems arise from experimental sciences. The mathematical formulations of those problems develop thanks to the cooperation of mathematicians and theoretical scientists that share the interest in formalizing nature's phenomena and general frameworks for the issues. In regard to the issue that concerns us here, the development of population balance models incorporating spatially inhomogeneous coagulation-condensation process in the particulate process occurred something analogous due to their broad impact in many scientific and engineering disciplines, namely, atmospheric science [1–3], astrophysics [4–6], chemical engineering [7–9] and physical sciences [10–12] etc. Few specific real-life applications of the process are vapor nucleation, the evolution of asteroids or planets or stars, aggregation of droplets in atmospheric clouds, polymer reaction, coagulation of colloidal clusters, the development of gas bubbles, etc. The space-inhomogeneous coagulation equation with

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condensation is described mathematically by the following governing equation:

$$\frac{\partial g(t,x,z)}{\partial t} + \frac{\partial}{\partial x}(p(x)g(t,x,z)) + \mathrm{div}_z(v(x,z)g(t,x,z))$$
$$= \frac{1}{2}\int_0^x K(x-y,y)\, g(t,x-y,z)\, g(t,y,z)\ \mathrm{d}y - g(t,x,z)\int_0^\infty K(x,y)\, g(t,y,z)\ \mathrm{d}y, \tag{1.1}$$

where $x > 0, t \geq 0$ and $z \in \mathbb{R}^3$. The equation (1.1) is commonly supplemented by an initial distribution

$$g(0,x,z) = g_{01}(x,z) \geq 0, \tag{1.2}$$

together with condensation germs of distribution function as

$$g(t,0,z) = g_{02}(t,z). \tag{1.3}$$

The model equations (1.1-1.3) illustrate the time evolution of particle growth in dispersing medium. The function $K(x,y)$ is symmetric and non-negative on $(0,\infty)\times(0,\infty)$, which defines the intensity of merging of smaller particles with property (e.g., mass, size, density, etc.) x and y to a newly formed particle characterized by property $x+y$. The unknown function $g(t,x,z)$ represents the distribution of particles with property characteristic $x > 0$ at the time $t \geq 0$ and the space position $z \in \mathbb{R}^3$. The function $v(x,z)$ is the space transfer velocity of the particles; $p(x)$ is a scalar speed of particle growth due to the condensation of molecules or clusters from an outer medium. This well-known phenomenon was first introduced mathematically by Levin, Sedunov and Berry [13–16] in the form of (1.1-1.3) to model the flow that occurs in a given velocity medium of spontaneous clusters [16]. The physical description of the governing rate equation (1.1) is as follows:

- The second expression of the left side describes evolution by evaporation (or by condensation).
- Third term depicts the change in g owing to motion in the physical field.
- Formation of new particles with property x on account of merging smaller particles with properties $x-y$ and y is characterized by the first integral of right side.
- The second integral of right-hand is the loss of particles having property x for the interaction of particles with properties y.

A few results on the study of equation (1.1) in the homogeneous space with $v \equiv 0$ have been reported in [17, 18]. Other mathematical results for the spatially inhomogeneous equation are as follows. Galkin [19] triumphed to achieve an existence-uniqueness result for bounded coagulation kernels. Dubovskiĭ [20] has proved the existence-uniqueness result for the coagulation model in spatially inhomogeneous velocity mean fields by considering the constant kernel, i.e., $K(x,y) \leq$ constant. Moreover, the solution in [20] belongs to the space

$$L_{1,\vartheta} = \{\text{All real valued functions } f \text{ defined on } \mathbb{R}_1 \times \mathbb{R}_3 \times \mathbb{R}_1^+ : \|f\|_\vartheta < \infty\},$$

where

$$\|f\|_{\vartheta} = \sup_{0\le t\le\infty} \exp(\vartheta t) \int_0^\infty \sup_{z\in\mathbb{R}^3} |f(t,x,z)|\mathrm{d}x,\ \vartheta \ge 0.$$

Studies on a few special types of unbounded coagulation kernels were shown in [21]. For the linear rate growth, i.e., $K(x, y) \le const(x+y)$, Dubovskiĭ [22] has investigated the existence-uniqueness result of the coagulation and particle fraction model. The existence result of the Smoluchowski equations with external sources and the condensation process taking into account has been demonstrated by Dubovskiĭ [23]. The type of the coagulation kernel in [23] is examined as

$$\overline{\lim_{x\to\infty}} K(x,y)x^{-\alpha} = \zeta(y),\ 0 < \alpha < 1,\ y \in \mathbb{R}_1^+,$$

where the function ζ is locally bounded. This kernel is a part of the kernel considered for this research work. Gajewski [24] has shown the existence of the stationary solution and its uniqueness of the mathematical model of an emulsion polymerization reactor. The time-dependent solutions and their asymptotic behavior are reported in [24]. The shape of the coagulation kernel, in [24], is $K(x, y) \le k_0(xy)^{-\beta_0}$ which is a specific form of

$$\frac{(x+y)^\theta}{(xy)^\mu} \text{ for } \theta = 0,\ \mu = \beta_0 \in [0, \tfrac{1}{2}]$$

The evolution of a coagulating system with the influence of condensation processes was investigated for the space homogeneous case by Gajewski and Zacharias [25]. The corresponding discrete formulation together with an effluxes term has been extensively studied by Chae and Dubovskii[26] and Herrero, Velázquez and Wrzosek [27], wherein the later the authors have considered the model excluding the scalar speed term. Over the years, the coagulation-condensation process incorporating several forms arises from practical applications have attracted many researchers, in this context, some notable works can be found in [28–40].

It is to be noted that the analysis of a spatially inhomogeneous coagulation model with condensation has not been undertaken so far. Also, extensive studies on singular coagulation-fragmentation equation is found to be a recent trend of research [41–48]. In this study, we thus aim at the study of the existence-uniqueness of solutions of (1.1) for sufficiently small initial datum with singular type coagulation kernels. The main novelty of this paper is studying the equation (1.1) for singular kernels. The form of the singular kernel in this study is

$$K(x,y) = \frac{(x+y)^\theta}{(xy)^\mu}, \tag{1.4}$$

where $\mu \in \left[0, \frac{1}{2}\right]$ and $\theta \in [0, 1]$. Furthermore, it is noteworthy that the singular kernels contemplated here include a substantial class of practical-oriented kernels, e.g., [1],

[1]We use the following relations (see [41]) for the inequalities

$$2^{\vartheta-1}(x^\vartheta + y^\vartheta) \le (x+y)^\vartheta \le x^\vartheta + y^\vartheta \text{ if } 0 \le \vartheta \le 1 \tag{1.5}$$

$$\text{and } 2^{\vartheta-1}(x^\vartheta + y^\vartheta) \ge (x+y)^\vartheta \ge x^\vartheta + y^\vartheta \text{ if } \vartheta \ge 1. \tag{1.6}$$

■ Kapur kernel (in granulation) [49]:

The form of the Kapur kernel is $k\dfrac{(x+y)^a}{(xy)^b}$. For any $a \in [0,1]$ and $b \in [0,\frac{1}{2}]$, the Kapur kernel is directly in the form of the singular kernel (1.4) of this study. For the peglow's kernel [50] $a = 0.7105$ and $b = 0.0621$, which is included in (1.4).

■ Shiloh et al. kernel (in nonlinear velocity profile) [51]:

$$\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^{\frac{7}{3}} \leq k2^{1-\frac{1}{3}}(x+y)^{\frac{7}{9}}, \text{ using (1.5).}$$

This is a particular form of (1.4) with $\theta = \frac{7}{9}$ and $\mu = 0$.

■ Hounslow equipartition kinetic energy kernel (in granulation) [52]:

$$\begin{aligned} k\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^2 \sqrt{\tfrac{1}{x}+\tfrac{1}{y}} &\leq k\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^2 (x+y)^{\frac{1}{2}}(xy)^{-\frac{1}{2}} \\ &\leq k\left[2^{1-\frac{1}{3}}(x+y)^{\frac{1}{3}}\right]^2 (x+y)^{\frac{1}{2}}(xy)^{-\frac{1}{2}}, \text{ using (1.5)} \\ &\leq k2^{\frac{4}{3}}(x+y)^{\frac{7}{6}}(xy)^{-\frac{1}{2}}, \end{aligned}$$

which is another particular case of (1.4) with $\theta = \frac{7}{6}$ and $\mu = \frac{1}{2}$.

■ Smoluchowski kernel (in Brownian diffusion kernel) [53]:

$$k\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)\left(x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\right) \leq k2^{\frac{4}{3}}(x+y)^{\frac{2}{3}}(xy)^{-\frac{1}{3}} \text{ using (1.5),}$$

which is the equivalent form of (1.4) with $\theta = \frac{2}{3}$ and $\sigma = \frac{1}{3}$.

