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RESEARCH ARTICLE

Conservation laws and exact solutions for the generalized Ostrovsky equation using symmetry analysis.

S. Sáez*

[1]Departamento de Matemáticas, Universidad de Cádiz, Puerto Real, Cádiz, 11510, Spain

Correspondence
*Sol Sáez, Departamento de Matemáticas, Avda. Universidad de Cádiz, nº 10, 11519 Campus Universitario de Puerto Real,, Spain. Email: sol.saez@uca.es

Abstract

In this work we consider a generalized Ostrovsky equation depending on two arbitrary functions and we make an in-depth study of this equation. We obtain the Lie symmetries which are admitted by this equation and some exact solutions as a periodic or solitary waves, obtained through ordinary and partial differential equations. Also, by means of the concept of multiplier, we obtain a wide range of conservation laws which preserve properties of the generalized Ostrovsky equation.

KEYWORDS:
Ostrovsky equation; conservation laws; symmetries; reductions

1 | INTRODUCTION

Ostrosvsky equation was introduced by Ostrovsky in[11] as a model for long waves wich are weakly nonlinear, to explain the propagation of the surface and internal waves in a fluid of reference which describe a rotating movement. The aim of this paper is to discuss the generalized Ostrovsky equation by using Lie's symmetry group method and low local conservation laws for this equation. Conservation laws are highly significant in the analysis of differential equations because they describe chemical and physical processes with conserved quantities. In order to evaluate conserved fluxes and densities, we study conservation laws for these equations and we resort to the invariance and multiplier perspective by using the Euler-Lagrange operator. In this paper, we have considered the generalized Ostrovsky equation

$$u_{tx} - \beta u_{xxxx} + \left(g(u)_x\right)_x = \alpha f(u) \tag{1}$$

where α and β are the real dispersion coefficients and the functions f and g are C^2 functions. This equation was introduced by Levandosky and Liu in[7] when f is the identity function and they proved the existence of solitary waves, called ground states, by employing variational methods. Specifically, in the case $g(u) = u^2$ and $f(u) = u$, we obtain the Ostrovsky equation

$$u_{tx} - \beta u_{xxxx} + \left((u^2)_x\right)_x = \alpha u \tag{2}$$

The function $u(t, x)$ denote the free surface of a liquid, $\alpha \in \mathbb{R}$ is a measure of rotational effects due to Coriolis force and $\beta \in \mathbb{R}$ determines the dispersion. Some special cases of the Ostrovsky equation has been investigated by several authors[6,17]. In[8] the authors prove that solutions of this equation converge to solutions of the Korteweg-de Vries equation. In[15] Varlamov and Liu proved that (1) has special characteristics in the space X_s for $s > 3/2$. In[14] extended the results of Varlamov and Liu for $s > -3/4$. In the absence of rotation, that is, when $\alpha = 0$, we integrate and we obtain the generalized Korteweg de Vries equation (KdV):

$$u_t - \beta u_{xxx} + (g(u))_x = 0. \tag{3}$$

In the current paper we have studied (1) using Lie symmetry reductions, symmetry group and a lot of conservative laws for them. Partial differential equations describe several scientific processes, such as the heat equation[2] and other chemical or physical processes[16,18]. Numerous solutions of differential equations have been obtained by several methods[1], such as the direct method[4], simplest equation method[5], the tanh method[9] or the homogeneous balanced method[19]. An important and efficient method to study differential equations is the Lie group method[12,13]. In Olver's[10] and Bluman and Kumei[3] we find a precise description about symmetry groups and Lie's theory. Symmetries help us to obtain invariable solutions of differential equations which have been reduced to other equations with fewer independent variables. By means of symmetry groups, we can obtain solutions of partial differential equations through other solutions.

This paper is ordered as follows: Firstly, we calculate local conservation laws of low order admitted by (1) by employing the multipliers theory and we obtain a wide variety of low order conservation laws for Equation (1). After that, we obtain the classical point symmetries admitted by Equation (1) in Section 3. By means of the point symmetries, we calculate symmetry reductions of the equation in the following section. In addition, we obtain others invariant solutions of the generalized equation and we obtain the Lie symmetry groups. Finally, we present some concluding comments.