■ Kernels consider in Friedlander [54] (in aerosol dynamics) :

$$k\left(\tfrac{1}{x}+\tfrac{1}{y}\right)^{\frac{1}{2}}(xy)^{\frac{2}{3}} \leq k(x+y)^{\frac{1}{2}}\left(\tfrac{1}{4}(x+y)^2\right)^{\frac{1}{6}} \leq (\tfrac{1}{4})^{\frac{1}{6}}k(x+y)^{\frac{5}{6}}$$

which is the case of (1.4) with $\theta = \frac{5}{6}$ and $\mu = 0$.

■ Coagulation kernel in Ding et al. [55]:

For a given constant $y_c \in \mathbb{R}$ and $q \in (0,3)$, consider

$$k\frac{\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^q}{1+\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{2y_c^{\frac{1}{3}}}\right)^3} \leq 2^{\frac{q-3}{3}}\ k\ y_c(xy)^{-\frac{3-q}{6}},$$

which is the case (1.4) with $\theta = 0$ and $\mu = \frac{3-q}{6}$.

Our strategy is based on the well-known approach for non-singular kernel in [56]. It turned out, however, that our modification is not always straightforward using a singular kernel. Chae and Dubovskiĭ [56] studied the non-singular unbounded coalescence kernel as $K(x, y) \leq const.(x + y)$. The following norm and space were established by Chae and Dubovskiĭ [56]

$$\|g\|_\lambda = \sup_{0\leq t\leq T} \int_0^\infty \exp(\lambda x) \sup_{z\in\mathbb{R}^3} |g(t, x, z)|\mathrm{d}x$$

and

$$\Omega_\lambda(T) = \{\text{all continuous functions } g(t, x, z) : \|g\|_\lambda < \infty\}.$$

We consider the singular kernels in this research paper. Due to this reason, singular integrals are coming into the picture, for instance, $\int_0^\infty x^{-\mu} f(x)dx$ is not bounded for all continuous functions $f(x)$ in $(0, \infty)$. If the function $f(x)$ complies with the property $\int_0^\infty x^{-r} f(x)dx$ exists finitely, where $r \in (0, 1]$, then the integration $\int_0^\infty x^{-\mu} f(x)dx$ satisfies the same property. Therefore, by defining new norm

$$\|g\|_{\lambda,r} := \sup_{0\leq t\leq T} \int_0^\infty \left(\exp(\lambda x) + \tfrac{\exp(\lambda)}{x^r}\right) \sup_{z\in\mathbb{R}^3} |g(t, x, z)|\mathrm{d}x$$

and the space

$$\Omega_{\lambda,r}(T) = \{\text{all continuous functions } g(t, x, z) : \|g\|_{\lambda,r} < \infty\},$$

we overcome the singular integral and difficulty to obtain the desired result of this paper.

The proposed work is organized in the following sequence. In §2, the existence of a solution is proved. The uniqueness of the solution is shown in §3. Finally, in §4, a brief conclusion and future directions of this study are provided.

2. Existence of solution

Let $T > 0$ be any given number and $0 < r \leq 1$. For a $\lambda > 0$, suppose $\Omega_{\lambda,r}(T)$ is the space of all continuous functions in $[0, T] \times \mathbb{R}^1_+ \times \mathbb{R}^3$ with the following finite norm

$$\|g\|_{\lambda,r} := \sup_{0\leq t\leq T} \int_0^\infty \left(\exp(\lambda x) + \tfrac{\exp(\lambda)}{x^r}\right) \sup_{z\in\mathbb{R}^3} |g(t, x, z)|\mathrm{d}x.$$

With the help of $\Omega_{\lambda,r}(T)$, we define another function space $\Omega_{.,r}(T)$ by

$$\Omega_{.,r}(T) := \bigcup_{\lambda>0} \Omega_{\lambda,r}(T).$$

The cone of nonnegative functions in $\Omega_{.,r}(T)$ is denoted by $\Omega^+_{.,r}(T)$.

Theorem 2.1. *Assume the kernel $K(x,y) = K(y,x) \geq 0$, for $(x,y) \in (0,\infty) \times (0,\infty)$, be a continuous function and $\lim_{(x,y)\to(0,0)} K(x,y) = +\infty$. Also, let $K(x,y) \leq \frac{(x+y)^\theta}{(xy)^\mu}$, for all $x,y \in (0,\infty)$, where k is a positive constant, $\mu \in \left[0, \frac{1}{2}\right]$ and $\theta \in [0,1]$. Let the function p be nonnegative and its second derivative p'' be continuous on $(0,\infty)$. We also assume the following inequality:*

$$div_z v(x,z) + p'(x) \geq \delta > 0. \tag{2.1}$$

Suppose the functions $v,\ g_{01},\ g_{02}$ are continuous and, in addition, g_{01} and g_{02} are nonnegative. We impose the following conditions ensuring smallness of the initial datum:

$$g_{01}(x,z) < A\exp(-ax) \ \text{ and } \ \sup_{0\leq t\leq T} (\exp(\delta t) g_{02}(t,z)) < A \tag{2.2}$$

for some positive numbers A and a so that

$$R = \max\left\{ \sup_{x\in(0,\infty)} p(x),\ \sup_{x\in(0,\infty)} |p'(x)| \right\} < \frac{\delta}{1+a}, \tag{2.3}$$

and

$$a > 2\sqrt{\frac{A}{\delta - R(1+a)}} + 2\frac{A\exp(a)a^r\Gamma(1-r)}{\delta(1-\eta)}. \tag{2.4}$$

Also, let $g_{01}(0,z) = g_{02}(z,0),\ z \in \mathbb{R}^3$. Then, there exists a nonnegative continuous solution g of the equation (1.1) in $\Omega^+_{\cdot,r}(T)$. This existing solution is differentiable along the characteristics of (1.1). Moreover, the solution g is unique in $\Omega^+_{\cdot,r}(T)$ provided

$$div_z v(x,z) + p'(x) \leq M(1+x),\ x \in \mathbb{R}^1_+,\ z \in \mathbb{R}^3. \tag{2.5}$$

for some constant M.

We proof our result step by step. To do so, following auxiliary result need to formulate.

Lemma 2.1. Assume Theorem 2.1 hold, and the coagulation kernel K has a compacta. Then, the governing equations (1.1)-(1.3) possesses a unique continuous solution $g \in \Omega_{\cdot,r}(T)$.

Proof. The proof follows a similar fashion to Lemma 2.1 of [56]. □

Now, following the kernel truncation idea of Camejo and Warnecke [41], Paul and Kumar [57], we approximate the unbounded singular kernel K by a sequence $\{K_n\}_{n=1}^\infty$ of continuous kernels with compact supports in the following technique:

$$K_n(x,y) \begin{cases} = K(x,y) & \text{if } (x,y) \in [\frac{1}{n}, n] \times [\frac{1}{n}, n] \\ \leq K(x,y) & \text{if } (x,y) \in (0,\infty) \times (0,\infty) \setminus [\frac{1}{n}, n] \times [\frac{1}{n}, n] \\ \searrow 0 & \text{in a finite domain of } (0,\infty) \times (0,\infty) \setminus [\frac{1}{n}, n] \times [\frac{1}{n}, n] \end{cases}$$

Notice that each K_n satisfies the conditions of Theorem 2.1. Now, recalling the Lemma 2.1, there exists a sequence $\{g_n\}_{n=1}^\infty$ of solutions to the problems (1.1)-(1.3) for the truncated kernels $\{K_n\}_{n=1}^{\infty}$.

With the help of the substitution

$$g_n(t,x,z) = (1-\tau)\hat{g}_n(\tau,x,z),\ \tau = 1-\exp(-\delta t),\ n = 1,2,3,\ldots.$$

we note that the problem (1.1)-(1.3) yields the following equivalent form:

$$\begin{aligned}
&\delta\frac{\partial}{\partial\tau}\hat{g}_n(\tau,x,z) + (1-\tau)^{-1}(v(x,z),\nabla_z\hat{g}_n(\tau,x,z)) + (1-\tau)^{-1}p(x)\frac{\partial}{\partial x}\hat{g}_n(\tau,x,z)\\
=\ &\frac{1}{2}\int_0^x K_n(x-y,y)\hat{g}_n(\tau,x-y,z)\hat{g}_n(y,z,\tau)\mathrm{d}y - \hat{g}_n(\tau,x,z)\int_0^\infty K_n(x,y)\hat{g}_n(\tau,y,z)\mathrm{d}y\\
&-[\mathrm{div}_z v(x,z) + p'(x) - \delta](1-\tau)^{-1}\hat{g}_n(\tau,x,z) \qquad (2.6)
\end{aligned}$$

supported by the following IC and BC

$$\left.\begin{aligned}\hat{g}_n(0,x,z) &= g_{01}(x,z)\\ \text{and } \hat{g}_n(\tau,0,z) &= (1-\tau)^{-1}g_{02}(t,z).\end{aligned}\right\} \qquad (2.7)$$

Lemma 2.2. Let the Theorem 2.1 holds and there exists a continuous function h that satisfies

$$\delta\frac{\partial h}{\partial\tau} + \frac{p(x)}{1-\tau}\frac{\partial h}{\partial x} = \frac{1}{2}\int_0^x \left((x-y)^{\theta-\mu}y^{-\mu} + (x-y)^{-\mu}y^{\theta-\mu}\right)h(\tau,x-y)h(\tau,y)\mathrm{d}y \qquad (2.8)$$

$$\text{with } h(0,x) = A\exp(-ax),\ \text{ where } A = h(\tau,0). \qquad (2.9)$$

Then, for each $n = 1,2,3,\ldots$,

$$\hat{g}_n(\tau,x,z) < h(\tau,x) \text{ for all } x>0,\ z\in\mathbb{R}^3,\ \tau\in[0,1).$$

Proof. Suppose (τ_0,x_0,z_0) is the first point where $\hat{g_n} = h$:

$$\hat{g}_n(\tau_0,x_0,z_0) = h(\tau_0,x_0), \hat{g}_n(\tau,x,z) < h(\tau,x), \tau\in[0,\tau_0), x\in[0,x(\tau)), z = z(\tau). \qquad (2.10)$$

Here $z(\tau)$ and $x(\tau)$ in (2.10) indicates on characteristic going through the point (τ_0,x_0,z_0) with $z(\tau_0) = z_0$ and $x(\tau_0) = x_0$. The existence of the point (τ_0,x_0,z_0) follows by continuity of $\hat{g}_n, h, r > 0$, and with the help of (2.2) and (2.9). By integrating (2.6) and (2.8) along characteristics, we get

$$\begin{aligned}
\hat{g}_n(\tau_0,x_0,z_0) &\le \frac{1}{2\delta}\int_0^{\tau_0}\int_0^{x_0} K_n(x(s)-y,y)\hat{g}_n(s,x(s)-y,z(s))\hat{g}_n(s,y,z(s))\mathrm{d}y\,\mathrm{d}s\\
&< \frac{1}{2\delta}\int\int_0^{\tau_0}\int_0^{x_0}\left((x-y)^{\theta-\mu}y^{-\mu} + (x-y)^{-\mu}y^{\theta-\mu}\right)h(s,x(s)-y)h(s,y)\mathrm{d}y\,\mathrm{d}s\\
&= h(\tau_0,x_0), \qquad (2.11)
\end{aligned}$$

which contradicts $\hat{g}_n(\tau_0,x_0,z_0) = h(\tau_0,x_0)$. Hence the Lemma 2.2 is proved.

□

In the next, we define

$$H(\tau,\lambda)=\int_0^\infty\left(\exp(\lambda x)+\frac{\exp(\lambda)}{x^r}\right)h(\tau,x)\mathrm{d}x.$$

Therefore, from (2.8), we obtain

$$\begin{aligned}
&\delta\frac{\partial H}{\partial\tau}-(1-\tau)^{-1}\int_0^\infty\left(\lambda\exp(\lambda x)-\frac{\exp(\lambda)r}{x^{r+1}}\right)r(x)h(\tau,x)\\
&-(1-\tau)^{-1}\int_0^\infty\left(\exp(\lambda x)+\frac{\exp(\lambda)}{x^r}\right)r'(x)h(\tau,x)\mathrm{d}x\\
&\leq\frac{1}{2}\int_0^\infty\left(\exp(\lambda x)+\frac{\exp(\lambda)}{x^r}\right)\int_0^x\left[(x-y)^{\theta-\mu}y^{-\mu}+(x-y)^{-\mu}y^{\theta-\mu}\right]h(\tau,x-y)h(\tau,y)\mathrm{d}y\,\mathrm{d}x.
\end{aligned}\tag{2.12}$$

For the expression in the right of (2.12) we calculate the following inequality

$$\begin{aligned}
&\int_{x=0}^\infty\left(\exp(\lambda x)+\frac{\exp(\lambda)}{x^r}\right)x^{\theta-\mu}h(\tau,x)\mathrm{d}x\\
=\ &\int_{x=0}^\infty\left(\exp(\lambda x)x^{\theta-\mu}+\exp(\lambda)x^{\theta-\mu-r}\right)h(\tau,x)\mathrm{d}x\\
=\ &\int_{x=0}^1\left(\exp(\lambda x)x^{\theta-\mu}+\exp(\lambda)x^{\theta-\mu-r}\right)h(\tau,x)dx+\int_{x=1}^\infty\left(\exp(\lambda x)x^{\theta-\mu}+\exp(\lambda)x^{\theta-\mu-r}\right)h(\tau,x)\mathrm{d}x\\
\leq\ &\int_{x=0}^1\left(\exp(\lambda)+\exp(\lambda)x^{-r}\right)h(\tau,x)\mathrm{d}x+\int_{x=1}^\infty\left(\exp(\lambda x)x+\exp(\lambda x)x\right)h(\tau,x)\mathrm{d}x\\
\leq\ &2\int_{x=0}^1\left(x\exp(\lambda x)+\exp(\lambda)x^{-r}\right)h(\tau,x)\mathrm{d}x+\int_{x=1}^\infty 2x\exp(\lambda x)h(\tau,x)\mathrm{d}x\\
\leq\ &2\int_{x=0}^1\left(x\exp(\lambda x)+\frac{\exp(\lambda)}{x^r}\right)h(\tau,x)\mathrm{d}x+\int_{x=1}^\infty 2\left(x\exp(\lambda x)+\frac{\exp(\lambda)}{x^r}\right)h(\tau,x)\mathrm{d}x\\
\leq\ &2\int_{x=0}^\infty\left(x\exp(\lambda x)+\frac{\exp(\lambda)}{x^r}\right)h(\tau,x)\mathrm{d}x\\
=\ &2\frac{\partial H}{\partial\lambda}.
\end{aligned}\tag{2.13}$$

Further, the following inequality is also true:

$$\begin{aligned}
& \int_{y=0}^{\infty} \exp(\lambda y) y^{-\mu} h(\tau, y) \mathrm{d}y \\
= & \int_{y=0}^{1} \exp(\lambda y) y^{-\mu} h(\tau, y) \mathrm{d}y + \int_{y=1}^{\infty} \exp(\lambda y) y^{-\mu} h(\tau, y) \mathrm{d}y \\
\leq & \exp(\lambda) \int_{y=0}^{1} y^{-\mu} h(\tau, y) \mathrm{d}y + \int_{y=1}^{\infty} \exp(\lambda y) h(\tau, y) \mathrm{d}y \\
\leq & \int_{y=0}^{1} \exp(\lambda) y^{-r} h(\tau, y) \mathrm{d}y + \int_{y=1}^{\infty} \exp(\lambda y) h(\tau, y) \mathrm{d}y \\
\leq & \int_{y=0}^{1} \left(\exp(\lambda y) + \frac{\exp(\lambda)}{y^r} \right) h(\tau, y) \mathrm{d}y + \int_{y=1}^{\infty} \left(\exp(\lambda y) + \frac{\exp(\lambda)}{y^r} \right) h(\tau, y) \mathrm{d}y \\
\leq & \int_{y=0}^{\infty} \left(\exp(\lambda y) + \frac{\exp(\lambda)}{y^r} \right) h(\tau, y) \mathrm{d}y \\
= & H(\tau, \lambda)
\end{aligned} \tag{2.14}$$

Thus, by (2.13) and (2.14), we get from (2.12) that

$$\delta \frac{\partial H}{\partial \tau} - \frac{1+a}{1-\tau} R H(\tau, \lambda) \leq 2H(\tau, \lambda) \frac{\partial H}{\partial \lambda}, \tag{2.15}$$

where R is given by (2.3) and

$$\begin{aligned}
H(0, \lambda) = & \int_0^{\infty} \left(\exp(\lambda x) + \frac{\exp(\lambda)}{x^r} \right) h(0, x) \mathrm{d}x \\
= & A \int_0^{\infty} \left(\exp(\lambda x - ax) + \exp(\lambda) \exp(-ax) x^{-r} \right) \mathrm{d}x \\
= & \frac{A}{a - \lambda} + A \exp(\lambda) a^r \Gamma(1-r),
\end{aligned} \tag{2.16}$$

where A is as given in the statement of Theorem 2.1.

Lemma 2.3. For a function H that satisfies the inequality (2.15) and the condition (2.16), there exists $\tilde{\lambda} \in (0, a)$ such that

$$H(\tau, \lambda) < F(\tau, \lambda), \ \tau \in [0, 1), \ 0 \leq \lambda \leq \tilde{\lambda}$$

where the real-valued function $F(\tau, \lambda)$ is a solution of the following equation

$$\delta \frac{\partial F}{\partial \tau} - \frac{1+a}{1-\tau} R F(\tau, \lambda) = 2F(\tau, \lambda) \frac{\partial F}{\partial \lambda} \tag{2.17}$$

with

$$F(0, \lambda) = \frac{D}{a - \lambda} + D \exp(a) a^r \Gamma(1-r), \tag{2.18}$$

where D is any positive number bigger than A.