2 | CONSERVATION LAWS

Conservations laws have been utilised to obtain solutions of partial differential equations and in the expansion of different numerical methods. They are used in proving the existence and uniqueness of solutions. To evaluate conserved fluxes and densities, we study conservation laws for Equation (1) and we have used the invariance and multiplier perspective with the use of the Euler-Lagrange operator. Then, we have obtained conservation laws for the Equation (1), namely, the following continuity equation

$$\left(D_t T + D_x X\right)|_{\varepsilon} = 0 \tag{4}$$

in terms of the total derivative operators holding for the solutions of (1). The spatial flux is noted by X, the conserved density is noted by T, depends of u, x, t and some derivatives of the function u. Also, D_x and D_t represent the total derivative functions with respect x and t respectively.

A multiplier Q is a function $Q(t, x, u, u_t, u_x, \dots)$ which satisfies that, for solutions of Equation (1),

$$(u_{tx} - \beta u_{xxxx} + \left(g(u)_x\right)_x - \alpha f(u))Q$$

is a divergence expression. Every significant conservation law emerges from multipliers. If we move away of the group of solutions of Equation (1), all significant conservation laws (4) are equivalent to conservation laws with the following characteristic form

$$\left(D_t \widehat{T} + D_x \widehat{X}\right) = (u_{tx} - \beta u_{xxxx} + \left(g(u)_x\right)_x - \alpha f(u))Q \tag{5}$$

where $(\widehat{T}, \widehat{X})$ varies from (T, X) by a banal conserved current.

We solve the following determining equation to find the set of multipliers:

$$\frac{\delta}{\delta u}(u_{tx} - \beta u_{xxxx} + \left(g(u)_x\right)_x - \alpha f(u))Q = 0 \tag{6}$$

where $\frac{\delta}{\delta u}$ represent the Euler Lagrange factor $\hat{E}[u]$ defined by

$$\hat{E}[u] := \frac{\partial}{\partial u} + \sum_{s\geq 1}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}}. \tag{7}$$

By considering (5)-(7), we derive the following local conservation laws for Equation (1) in the case $g(u) = \frac{1}{n+1}u^{n+1}$:

Theorem 1. Under some arbitrary constants, the generalized Ostrovske equation (1) with $g(u) = \frac{1}{n+1}u^{n+1}$, admits the following local conservation laws of low order

Case 1. In the case $n \neq 0$, $\alpha \neq 0$ and $f(u) = c_1 u^{n+1} + c_2 u + c_3$ with $c_1 \neq 0$, we obtain local conservation laws under the followings conservation law multipliers :

(a) $Q_1 = (n+1)\alpha c_1 u_x + u_{xxx} - \frac{1}{\beta}u^n u_x - \frac{1}{\beta}u_t$, give us the following local conservation law:

$$
\begin{aligned}
&D_t\left(\frac{\alpha}{\beta}\left(\frac{n+1}{2}c_1u_x^2\beta + \frac{1}{n}c_1u^n + \frac{1}{2}c_2u^2 + c_3u\right)\right)\\
&+D_x\Big(-(n+1)\alpha\beta c_1u_xu_{xxx} - \frac{1}{2}\beta u_{xxx}^2 + u^nu_xu_{xxx} + u_tu_{xxx} + \frac{n+1}{2}\alpha\beta c_1u_{xx}^2\\
&\quad -\alpha c_1u^{n+1}u_{xx} - \alpha c_2uu_{xx} - \alpha c_3u_{xx} + (n+1)\alpha c_1u^nu_x^2 + \tfrac{1}{2}\alpha c_2u_x^2\\
&\quad -\tfrac{1}{2\beta}u^{2n}u_x^2 - \tfrac{1}{\beta}u^nu_tu_x - \tfrac{1}{2\beta}u_t^2 - \tfrac{n+1}{n+2}\alpha^2c_1^2u^{n+2} - \left(\tfrac{n+1}{2}\right)\alpha^2c_1c_2u^2\\
&\quad -(n+1)\alpha^2c_1c_3u + \tfrac{\alpha c_1}{2\beta(n+1)}u^{2(n+1)} + \tfrac{\alpha c_2}{\beta(n+2)}u^{n+2} + \tfrac{\alpha c_3}{\beta(n+1)}u^{n+1}\Big) = 0.
\end{aligned}
\tag{8}
$$

(b) $Q_2 = \left(e^{\sqrt{n+1}}\alpha^{3/2}c_1^{3/2}\beta t\right)^{n+1} e^{x\sqrt{(n+1)\alpha c_1}} e^{\frac{\sqrt{n+1}\sqrt{\alpha}c_2 t}{(n+1)\sqrt{c_1}}}$, with the conservation law:

$$
\begin{aligned}
&D_t\left(u_x e^{\frac{\sqrt{(n+1)\alpha}}{(n+1)\sqrt{c_1}}((n+1)^2\alpha\beta c_1^2t + (n+1)c_1x + c_2t)}\right)\\
&+D_x\Bigg(-\frac{n+1}{\sqrt{c_1}}e^{\frac{\sqrt{(n+1)\alpha}}{(n+1)\sqrt{c_1}}((n+1)^2\alpha\beta c_1^2t + (n+1)c_1x + c_2t)}\\
&\quad\cdot\left(c_1^{3/2}\alpha\beta u_x + \tfrac{\sqrt{c_1}}{n+1}(\beta u_{xxx} - u_xu^n) - \tfrac{\sqrt{\alpha(n+1)}}{n+1}(c_1\beta u_{xx} - \tfrac{c_1u^{n+1}}{n+1} - \tfrac{c_2u}{n+1})\right)\Bigg) = 0.
\end{aligned}
\tag{9}
$$

(c) $Q_3 = \left(e^{-\sqrt{n+1}}\alpha^{3/2}c_1^{3/2}\beta t\right)^{n+1} e^{-x\sqrt{(n+1)\alpha c_1}} e^{-\frac{\sqrt{n+1}\sqrt{\alpha}c_2 t}{(n+1)\sqrt{c_1}}}$, with the conservation law:

$$
\begin{aligned}
&D_t\left(u_x e^{-\frac{\sqrt{(n+1)\alpha}}{(n+1)\sqrt{c_1}}((n+1)^2\alpha\beta c_1^2t + (n+1)c_1x + c_2t)}\right)\\
&+D_x\Bigg(-\frac{n+1}{\sqrt{c_1}}e^{-\frac{\sqrt{(n+1)\alpha}}{(n+1)\sqrt{c_1}}((n+1)^2\alpha\beta c_1^2t + (n+1)c_1x + c_2t)}\\
&\quad\cdot\left(c_1^{3/2}\alpha\beta u_x + \tfrac{\sqrt{c_1}}{n+1}(\beta u_{xxx} - u_xu^n) - \tfrac{\sqrt{\alpha(n+1)}}{n+1}(c_1\beta u_{xx} - \tfrac{c_1u^{n+1}}{n+1} - \tfrac{c_2u}{n+1})\right)\Bigg) = 0.
\end{aligned}
\tag{10}
$$

Case 2. In the case $n \neq 0$, $\alpha \neq 0$ and $f(u) = c_2u + c_3$, with $c_2 \neq 0$, under the conservation law multiplier $Q = -u^nu_x + \beta u_{xxx} - u_t$, we obtain one local conservation law:

$$
\begin{aligned}
&D_t\left(\frac{c_2\alpha}{2}u^2 + \alpha c_3u\right) + D_x\Big(-\frac{1}{2}u_t^2 + \beta\left(\frac{1}{2}c_2u_x^2\alpha - c_3\alpha u_{xx} + u_tu_{xxx}\right) - \frac{\beta}{2}u_{xxx}^2\\
&-\alpha\beta c_2uu_{xx} + u_x(\beta u_{xxx} - u_t)u^n + \tfrac{\alpha c_3}{n+1}u^{n+1} + \tfrac{\alpha c_2}{n+1}u^{n+2} - \tfrac{1}{2}u_x^2u^{2n}\Big) = 0.
\end{aligned}
\tag{11}
$$

Case 3. In the case $n \neq 0$, $\alpha \neq 0$, $f(u) = c_3$, with $c_3 \neq 0$, under the conservation law multiplier $Q = F_2(t,\gamma) + F_1(-\gamma)$, with $\gamma = u^nu_x - \beta u_{xxx} + u_t$, we obtain the following local conservations law:

$$
D_t(0) + D_x\left(\int -\beta(F_2(t,\gamma) + F_1(-\gamma))du_{xxx}\right) = 0. \tag{12}
$$