Proof. The strategy of the proof is on contradictory way. For the PDE (2.17) with the initial condition (2.18) we consider the family of characteristics. In the next, we define

$$\Xi(\tau_0,\lambda_0)=\{(\tau,\lambda):\tau\in[0,\tau_0],\ \lambda\in[0,\lambda(\tau))\},\tag{2.19}$$

where $\lambda(\tau)$ indicates the value of λ on the characteristic curve $\Theta(\tau_0,\lambda_0)$ of (2.17) which passes through (τ_0,λ_0). Also, we assume $\lambda(0)\in(0,a)$. We take (τ_0,λ_0) such a way that

$$F(\tau_0,\lambda_0)=H(\tau_0,\lambda_0),\ \text{but } H(\tau,\lambda)<F(\tau,\lambda) \text{ if } (\tau,\lambda)\in\Xi(\tau_0,\lambda_0).$$

It is notable that $\tau_0>0$ since $D>A$. In the next, we consider the characteristic curve $\tilde{\Theta}(\tau_0,\lambda_0)$ for the problem (2.15) with the initial condition (2.16) and also we consider $\tilde{\Theta}(\tau_0,\lambda_0)\in\Xi(\tau_0,\lambda_0)$. Therefore,

$$\begin{aligned}
H(\tau_0,\lambda_0)&\leq H(0,\bar{\lambda}_0)+\int_{\tilde{\Theta}(\tau_0,\lambda_0)}\frac{R(1+a)}{1-\tau}H(\tau,\lambda(\tau))\mathrm{d}\tau\\
&< H(0,\hat{\lambda}_0)+\int_{\Theta(\tau_0,\lambda_0)}\frac{R(1+a)}{1-\tau}H(\tau,\lambda(\tau))\mathrm{d}\tau\\
&< F(0,\hat{\lambda}_0)+\int_{\Theta(\tau_0,\lambda_0)}\frac{R(1+a)}{1-\tau}F(\tau,\lambda(\tau))\mathrm{d}\tau\\
&= F(\tau_0,\lambda_0)
\end{aligned}$$

where $\hat{\lambda}_0$ and $\bar{\lambda}_0$ are beginnings of the characteristic curve $\Theta(\tau_0,\lambda_0)$ and $\tilde{\Theta}(\tau_0,\lambda_0)$ and $\hat{\lambda}_0>\bar{\lambda}_0$. We have used $a>\lambda$ and H is a increasing function w.r.t. λ. Therefore $H(\tau_0,\lambda_0)<F(\tau_0,\lambda_0)$ which is a contradiction of $H(\tau_0,\lambda_0)=F(\tau_0,\lambda_0)$. Hence the Lemma 2.3 proved. $\square$

In the next, we focus on the function $F(\tau,\lambda)$. By the substitution $F(\tau,\lambda)=(1-\tau)^{-\eta}L(\tau,\lambda)$ for all $\lambda\in[0,\tilde{\lambda}],\ \tau\in[0,1)$, where $\eta=\frac{R(1+a)}{\delta}$, the equation (2.17) reduces to

$$\delta\frac{\partial L}{\partial\tau}-2(1-\tau)^{-\eta}L(\tau,\lambda)\frac{\partial L}{\partial\lambda}=0\tag{2.20}$$

with

$$L(0,\lambda)=\frac{D}{a-\lambda}+D\exp(a)a^r\Gamma(1-r).$$

Now, the characteristic equations of (2.20) gives

$$\frac{d\lambda}{d\tau}=-\delta^{-1}2(1-\tau)^{-\eta}L(\tau,\lambda)\text{ and }L(\tau,\lambda)=L(0,\lambda_0),$$

where λ_0 is the value of $\lambda(\tau)$ at $\tau=0$. Thus,

$$L(\tau,\lambda)=\frac{D}{a-\lambda_0}+D\exp(a)a^r\Gamma(1-r),$$

and

$$\frac{d\lambda}{d\tau} = -2\delta^{-1}(1-\tau)^{-\eta}\left(\frac{D}{a-\lambda_0} + D\exp(a)a^r\Gamma(1-r)\right). \tag{2.21}$$

Hence,

$$\lambda(\tau) = \lambda_0 - 2D\delta^{-1}\left(\frac{1}{a-\lambda_0} + \exp(a)a^r\Gamma(1-r)\right)\frac{1-(1-\tau)^{1-\eta}}{1-\eta}. \tag{2.22}$$

Next, we find the condition under which the characteristics given by (2.22) for different starting point λ_0^1 and λ_0^2 intersect. If they intersect, then

$$\lambda_0^1 - \lambda_0^2 = \frac{2D}{\delta(1-\eta)}(1-(1-\tau)^{1-\eta})(\lambda_0^1 - \lambda_0^2)\frac{1}{(a-\lambda_0^1)(a-\lambda_0^2)},$$

which implies

$$1-(1-\tau)^{1-\eta} = \frac{\delta(1-\eta)}{2D}\ (a-\lambda_0^1)(a-\lambda_0^2). \tag{2.23}$$

Consequently, the characteristic curves of the problem (2.20) have no intersection if

$$2D < a^2\delta\left(1 - R\frac{(1+a)}{\delta}\right) \tag{2.24}$$

and the initial datum are sufficiently small:

$$0 < \lambda_0^1, \lambda_0^2 < a - \sqrt{\frac{D}{\delta - R(1+a)}}. \tag{2.25}$$

We remark here that (2.24) and (2.25) implies the right of (2.23) is bigger than 1, and hence (2.23) will not hold if (2.24) and (2.25) are true.

Since $D > 0$, the inequality (2.24) gives $R < \frac{\delta}{1+a}$.

Let us now find when the problem (2.20) will have a smooth solution for a small $\lambda > 0$ and for all $\tau \in [0,1)$. Note that if the characteristics have no intersection and $\lambda(1) > 0$, then (2.20) has a smooth solution. At $\tau = 1$, we obtain from (2.22) that

$$\lambda_0 - 2\delta^{-1}D\left(\frac{1}{a-\lambda_0} + \exp(a)a^r\Gamma(1-r)\right)\frac{1}{1-\eta} > 0. \tag{2.26}$$

Hence,

$$\frac{a}{2} + \frac{D\exp(a)a^r\Gamma(1-r)}{\delta(1-\eta)} - C < \lambda_0 < \frac{a}{2} + \frac{D\exp(a)a^r\Gamma(1-r)}{\delta(1-\eta)} + C, \tag{2.27}$$

where

$$C = \frac{1}{2}\sqrt{\left(a + \frac{2D\exp(a)a^r\Gamma(1-r)}{\delta(1-\eta)}\right)^2 - 8\frac{D}{\delta(1-\eta)}(1 + a\exp(a)a^r\Gamma(1-r))} \tag{2.28}$$

To obtain a suitable $\lambda_0 > 0$, the equations (2.25) and (2.27) need to be compatible, which indicates

$$\frac{a}{2} + \frac{A\exp(a)a^r\Gamma(1-r)}{\delta(1-\eta)} - C < a - \sqrt{\frac{A}{\delta - R(1+a)}} \tag{2.29}$$

since $D > A$. Hence, (2.29) holds, provided

$$a > 2\sqrt{\frac{A}{\delta - R(1+a)}} + 2\frac{A\exp(a)a^r\Gamma(1-r)}{\delta(1-\eta)}. \tag{2.30}$$

We observe that the estimate inside the square root of (2.28) is positive if (2.30) holds. Furthermore, the inequality (2.30) ensures that (2.25) and (2.29) holds due to the condition (2.4) of Theorem 2.1. Thus, for sufficiently small enough $\lambda > 0$, and $0 \le \tau \le 1$, there exists a continuous function $F(\tau, \lambda)$ and the supremum $\sup_{0\le\tau\le 1} F(\tau,\lambda)$ is an upper bound of the integrals

$$\sup_{t\in[0,\infty)} \int_0^\infty \left(\exp(\lambda x) + \frac{\exp(\lambda)}{x^r}\right) \sup_{z\in\mathbb{R}^3} g_n(t,x,z)\mathrm{d}x$$

uniformly with respect to $n \ge 1$. Subsequently, the proof of the following lemma follows.

Lemma 2.4. Under the conditions of Theorem 2.1, there exist positive constants E and $\tilde{\lambda}$ so that

$$\sup_{0\le\lambda\le\tilde{\lambda}} \|g_n\|_{\lambda,r} \le E < +\infty,\ n \ge 1.$$

Proof of Theorem 2.1

Lemma 2.5. The sequence $\{g_n\}_{n=1}^\infty$ is uniformly bounded and equicontinuous on each compact set in $(0,\infty)\times\mathbb{R}^3\times[0,T]$ under the conditions of the Theorem 2.1.

Proof. The proof follows from Lemma 4.1 in [56]. □

By that standard diagonal process we can derive a subsequence $\{g_{n'}\}_{n'=1}^\infty$, from the sequence $\{g_n\}_{n=1}^\infty$ which converges on each compact domain to a continuous function $g \ge 0$, thanks to Lemma 2.5 [58].