Case 4. In the case $n \neq 0$ and $\alpha = 0$, under the conservation law multipliers $Q_1 = x$, $Q_2 = F(t)$ and $Q_3 = u_{xxx} - \frac{1}{b}u_t - \frac{1}{b}u^n u_x$, we obtain the followings local conservations laws respectively:

$$D_t(-u) + D_x\left(xu^n u_x - \beta x u_{xxx} + \beta u_{xx} + xu_t - \tfrac{1}{n+1}u^{n+1}\right) = 0,$$

$$D_t\left(F(t)u_x\right) + D_x\left(-F(t)\beta u_{xxx} + F(t)u^n u_x - F'(t)u\right) = 0, \tag{13}$$

$$D_t(0) + D_x\left(-\tfrac{1}{2\beta}(-u^n u_x + \beta u_{xxx} - u_t)^2\right) = 0.$$

and the multiplier $Q_4 = 1$ with the local conservation laws:

$$\begin{aligned} &D_t(u_x) + D_x\left(-\beta u_{xxx} + u^n u_x\right) = 0,\\ &D_t(0) + D_x\left(-\beta u_{xxx} + u^n u_x + u_t\right) = 0. \end{aligned} \tag{14}$$

Case 5. In the case $n = 0$, $\alpha \neq 0$, $f(u)$ a C^2 function, under the conservation law multipliers $Q_1 = u_x$ and $Q_2 = u_t$, we obtain the followings local conservations laws respectively:

$$\begin{aligned} &D_t\left(\tfrac{1}{2}u_x^2\right) + D_x\left(-\beta u_x u_{xxx} + \tfrac{\beta}{2}u_{xx}^2 + \tfrac{1}{2}u_x^2 - \alpha\int f(u)du\right) = 0,\\ &D_t\left(-\tfrac{\beta}{2}u_{xx}^2 - \tfrac{1}{2}u_x^2 - \alpha\int f(u)du\right) + D_x\left(-\beta u_t u_{xxx} + \beta u_{tx}u_{xx} + u_t u_x + \tfrac{1}{2}u_t^2\right) = 0. \end{aligned} \tag{15}$$

Case 6. In the case $n = 0$, $\alpha \neq 0$, $f(u) = c_3 \in \mathbb{R}$, we obtain local conservation laws under the followings conservation law multipliers :

(a) $Q_1 = \beta u_{xxx} + u_t$, give us the following local conservation law:

$$\begin{aligned} &D_t\left(-\beta u_{xx}^2 - \frac{1}{2}u_x^2 - c_3\alpha u\right)\\ &+D_x\left(-\frac{\beta^2}{2}u_{xxx}^2 + \frac{\beta}{2}(u_{xx}^2 + 4u_{tx}u_{xx} - 2u_t u_{xxx}) + \frac{1}{2}u_t^2 + u_t u_x - c_3\alpha\beta u_{xx}\right) = 0. \end{aligned} \tag{16}$$

(b) $Q_2 = u_x$, give us the following local conservation law:

$$D_t\left(\frac{1}{2}u_x^2\right) + D_x\left(\frac{\beta}{2}(-2u_x u_{xxx} + u_{xx}^2) + \frac{1}{2}u_x^2 - c_3\alpha u\right) = 0. \tag{17}$$

(c) $Q_3 = u_t$, give us the following local conservation law:

$$D_t\left(-\frac{\beta}{2}u_{xx}^2 - \frac{1}{2}u_x^2 - \alpha c_3 u\right) + D_x\left(\beta - u_t u_{xxx} + \beta u_{tx}u_{xx} + u_t u_x + \frac{1}{2}u_t^2\right) = 0. \tag{18}$$

Case 7. In the case $n = 0$, $\alpha = 0$, we obtain local conservation laws under the followings conservation law multipliers :

(a) $Q_1 = x$, give us the following local conservation law:

$$D_t\left(xu_x\right) + D_x\left(-\beta x u_{xxx} + \beta u_{xx} + xu_x - u\right) = 0. \tag{19}$$

(b) $Q_2 = \frac{1}{2}x^2 - tx$, with the conservation law:

$$\begin{aligned} &D_t\left(-\frac{1}{2}(x(2t - x)u_x)\right)\\ &+D_x\left(\frac{\beta}{2}(-u_{xxx}x^2 + 2x(tu_{xxx} + u_{xx}) - 2tu_{xx} - 2u_x) - txu_x + tu\right) = 0. \end{aligned} \tag{20}$$

(c) $Q_3 = -\frac{x}{6}(3t^2 - 3tx + x^2)$, with the conservation law:

$$
\begin{aligned}
&D_t\left(-\frac{x}{6}(3t^2 - 3tx + x^2)u_x\right)\\
&+D_x\left(\frac{\beta}{6}(u_{xxx}x^3 + x^2(-3tu_{xxx} - 3u_{xx}) + x(3t^2u_{xxx} + 6tu_{xx} + 6u_x))\right.\\
&\quad\left.+\beta\left(-\frac{1}{2}u_{xx}t^2 - tu_x - u\right) - \frac{1}{2}u_xt^2x + \frac{1}{2}u_xtx^2 - \frac{1}{6}u_xx^3 + \frac{1}{2}ut^2\right) = 0.
\end{aligned}
\tag{21}
$$