Note that from the corollary of Lemma 2.4 we have

$$\sup_{0\le\lambda\le\tilde{\lambda}} \|g\|_{\lambda,r} \le E.$$

Thus, the result in Theorem 2.1 follows.

3. Uniqueness of solution

In order to demonstrate the uniqueness of solution, let there are two solutions $d_1,\ d_2 \in \Omega_{\cdot,r}(T)$ to the problem (1.1)–(1.3) under the hypothesis in Theorem 2.1.

Let us denote $u = d_1 - d_2$ and $\psi = d_1 + d_2$.

Then, from (1.1), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}u(t,x,z) + p(x)\frac{\partial}{\partial x}u(t,x,z) + v(x,z)\frac{\partial}{\partial z}u(t,x,z) \\
=\ & \frac{1}{2}\int_0^x K(x-y,y)u(t,x-y,z)\psi(t,y,z)\mathrm{d}y - \frac{1}{2}\int_0^\infty K(x,y)u(t,x,z)\psi(t,y,z)\mathrm{d}y \\
&- \frac{1}{2}\int_0^\infty K(x,y)u(t,y,z)\psi(t,x,z)dy - (p'(x) + v_z(x,z))u(t,x,z). \qquad (3.1)
\end{aligned}$$

In the next, we define two functions $U(t,\lambda)$ and $\Psi(t,\lambda)$ in the following way:

$$\begin{aligned}
U(t,\lambda) &= \sup_{z\in\mathbb{R}^3}\int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)|u(t,x,z)|\mathrm{d}x \qquad (3.2)\\
\text{and } \Psi(t,\lambda) &= \sup_{z\in\mathbb{R}^3}\int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\psi(t,x,z)\mathrm{d}x.
\end{aligned}$$

Now, multiplying (3.1) by $\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)$, where $0 < \nu < r - \mu$, and then integrating w.r.t x in the range $(0,\infty)$ and taking supremum over $z \in \mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}U(t,\lambda) + \sup_{z\in\mathbb{R}^3}\int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)p(x)\,\mathrm{sgn}(u(t,x,z))\frac{\partial u}{\partial x}\,\mathrm{d}x \\
&+ \sup_{z\in\mathbb{R}^3}\int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)v(x,z)\,\mathrm{sgn}(u(t,x,z))\frac{\partial u}{\partial z}\,\mathrm{d}x \\
\le\ & \sup_{z\in\mathbb{R}^3}\Bigg\{\frac{1}{2}\int_0^\infty \mathrm{sgn}(u(t,x,z))\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\int_0^x K(x-y,y)u(t,x-y,z)\psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \frac{1}{2}\int_0^\infty \mathrm{sgn}(u(t,x,z))\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)u(t,x,z)\int_0^\infty K(x,y)\psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \frac{1}{2}\int_0^\infty \mathrm{sgn}(u(t,x,z))\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\psi(t,x,z)\int_0^\infty K(x,y)u(t,y,z)\,\mathrm{d}y\,\mathrm{d}x\Bigg\} \\
&+ \sup_{z\in\mathbb{R}^3}\int_0^\infty (p'(x) + v_z(x,z))u(t,x,z)\,\mathrm{sgn}(u(t,x,z))\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\mathrm{d}x. \qquad (3.3)
\end{aligned}$$

For the first expression on the right side of (3.3), we have

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \mathrm{sgn}(u(t,x,z))\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\int_0^x K(x-y,y)u(t,x-y,z)\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x \\
=\ & \frac{1}{2}\int_0^\infty\int_0^\infty \mathrm{sgn}(u(x+y,z,s))\left(\exp(\lambda(x+y)) + \frac{1}{(x+y)^\nu}\right)K(x,y)u(t,x,z)\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$
(3.4)

Therefore, first three terms on the right of (3.3) yield

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty\int_0^\infty \left[\operatorname{sgn}(u(x+y,z,t))\left(\exp(\lambda(x+y))+\frac{1}{(x+y)^\nu}\right)-\operatorname{sgn}(u(t,x,z))\left(\exp(\lambda x)+\frac{1}{x^\nu}\right)\right.\\
&\left.-\operatorname{sgn}(u(t,y,z))\left(\exp(\lambda y)+\frac{1}{y^\nu}\right)\right]K(x,y)u(t,x,z)\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x\\
\le\ &\frac{1}{2}\int_0^\infty\int_0^\infty\left[\left(\exp(\lambda(x+y))+\frac{1}{(x+y)^\nu}\right)+\left(\exp(\lambda y)+\frac{1}{y^\nu}\right)\right]K(x,y)|u(t,x,z)|\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x\\
\le\ &\frac{1}{2}\int_0^\infty\int_0^\infty\left[\left(\exp(\lambda(x+y))+\frac{1}{y^\nu}\right)+\left(\exp(\lambda y)+\frac{1}{y^\nu}\right)\right]K(x,y)|u(t,x,z)|\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x\\
\le\ &\frac{1}{2}\int_0^\infty\int_0^\infty 2\exp(\lambda x)\left[\left(\exp(\lambda y)+\frac{1}{y^\nu}\right)\right]K(x,y)|u(t,x,z)|\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x\\
\le\ &\frac{1}{2}\int_0^\infty\int_0^\infty 2\exp(\lambda x)\left(\exp(\lambda y)+\frac{1}{y^\nu}\right)\frac{x^\theta+y^\theta}{(xy)^\mu}|u(t,x,z)|\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x \qquad (3.5)
\end{aligned}
$$

In the next, we recall the following inequalities from [45]:

$$
\begin{aligned}
&\sup_{z\in\mathbb{R}^3}\int_0^\infty \exp(\lambda x)x^{-\mu}|u(t,x,z)|\mathrm{d}x \le U(t,\lambda)(1+\exp(\lambda)),\\
&\sup_{z\in\mathbb{R}^3}\int_0^\infty \exp(\lambda x)x^{\theta-\mu}|u(t,x,z)|\mathrm{d}x \le \chi_0 U(t,\lambda)+\frac{\partial U}{\partial\lambda},\\
&\sup_{z\in\mathbb{R}^3}\int_0^\infty \left(\exp(\lambda y)+\frac{1}{y^\nu}\right)y^{-\mu}\psi(t,y,z)\mathrm{d}y \le \chi_0\bar{N}_{-\nu-\mu}+\Psi(t,\lambda),\\
\text{and }&\sup_{z\in\mathbb{R}^3}\int_0^\infty \left(\exp(\lambda y)+\frac{1}{y^\nu}\right)y^{\theta-\mu}\psi(t,y,z)dy \le \Psi(t,\lambda)+\frac{\partial\Psi}{\partial\lambda}+\bar{M}, \qquad (3.6)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{N}_{-\mu-\nu} &= \sup_{z\in\mathbb{R}^3}\int_0^\infty x^{-\mu-\nu}(c_1(t,x,z)+c_2(t,x,z))\mathrm{d}x < +\infty,\\
\bar{M} &= \sup_{z\in\mathbb{R}^3}\int_0^\infty x(c_1(t,x,z)+c_2(t,x,z))\mathrm{d}x < +\infty,\\
\text{and }\ \chi_0 &= (1+\exp(\lambda)).
\end{aligned}
$$

Therefore, from (3.5), we get

$$
\begin{aligned}
&\sup_{z\in\mathbb{R}^3}\int_0^\infty\int_0^\infty \exp(\lambda x)\left(\exp(\lambda y)+\frac{1}{y^\nu}\right)\frac{x^\theta+y^\theta}{(xy)^\mu}|u(t,x,z)|\psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}x\\
\le\ &\sup_{z\in\mathbb{R}^3}\int_0^\infty \exp(\lambda x)x^{\theta-\mu}|u(t,x,z)|dx\int_0^\infty\left(\exp(\lambda y)+\frac{1}{y^\nu}\right)y^{-\mu}\psi(t,y,z)\mathrm{d}y\\
&+\sup_{z\in\mathbb{R}^3}\int_0^\infty \exp(\lambda x)x^{-\mu}|u(t,x,z)|dx\int_0^\infty\left(\exp(\lambda y)+\frac{1}{y^\nu}\right)y^{\theta-\mu}\psi(t,y,z)\mathrm{d}y\\
\le\ &\left(\chi_0 U+\frac{\partial U}{\partial\lambda}\right)(\chi_0\bar{N}_{-\nu-\mu}+\Psi)+U\chi_0\left(\Psi+\frac{\partial U}{\partial\lambda}+\bar{M}\right). \qquad (3.7)
\end{aligned}
$$