(d) $Q_4 = \frac{2}{3}xu_x + t\beta u_{xxx} + u_t + \frac{1}{3}u_x$, with the conservation law:

$$
\begin{aligned}
&D_t\left(\frac{1}{3}xu_x^2 + \frac{1}{3}t(-3\beta u_{xx}^2 - u_x^2)\right)\\
&+D_x\left(\frac{t}{6}\left(-3\beta^2u_{xxx}^2 + \beta\left(-2u_xu_{xxx} - 6u_tu_{xxx}\right.\right.\right.\\
&\quad\left.\left.\left.-16\left(-\frac{u_{xx}}{4} - \frac{3}{4}u_{tx}\right)u_{xx}\right) + u_x^2 + 6u_tu_x + 3u_t^2\right)\right) = 0.
\end{aligned}
\tag{22}
$$

(e) $Q_5 = F_2(t,\delta) + F_1(t)$, with $\delta = -\beta u_{xxx} + u_t + u_x$ give us the conservation law:

$$
D_t\left(F_1(t)u_x\right) + D_x\left(\int -\beta(F_2(t,\delta) + F_1(t))du_{xxx} + F_1(t)u_x - F_1'(t)u\right) = 0. \tag{23}
$$

(f) $Q_6 = \frac{1}{3}(2t + x)u_x + tu_t$, with the conservation law:

$$
\begin{aligned}
&D_t\left(\frac{1}{6}(x - t)u_x^2 - \frac{1}{2}\beta tu_{xx}^2\right)\\
&+D_x\left(\frac{1}{6}\beta t(-6u_tu_{xxx}) - 4u_xu_{xxx} + 6u_{tx}u_{xx} + 2u_{xx}^2\right)\\
&\quad+\tfrac{1}{6}\beta((-2xu_{xxx} + 2u_{xx})u_x + xu_{xx}^2) + t\left(\tfrac{u_t^2}{2} + u_tu_x + \tfrac{u_x^2}{3}\right) + \tfrac{u_x^2x}{6}\Big) = 0.
\end{aligned}
\tag{24}
$$

They are the absolute collection of local conservation laws accepted by Equation (1) under the constants α and β.

Proof. Equation (4) is satisfied when the generalized Ostrovsky equation (1) holds. The usual form of low order multipliers of Equation (1) is represented by

$$
Q(t, x, u, u_t, u_x, u_{xx}u_{xxx}).
$$

The determining equation (6) give us determining systems. We solve them and we obtain the previous solution multipliers given in each case, which provide us conserved fluxes and densities of Equation (1). □

3 | POINT SYMMETRIES OF THE GENERALIZED OSTROVSKY EQUATION

A Lie symmetry of determined partial differential equations is a operator which transform solutions into other solutions. The mathematician, Sophus Lie, elaborated a technique in the 80s, to find the point symmetries for partial differential equations. Now, we apply the classical Lie method to calculate symmetry reductions of the generalized Ostrovsky equation (1) and we consider the following one parameter group:

$$\begin{aligned}
\hat{t} &= t + \varepsilon\tau(t,x,u) + \Theta(\varepsilon^2),\\
\hat{x} &= x + \varepsilon\xi(t,x,u) + \Theta(\varepsilon^2),\\
\hat{u} &= u + \varepsilon\eta(t,x,u) + \Theta(\varepsilon^2),\\
\frac{\partial\hat{u}}{\partial\hat{t}} &= \frac{\partial u}{\partial t} + \varepsilon\eta^t(t,x,u) + \Theta(\varepsilon^2),\\
\frac{\partial\hat{u}}{\partial\hat{x}} &= \frac{\partial u}{\partial x} + \varepsilon\eta^x(t,x,u) + \Theta(\varepsilon^2),\\
\frac{\partial^2\hat{u}}{\partial\hat{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \varepsilon\eta^{xx}(t,x,u) + \Theta(\varepsilon^2),\\
&\vdots
\end{aligned} \tag{25}$$

where ε is a small group parameter, η, τ and ξ represent the infinitesimals of symmetry transformations, corresponding to the dependent and independent variables respectively, where

$$\begin{aligned}
\eta^t &= D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau),\\
\eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
\eta^{xx} &= D_x(\eta^x - u_{xx} D_x(\xi)) - u_{tx} D_x(\tau),\\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxx} D_x(\xi) - u_{txx} D_x(\tau),\\
\eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxx} D_x(\xi) - u_{txxx} D_x(\tau),\\
&\vdots
\end{aligned}$$