By using (2.5), the last term in (3.3) gives

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^3}\int_0^\infty (p'(x)+v_z(x,z))u(t,x,z)\,\mathrm{sgn}(u(t,x,z))\left(\exp(\lambda x)+\frac{1}{x^\nu}\right)\mathrm{d}x\\
\le\ &\sup_{z\in\mathbb{R}^3}\int_0^\infty M(1+x)\,\mathrm{sgn}(u(t,x,z))\left(\exp(\lambda x)+\frac{1}{x^\nu}\right)u(t,x,z)\mathrm{d}x\\
=\ &MU(t,\lambda)+MU_\lambda(t,\lambda)+M\int_0^\infty x^{1-\nu}|u(t,x,z)|\mathrm{d}x. \qquad (3.8)
\end{aligned}$$

For the last term of (3.8), we obtain

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^3}\int_0^\infty x^{1-\nu}|u(t,x,z)|\mathrm{d}x\\
\le\ &\sup_{z\in\mathbb{R}^3}\int_0^1 x^{1-\nu}|u(t,x,z)|\mathrm{d}x+\sup_{z\in\mathbb{R}^3}\int_1^\infty x^{1-\nu}|u(t,x,z)|\mathrm{d}x\\
\le\ &\sup_{z\in\mathbb{R}^3}\int_0^1\left(\exp(\lambda x)+\frac{1}{x^\nu}\right)|u(t,x,z)|dx+\sup_{z\in\mathbb{R}^3}\int_1^\infty x\exp(\lambda x)|u(t,x,z)|\mathrm{d}x\\
\le\ &U(t,\lambda)+\frac{\partial U}{\partial\lambda} \qquad (3.9)
\end{aligned}$$

Thus, by using (3.7), (3.8) and (3.9), we obtain from (3.3) that

$$\begin{aligned}
\frac{\partial U}{\partial t}\ \le\ &\left(\chi_0 U+\frac{\partial U}{\partial\lambda}\right)(\chi_0\bar{N}_{-\nu-\mu}+\Psi)+U\chi_0\left(\Psi+\bar{M}\frac{\partial\Psi}{\partial\lambda}\right)+2M\left(U+\frac{\partial U}{\partial\lambda}\right)\\
=\ &U\left[(\chi_0^2\bar{N}_{-\nu-\mu}+\chi_0\Psi)+\chi_0\left(\Psi+\frac{\partial\Psi}{\partial\lambda}+\bar{M}\right)+2M\right]+\frac{\partial U}{\partial\lambda}(\chi_0\bar{N}_{-\nu-\mu}+\Psi+2M)
\end{aligned} \qquad (3.10)$$

Multiplying (3.1) by $x\exp(\lambda x)$, and taking supremum on $z\in\mathbb{R}^3$ after integrating over x in the range $(0,\infty)$, we attain

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial U}{\partial\lambda}+\sup_{z\in\mathbb{R}^3}\int_0^\infty x\exp(\lambda x)p(x)\,\mathrm{sgn}(u(t,x,z))\frac{\partial u}{\partial x}\mathrm{d}x\\
&+\sup_{z\in\mathbb{R}^3}\int_0^\infty x\exp(\lambda x)v(x,z)\,\mathrm{sgn}(u(t,x,z))\frac{\partial u}{\partial z}\mathrm{d}x\\
\le\ &\sup_{z\in\mathbb{R}^3}\left\{\frac{1}{2}\int_0^\infty \mathrm{sgn}(u(t,x,z))x\exp(\lambda x)\int_0^x K(x-y,y)u(t,x-y,z)\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x\right.\\
&\qquad-\frac{1}{2}\int_0^\infty \mathrm{sgn}(u(t,x,z))x\exp(\lambda x)u(t,x,z)\int_0^\infty K(x,y)\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x\\
&\qquad\left.-\frac{1}{2}\int_0^\infty \mathrm{sgn}(u(t,x,z))x\exp(\lambda x)\psi(t,x,z)\int_0^\infty K(x,y)u(t,y,z)\mathrm{d}y\,\mathrm{d}x\right\}\\
&+\sup_{z\in\mathbb{R}^3}\int_0^\infty\left(p'(x)+\frac{\partial v}{\partial z}\right)u(t,x,z)\,\mathrm{sgn}(u(t,x,z))x\exp(\lambda x)\mathrm{d}x. \qquad (3.11)
\end{aligned}$$

From the expression under the braces on the right of (3.11), we get

$$
\begin{aligned}
& \frac{1}{2}\int_0^\infty \int_0^\infty [\operatorname{sgn}(u(x+y,z,t))(x+y)\exp(\lambda(x+y)) - \operatorname{sgn}(u(t,x,z))x\exp(\lambda x) \\
& \qquad\qquad - \operatorname{sgn}(u(t,y,z))y\exp(\lambda,y)]K(x,y)u(t,x,z)\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x \\
\leq\ & \frac{1}{2}\int_0^\infty \int_0^\infty [(x+y)\exp(\lambda(x+y)) - x\exp(\lambda x) \\
& \qquad\qquad + y\exp(\lambda,y)]K(x,y)|u(t,x,z)|\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x \\
\leq\ & \frac{1}{2}\int_0^\infty \int_0^\infty [(x+y)\exp(\lambda(x+y)) + y\exp(\lambda,y)]K(x,y)|u(t,x,z)|\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x \\
=\ & \frac{1}{2}\int_0^\infty \int_0^\infty [(x+y)\exp(\lambda(x+y)) + y\exp(\lambda,y)](x^{\theta-\mu}y^{-\mu} + x^{-\mu}y^{\theta-\mu}) \\
& \qquad\qquad |u(t,x,z)|\psi(t,y,z)\mathrm{d}y\,\mathrm{d}x
\end{aligned}
\tag{3.12}
$$

Next, we call back the following inequalities from [45]:

$$
\left.
\begin{aligned}
& \sup_{z\in\mathbb{R}^3}\int_0^\infty \exp(\lambda x)x^{-\mu}|u(t,x,z)|\mathrm{d}x \leq \chi_0 U(t,\lambda), \\
& \sup_{z\in\mathbb{R}^3}\int_0^\infty \exp(\lambda x)x^{\theta-\mu}|u(t,x,z)|\mathrm{d}x \leq \chi_0 U(t,\lambda) + \frac{\partial U}{\partial \lambda}, \\
& \sup_{z\in\mathbb{R}^3}\int_0^\infty x\exp(\lambda x)x^{-\mu}|u(t,x,z)|\mathrm{d}x \leq \chi_0 U(t,\lambda) + \frac{\partial U}{\partial \lambda}, \\
& \sup_{z\in\mathbb{R}^3}\int_0^\infty x\exp(\lambda x)x^{\theta-\mu}|u(t,x,z)|\mathrm{d}x \leq \frac{\partial U}{\partial \lambda} + \frac{\partial^2 U}{\partial \lambda^2}, \\
& \sup_{z\in\mathbb{R}^3}\int_0^\infty \exp(\lambda x)x^{-\mu}\psi(t,x,z)\mathrm{d}x \leq \chi_0 \Psi(t,\lambda), \\
& \sup_{z\in\mathbb{R}^3}\int_0^\infty \exp(\lambda x)x^{\theta-\mu}\psi(t,x,z)\mathrm{d}x \leq \chi_0 \Psi(t,\lambda) + \frac{\partial \Psi}{\partial \lambda}, \\
& \sup_{z\in\mathbb{R}^3}\int_0^\infty x\exp(\lambda x)x^{-\mu}\psi(t,x,z)\mathrm{d}x \leq \chi_0 \Psi(t,\lambda) + \frac{\partial \Psi}{\partial \lambda}, \\
& \sup_{z\in\mathbb{R}^3}\int_0^\infty x\exp(\lambda x)x^{\theta-\mu}\psi(t,x,z)\mathrm{d}x \leq \frac{\partial \Psi}{\partial \lambda} + \frac{\partial^2 \Psi}{\partial \lambda^2}, \\
& \sup_{z\in\mathbb{R}^3}\int_0^\infty x^{-\mu}|u(t,x,z)|\mathrm{d}x \leq U(t,\lambda) \\
\text{and}\ & \sup_{z\in\mathbb{R}^3}\int_0^\infty x^{\theta-\mu}|u(t,x,z)|\mathrm{d}x \leq U(t,\lambda),
\end{aligned}
\right\}
\tag{3.13}
$$

where $\chi_0 = 1 + \exp(\lambda)$. Therefore, from (3.11), using (3.13), we get

$$\begin{aligned}
\frac{\partial^2 U}{\partial\lambda\partial t} \le \frac{1}{2}\ \Bigg[& \left(\frac{\partial U}{\partial\lambda} + \frac{\partial^2 U}{\partial\lambda^2}\right)\chi_0\Psi + \left(\chi_0 U + \frac{\partial U}{\partial\lambda}\right)\left(\chi_0\Psi + \frac{\partial\Psi}{\partial\lambda}\right) \\
& + \left(\chi_0 U + \frac{\partial U}{\partial\lambda}\right)\left(\chi_0\Psi + \frac{\partial\Psi}{\partial\lambda}\right) + U\chi_0\left(\frac{\partial\Psi}{\partial\lambda} + \frac{\partial^2\Psi}{\partial\lambda^2}\right) \\
& + U\left(\chi_0\Psi + \frac{\partial\Psi}{\partial\lambda}\right) + U\left(\Psi + \frac{\partial^2\Psi}{\partial\lambda^2}\right) \\
& + \bar{M}\left(U + \frac{\partial U}{\partial\lambda}\right) + R\left(2 + \tilde{\lambda}\right)U\Bigg]. \qquad (3.14)
\end{aligned}$$