The operators D_t and D_x represent the total derivative functions with respect t and x respectively, which are defined as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \dots\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \dots
\end{aligned} \tag{26}$$

The generators associated to the Lie algebra are given by the generator X, represented by

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{27}$$

where

$$\tau = \left.\frac{d\hat{t}}{d\varepsilon}\right|_{\varepsilon=0}, \quad \xi = \left.\frac{d\hat{x}}{d\varepsilon}\right|_{\varepsilon=0}, \quad \eta = \left.\frac{d\hat{u}}{d\varepsilon}\right|_{\varepsilon=0}.$$

If the generator (27) give us a symmetry of (1), then X satisfies the symmetry condition

$$pr^{(4)}X(\Delta)|_{\varepsilon=0} = 0, \tag{28}$$

where $\Delta = u_{tx} - \beta u_{xxxx} + \left((g(u))_x\right)_x - \alpha f(u)$ and $pr^{(4)}X$ represents the fourth prolongation of (27)

$$pr^{(4)}X = X + \sum_J \phi^J(t,x,u^{(4)})\frac{\partial}{\partial u_J}, \tag{29}$$

where $\phi^J(t,x,u^{(4)}) = D_J(\phi - \tau u_t - \xi x) + \xi u_{Jx} + \tau u_{Jt}$, and $J = (j_1,\dots,j_k)$, for $1 \le j_k \le 2$ and $1 \le k \le 4$.

Precisely, the fourth prolongation of the vector field X is given by

$$pr^{(4)}X = X + \phi^x\frac{\partial}{\partial u_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{tx}\frac{\partial}{\partial u_{tx}} + \phi^{xxxx}\frac{\partial}{\partial u_{xxxx}}. \tag{30}$$

where ϕ^x, ϕ^{xx}, ϕ^{tx}, ϕ^{xxxx} are expressed as a function of τ, ξ, η and the derivatives of η. By using (28) we obtain the infinitesimals $\tau(t,x,u)$, $\xi(t,x,u)$ and $\eta(t,x,u)$. From (28) and (30), the invariance condition reads as

$$(g''(u)u_{xx} + g'''(u)u_x^2 - \alpha f'(u))\eta + 2g''(u)u_x\phi^x + g'(u)\phi^{xx} + \phi^{tx} - \beta\phi^{xxxx} = 0. \tag{31}$$

where

$$
\begin{aligned}
\phi^{x} &= D_x\eta - u_t D_x\tau - u_x D_x\xi \\
\phi^{tx} &= D_x\phi^{t} - u_{tt} D_x\tau - u_{tx} D_x\xi \\
\phi^{xx} &= D_x\phi^{x} - u_{tx} D_x\tau - u_{xx} D_x\xi \\
\phi^{xxxx} &= D_x\phi^{xxx} - u_{txxx} D_x\tau - u_{xxxx} D_x\xi
\end{aligned}
\tag{32}
$$

and D_x and D_t are the total differential operators with respect to x and t respectively given by (26).

By using (28) we calculate the point symmetries of Equation (1). A point symmetry of Equation (1) is a one-parameter group of transformations depending on (t, x, u) with generator (27). The prolongation of the Lie group leaves invariant equation (1). The vector field (27) yields a point symmetry of Equation (1) when satisfies the Lie's symmetry condition(28), which give rise to the following linear system of determining differential equations for the infinitesimals $\xi(t, x, u)$, $\tau(t, x, u)$, $\eta(t, x, u)$, the functions $g(u)$, $f(u)$ and the real parameters α and β:

$$
\begin{aligned}
&\tau_x = 0, \quad \tau_u = 0, \quad \eta_{xu} = 0, \quad \eta_{uu} = 0, \quad \xi_u = 0, \\
&\xi_{xx} = 0, \quad 3\xi_x - \tau_t = 0, \quad 2\xi_x g'(u) + \eta g'(u) - \xi_t = 0, \\
&\eta_{tu} - \xi_{tx} + 2\eta_x g''(u) = 0, \quad \eta_u g''(u) + 2\xi_x g''(u) + \eta g'''(u) = 0, \\
&-\eta_{tx} + \alpha\eta f'(u) - \eta_{xx} g'(u) - \alpha\eta_u f(u) + 4\alpha\xi_x f(u) + \beta\eta_{xxxx} = 0,
\end{aligned}
\tag{33}
$$