From (3.10) and (3.14), we obtain

$$\begin{aligned}
\frac{\partial U}{\partial t} &\le b_1(t,\lambda)\frac{\partial U}{\partial\lambda} + b_2(t,\lambda)U(t,\lambda) \\
\frac{\partial^2 U}{\partial\lambda\partial t} &\le \frac{\partial}{\partial\lambda}\left[b_1(t,\lambda)\frac{\partial U}{\partial\lambda} + b_2(t,\lambda)U(t,\lambda)\right], \qquad (3.15)
\end{aligned}$$

where

$$\begin{aligned}
b_1(t,\lambda) &= 2\bar{M} + \chi_0\bar{N}_{-\mu-\nu} + \chi_0\Psi \\
\text{and } b_2(t,\lambda) &= \lambda(2+\tilde{\lambda})R + \lambda\bar{M} + \chi_0^2\bar{N}_{-\mu-\nu} + (\chi_0+2)\bar{M} \\
&\quad + \Psi(3\chi_0 + \lambda(2\chi_0^2 + \chi_0 + 1) + (3\chi_0 + 1)) + (1+\chi_0)\frac{\partial\Psi}{\partial\lambda}.
\end{aligned}$$

In the below, we recall Lemma 5.2. from [56].

Lemma 3.1. (See [56]). Let $v(t,\lambda)$ be a real-valued continuous function possesses continuous partial derivatives $\frac{\partial v}{\partial\lambda}$ and $\frac{\partial^2 v}{\partial\lambda^2}$ on $D = \{(t,\lambda) : 0 \le \lambda \le \lambda_0,\ 0 \le t \le T\}$. Assume that the real-valued function $\alpha(\lambda)$, $\beta(t,\lambda)$, $\gamma_1(t,\lambda)$, $\gamma_2(t,\lambda)$, $\theta_1(t,\lambda)$ and $\theta_2(t,\lambda)$ and their respective partial partial derivatives with respect to λ are continuous on D. Further, let the functions $v, \frac{\partial v}{\partial\lambda}, \beta, \gamma_1, \gamma_2$ are non-negative in D. Assume the following results are true on D :

$$v(t,\lambda) \le \alpha(\lambda) + \int_0^t \left(\beta(\lambda,s)\frac{\partial v}{\partial\lambda}(\lambda,s) + \gamma_1(\lambda,s)v(\lambda,s) + \theta_1(\lambda,s)\right)\mathrm{d}s$$

and

$$\begin{aligned}
\frac{\partial v}{\partial\lambda}(t,\lambda) \le \frac{d\alpha}{d\lambda} + \int_0^t \Bigg(&\frac{\partial}{\partial\lambda}\left(\beta(\lambda,s)\frac{\partial v}{\partial\lambda}(\lambda,s) + \gamma_1(\lambda,s)v(\lambda,s) + \theta_1(\lambda,s)\right) \\
&+ \gamma_2(\lambda,s)v(\lambda,s) + \theta_2(\lambda,s)\Bigg)\mathrm{d}s. \qquad (3.16)
\end{aligned}$$

Then, denoting $c_0 = \sup_{0\le\lambda\le\lambda_0}\alpha,\ c_1 = \sup_D \beta,\ c_2 = \sup_D(\beta\gamma_2 + \gamma_1)$ and $c_3 = \sup_D(\beta\theta_2 + \theta_1)$, we have

$$v(t,\lambda) \le c_0\exp(c_2 t) + \tfrac{c_3}{c_2}(\exp(c_2 t) - 1) \qquad (3.17)$$

in any region

$$D_1 = \{(\lambda, t) : 0 \leq t \leq t' < T',\ \lambda_1 - c_1 t \leq \lambda \leq \lambda_0 - c_1 t,\ 0 < \lambda_1 < \lambda_0\} \subset D,$$

where $T' = \min\left\{\frac{\lambda_1}{c_1}, T\right\}$.

By comparing (3.15) with (3.16) of Lemma 3.1, we obtain $c_0 = 0$ and $c_3 = 0$. Therefore, (3.17) yields $U(t, \lambda) \leq 0$. However, from the definition (3.2) of U, we have $U(t, \lambda) \geq 0$. Therefore,

$$\sup_{z \in \mathbb{R}^3} \int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\mu} \right) |u(t, x, z)| \mathrm{d}x = 0 \tag{3.18}$$

in the region D_1 as defined in Lemma 3.1. Since $u(t, x, z)$ is continuous, $u(t, x, z) = 0$ for $0 \leq t \leq t',\ x > 0$. Thanks to

$$u(x, 0) \equiv u(0, t) \equiv 0.$$

Therefore, the integral (3.18) is equal to zero in D_1, as well as for all $0 \leq \lambda \leq \tilde{\lambda},\ 0 \leq t \leq t'$. Applying the same logic in the extended interval $[t', 2t']$, we conclude that

$$u(t, x, z) = 0 \quad \text{for} \ \ 0 \leq t \leq 2t',\ 0 \leq x < \infty.$$

Repeating this procedure, we arrive at

$$u(t, x, z) \equiv 0.$$

Hence, the uniqueness result of Theorem 2.1 is proved.

4. Conclusions

In this scientific report, the existence-uniqueness of solution to the space inhomogeneous coagulation condensation model has been derived. To demonstrate the existence of the solution, we notice that some fundamental inequalities on improper integrals and the concept of characteristics of partial differential equations play essential roles. Furthermore, it is shown that the existing solution is differentiable along the characteristics curve of the model. The theory presented here could shed some light on the development of the theoretical aspects for many coagulation kernels in several physical directions e.g., granulation field, Brownian diffusion motion, nonlinear velocity profile, aerosol dynamics, activated sludge flocculation, etc. A beautiful open problem is how to include $\mu > \dfrac{1}{2}$ in the existence-uniqueness theory. In future, one may attempt to study the equations with a fragmentation term and explore the equilibrium and stability analysis, i.e., the mathematical analysis of the time-dependent solutions to the equilibrium state.

References

[1] C. Li, R. Signorell, Understanding vapor nucleation on the molecular level: A review, Journal of Aerosol Science (2020) 105676.

[2] M. M. Gogoi, R. C. Thakur, S. Gazi, V. S. Nair, R. Mohan, S. S. Babu, Vertical distributions of the microscopic morphological characteristics and elemental composition of aerosols over india, Journal of Atmospheric Chemistry 77 (4) (2020) 117–140.

[3] Y. Zhou, S. Gong, C. Zhou, L. Zhang, J. He, Y. Wang, D. Ji, J. Feng, J. Mo, H. Ke, A new parameterization of uptake coefficients for heterogeneous reactions on multi-component atmospheric aerosols, Science of The Total Environment 781 (2021) 146372.

[4] M. Gárate Silva, Protoplanetary disk dynamics in high dust-to-gas ratio environments, Ph.D. thesis, lmu (2020).

[5] M. E. Steinrueck, A. P. Showman, P. Lavvas, T. Koskinen, X. Tan, X. Zhang, 3d simulations of photochemical hazes in the atmosphere of hot jupiter hd 189733b, Monthly Notices of the Royal Astronomical Society 504 (2) (2021) 2783–2799.

[6] C. W. Ormel, A. Vazan, M. G. Brouwers, How planets grow by pebble accretion-iii. emergence of an interior composition gradient, Astronomy & Astrophysics 647 (2021) A175.

[7] S. Anand, Y. Mayya, Coagulation in a spatially inhomogeneous plume: Formation of bimodal size distribution, Journal of Aerosol Science 84 (2015) 9–13.

[8] M. Mu”ller, V. Abetz, Nonequilibrium processes in polymer membrane formation: Theory and experiment, Chemical Reviews.

[9] P. Mou, J. Pan, Q. Niu, Z. Wang, Y. Li, D. Song, Coal pores: Methods, types, and characteristics, Energy & Fuels 35 (9) (2021) 7467–7484.

[10] L. Chen, J. Wu, Z. Sun, Effect of cationic collector on the attachment of glass beads to a stationary bubble, Colloids and Surfaces A: Physicochemical and Engineering Aspects (2021) 126979.

[11] E. S. Kite, L. J. Steele, M. A. Mischna, M. I. Richardson, Warm early mars surface enabled by high-altitude water ice clouds, Proceedings of the National Academy of Sciences 118 (18).

[12] P. G. Debenedetti, Metastable liquids, Princeton university press, 2021.

[13] E. X. Berry, A mathematical framework for cloud models, Journal of the Atmospheric Sciences 26 (1) (1969) 109–111.