Solving (33), we obtain the following theorem with the infinitesimals generators of the generalized Ostrovsky equation (1):

Theorem 2. The point symmetries of Equation (1) are defined by the the following independent operators

Case 1. In the case $g(u) = \frac{1}{n+1}u^{n+1}$ and $f(u) = cu^{2n+1}$ with $n \neq 0$, $\alpha \neq 0$, the point symmetries of Equation (1) are defined by the generators

$$
V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_3 = 3t\partial_t + x\partial_x - \frac{2}{n}u\partial_u \tag{34}
$$

Case 2. In the case $g(u) = \frac{1}{n+1}u^{n+1}$ and $f(u) = cu$, with $n \neq 0$, $\alpha \neq 0$, the point symmetries of Equation (1) are defined by the generators $V_1 = \partial_t$ and $V_2 = \partial_x$.

Case 3. In the case $g(u) = \frac{1}{n+1}u^{n+1}$, $n \neq 0$, $\alpha = 0$, the point symmetries of Equation (1) are defined by V_1, V_2 and V_3.

Case 4. In the case $g(u) = u$, there are point symmetries of Equation (1) in the case $\alpha \neq 0$ and $f(u) = cu$, or in the case $\alpha = 0$, and the generators are given by:

$$
V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_4 = u\partial_u, \tag{35}
$$

Case 5. In the case $g(u) = c_1 e^u$ and $f(u) = c_2 e^{2u}$ the point symmetries are given by:

$$
V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_5 = 3t\partial t + x\partial x - 2\partial_u, \tag{36}
$$

Case 6. In the general case for $g(u)$ and $f(u)$ the point symmetries are given by V_1 and V_2.

□

We can easily check that the generators (34) - (36) are closed under the Lie bracket.

Now, we calculate the adjoint table for V_i and V_j, for $i, j = 1, \dots, 5$, given by the following expression

$$
\begin{aligned}
\mathrm{Ad}(\exp(\varepsilon V))W_0 &= \textstyle\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\mathrm{com}V)^n(W_0) \\
&= W_0 - \varepsilon[V, W_0] + \frac{\varepsilon^2}{2}[V, [V, W_0]] - \dots
\end{aligned}
$$

By using the adjoint representation, we obtain conjugated subalgebras through equivalence classes, which represent every subalgebra of the Lie algebra.

By means of the optimal one-dimensional system, we calculate reduced equations from the equation (1). The optimal systems in each case are given by: $\{V_1 + \lambda V_2,\ V_3\}$ in the first and third cases, $\{V_1 + \lambda V_2\}$ in the second and sixth cases, $\{V_1 + \lambda_1 V_2 + \lambda_2 V_4\}$ in the fourth case and $\{V_1 + \lambda_1 V_2, V_5\}$ in the fifth case.

Now, we use (34) - (36) to determinate the symmetry groups $g(t, x, u)$ associated to the generator (27) and calculate new solutions for Equation (1) associated with them. To obtain the symmetry group, we use the following system of initial problems

$$\frac{\partial \hat{t}}{\partial \epsilon} = \tau(t,x,u),\ \frac{\partial \hat{x}}{\partial \epsilon} = \xi(t,x,u),\ \frac{\partial \hat{u}}{\partial \epsilon} = \eta(t,x,u) \tag{37}$$

and

$$(\hat{t},\hat{x},\hat{u})|_{\epsilon=0} = (t,x,u). \tag{38}$$

where $g(t,x,u) = (\hat{t},\hat{x},\hat{u})$. From (37) and (38) we get the following theorem with the corresponding Lie symmetry group.