[14] M. Slemrod, A. Qi, M. Grinfeld, I. Stewart, A discrete velocity coagulation-fragmentation model, Mathematical methods in the applied sciences 18 (12) (1995) 959–993.

[15] M. Slemrod, Metastable fluid flow described via a discrete-velocity coagulation-fragmentation model, Journal of statistical physics 83 (5) (1996) 1067–1108.

[16] F. Da Costa, Mathematical aspects of coagulation-fragmentation equations, in: Mathematics of energy and climate change, Springer, 2015, pp. 83–162.

[17] J. Ball, J. Carr, The discrete coagulation-fragmentation equations: existence, uniqueness, and density conservation, Journal of Statistical Physics 61 (1) (1990) 203–234.

[18] I. Stewart, E. Meister, A global existence theorem for the general coagulation–fragmentation equation with unbounded kernels, Mathematical Methods in the Applied Sciences 11 (5) (1989) 627–648.

[19] V. Galkin, Smoluchowski equation of the kinetic theory of coagulation for spatially nonuniform systems, Soviet Physics Doklady 20 (1985) 1012–1014.

[20] P. Dubovskiĭ, An iterative method for solving the coagulation equation with spatially inhomogeneous velocity fields, Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki 30 (11) (1990) 1755–1757.

[21] V. Galkin, Generalized solution of the smoluchowski kinetic equation for spatially inhomogeneous systems, Doklady Akademii Nauk 293 (1) (1987) 74–77.

[22] P. Dubovskiĭ, Solutions of a spatially inhomogeneous coagulation equation with particle fractionation taken into account, Differential Equations 26 (3) (1990) 380–384.

[23] P. Dubovskiĭ, Generalized solutions of coagulation equations, Funktsional'nyi Analiz i ego Prilozheniya 25 (2) (1991) 62–64.

[24] H. Gajewski, On a first order partial differential equation with nonlocal nonlinearity, Mathematische Nachrichten 111 (1) (1983) 289–300.

[25] H. Gajewski, K. Zacharias, On an initial value problem for a coagulation equation with growth term, Mathematische Nachrichten 109 (1) (1982) 135–156.

[26] D. Chae, P. Dubovskiĭ, Existence and uniqueness for spatially inhomogeneous coagulation equation with sources and effluxes, Zeitschrift für Angewandte Mathematik und Physik 46 (4) (1995) 580–594.

[27] M. A. Herrero, J. Velázquez, D. Wrzosek, Sol–gel transition in a coagulation–diffusion model, Physica D: Nonlinear Phenomena 141 (3-4) (2000) 221–247.

[28] N. Igbida, A partial integrodifferential equation in granular matter and its connection with a stochastic model, SIAM Journal on Mathematical Analysis 44 (3) (2012) 1950–1975.

[29] J. A. Canizo, B. Lods, Exponential convergence to equilibrium for subcritical solutions of the becker–döring equations, Journal of Differential Equations 255 (5) (2013) 905–950.

[30] F. da Costa, J. T. Pinto, R. Sasportes, The redner–ben-avraham–kahng coagulation system with constant coefficients: the finite-dimensional case, Zeitschrift für angewandte Mathematik und Physik 66 (4) (2015) 1375–1385.

[31] J. Pavlova, A. Fasano, A. Sequeira, Numerical simulations of a reduced model for blood coagulation, Zeitschrift für angewandte Mathematik und Physik 67 (2) (2016) 28.

[32] S. E. Wawra, L. Pflug, T. Thajudeen, C. Kryschi, M. Stingl, W. Peukert, Determination of the two-dimensional distributions of gold nanorods by multiwavelength analytical ultracentrifugation, Nature communications 9 (1) (2018) 1–11.

[33] R. Alonso, V. Bagland, Y. Cheng, B. Lods, One-dimensional dissipative boltzmann equation: measure solutions, cooling rate, and self-similar profile, SIAM Journal on Mathematical Analysis 50 (1) (2018) 1278–1321.

[34] L. Desvillettes, S. Lorenzani, Homogenization of the discrete diffusive coagulation–fragmentation equations in perforated domains, Journal of Mathematical Analysis and Applications 467 (2) (2018) 1100–1128.

[35] G. Baird, E. Süli, A mixed discrete-continuous fragmentation model, Journal of Mathematical Analysis and Applications 473 (1) (2019) 273–296.

[36] M. Singh, H. Y. Ismail, T. Matsoukas, A. B. Albadarin, G. Walker, Mass-based finite volume scheme for aggregation, growth and nucleation population balance equation, Proceedings of the Royal Society A 475 (2231) (2019) 20190552.

[37] L. Hoessly, C. Mazza, Stationary distributions and condensation in autocatalytic reaction networks, SIAM Journal on Applied Mathematics 79 (4) (2019) 1173–1196.

[38] A. Lambert, E. Schertzer, Coagulation-transport equations and the nested coalescents, Probability Theory and Related Fields 176 (1) (2020) 77–147.

[39] R. M. Harrison, Airborne particulate matter, Philosophical Transactions of the Royal Society A 378 (2183) (2020) 20190319.

[40] M. Woo, R. T. Nishida, M. A. Schriefl, M. E. Stettler, A. M. Boies, Open-source modelling of aerosol dynamics and computational fluid dynamics: Nodal method for nucleation, coagulation, and surface growth, Computer physics communications 261 (2021) 107765.

[41] C. Camejo, G. Warnecke, The singular kernel coagulation equation with multifragmentation, Mathematical Methods in the Applied Sciences 38 (14) (2015) 2953–2973.

[42] J. Saha, J. Kumar, The singular coagulation equation with multiple fragmentation, Zeitschrift für angewandte Mathematik und Physik 66 (3) (2015) 919–941.

[43] D. Ghosh, J. Kumar, Existence of equilibrium solution of the coagulation-fragmentation equation with linear fragmentation kernel, in: Mathematics and Computing, Proceedings of Mathematics and Statistics, Vol. 253, Springer, 2018, pp. 295–304.

[44] D. Ghosh, J. Kumar, Existence of mass conserving solution for the coagulation–fragmentation equation with singular kernel, Japan Journal of Industrial and Applied Mathematics 35 (3) (2018) 1283–1302.

[45] D. Ghosh, J. Kumar, Uniqueness of solutions to the coagulation–fragmentation equation with singular kernel, Japan Journal of Industrial and Applied Mathematics 37 (2020) 487–505.

[46] D. Ghosh, J. Saha, J. Kumar, Existence and uniqueness of steady-state solution to a singular coagulation-fragmentation equation, Journal of Computational and Applied Mathematics 380 (2020) 112992.

[47] P. K. Barik, A. K. Giri, P. Laurençot, Mass-conserving solutions to the smoluchowski coagulation equation with singular kernel, Proceedings of the Royal Society of Edinburgh Section A: Mathematics 150 (4) (2020) 1805–1825.

[48] P. K. Barik, A. K. Giri, Existence and uniqueness of weak solutions to the singular kernels coagulation equation with collisional breakage, Nonlinear Differential Equations and Applications NoDEA 28 (3) (2021) 1–23.

[49] P. Kapur, Kinetics of granulation by non-random coalescence mechanism, Chemical Engineering Science 27 (10) (1972) 1863–1869.

[50] M. Peglow, Beitrag zur Modellbildung von eigenschaftsverteilten dispersen Systemen am Beispiel der Wirbelschicht-Sprühagglomeration, PhD Thesis, Otto-von-Guericke-e-Universität Magdeburg, 2005.

[51] K. Shiloh, S. Sideman, W. Resnick, Coalescence and break-up in dilute polydispersions, The Canadian Journal of Chemical Engineering 51 (5) (1973) 542–549.

[52] M. Hounslow, The population balance as a tool for understanding particle rate processes, KONA Powder and Particle Journal 16 (1998) 179–193.

[53] M. Smoluchowski, An experiment on mathematical theorization of coagulation kinetics of the colloidal solutions, Zeitschrift für Physikalisch Chemie 92 (1917) 129–168.

[54] S. Friedlander, Smoke, dust, and haze, Oxford University Press, New York, 2000.

[55] A. Ding, M. Hounslow, C. Biggs, Population balance modelling of activated sludge flocculation: Investigating the size dependence of aggregation, breakage and collision efficiency, Chemical Engineering Science 61 (1) (2006) 63–74.

[56] D. Chae, P. Dubovskiĭ, Existence and uniqueness for spatially inhomogeneous coagulation-condensation equation with unbounded kernels, The Journal of Integral Equations and Applications 9 (3) (1997) 219–236.

[57] J. Paul, J. Kumar, An existence-uniqueness result for the pure binary collisional breakage equation, Mathematical Methods in the Applied Sciences 41 (7) (2018) 2715–2732.

[58] P. B. Dubovskiĭ, I. W. Stewart, Existence, uniqueness and mass conservation for the coagulation-fragmentation equation, Mathematical Methods in the Applied Sciences 19 (7) (1996) 571–591.