Theorem 3. The Lie symmetry groups G_i, $i = 1,\dots,5$, generated by V_i, $i = 1,\dots,5$ are specified by :

$$G_1(t,x,u) = (t+\epsilon,x,u) \qquad \text{time-translation} \tag{39}$$

$$G_2(t,x,u) = (t,x+\epsilon,u) \qquad \text{space-translation across the x-axis} \tag{40}$$

$$G_3(t,x,u) = (e^{3\epsilon}t, e^{\epsilon}x, e^{-\frac{2\epsilon}{n}}u) \qquad \text{scaling group} \tag{41}$$

$$G_4(t,x,u) = (t,x,e^{\epsilon}u) \qquad \text{exponential dilation} \tag{42}$$

$$G_5(t,x,u) = (e^{3\epsilon}t, e^{\epsilon}x, u-2\epsilon) \qquad \text{shift} \tag{43}$$

where ϵ is the parameter of the group. □

Solutions of Equation (1) are transformed into solution through the use of symmetry groups. So, on the assumption that $u = f(t,x)$ is a solution of the generalized equation (1), we obtain new solutions for Equation (1) by means of diverse symmetry groups . Then, by using the previous groups G_i, $i = 1,\dots 5$, we calculate the appropiate new solutions:

$$\hat{u}_1 = f(t-\epsilon,x) \tag{44}$$

$$\hat{u}_2 = f(t,x-\epsilon) \tag{45}$$

$$\hat{u}_3 = f(te^{-3\epsilon}, xe^{-\epsilon})e^{-\frac{2\epsilon}{n}} \tag{46}$$

$$\hat{u}_4 = f(te^{-\epsilon}, x)e^{\epsilon} \tag{47}$$

$$\hat{u}_5 = f(te^{-3\epsilon}, xe^{-\epsilon}) - 2\epsilon. \tag{48}$$

4 | SYMMETRY REDUCTIONS AND EXACT SOLUTIONS

In this part, we use the optimal system computed in the anterior subsection and we get the following symmetry reductions of the Equation (1).

Case 1a. By using the generator V_3, we get

$$u = h(w)t^{-\frac{2}{3n}} \tag{49}$$

where $w = xt^{-\frac{1}{3}}$. By considering (49) and (1) we have a reduced ordinary reduced equation given by

$$-\frac{1}{3}wh'' + nh^{n-1}(h')^2 - \beta h^{iv)} + h^n h'' - \frac{2+n}{3n}h' = 0. \tag{50}$$

Case 1b. From the generator $\lambda V_1 + V_2$, one has

$$u = h(w) \tag{51}$$

where $w = x - \lambda t$. By the substitution of (51) into (1), we have an ordinary differential equation given by

$$-\alpha h^{2n+1} - \lambda h'' + nh^{n-1}(h')^2 - \beta h^{iv)} + h^n h'' = 0. \tag{52}$$

Case 2. By using the generator $\lambda V_1 + V_2$, we get the similarity variables

$$w = x - \lambda t, \quad u = h(w) \tag{53}$$

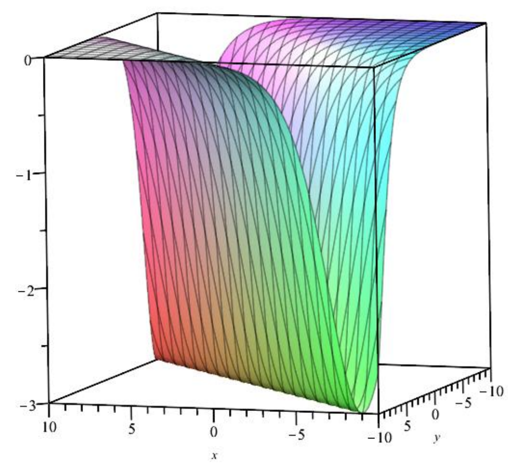

FIGURE 1 Exact solution of (59) for $\lambda = -1, k = 1$ and $b = 1$

and the ODE

$$-\lambda h'' + nh^{n-1}(h')^2 - \beta h^{iv)} + h^n h'' - \alpha h = 0 \tag{54}$$

Case 3a. By using the generator $\lambda V_1 + V_2$, we get the similarity variables

$$w = x - \lambda t, \quad u = h(w) \tag{55}$$

and the ODE

$$-\lambda h'' + nh^{n-1}(h')^2 - \beta h^{iv)} + h^n h'' = 0 \tag{56}$$

which give us the following ordinary differential equation

$$\frac{\lambda}{2}(n+1)(n+2)h^2 - h^{n+2} + \beta(n+1)(n+2)(h')^2 = 0. \tag{57}$$

with solution for $n = 1$

$$h(w) = 3\lambda\left(1 + \tan\left((k-w)\frac{\sqrt{18b\lambda}}{12b}\right)^2\right) \tag{58}$$

where $k \in \mathbb{R}$. This give us the following solution for Equation (1)

$$u(t,x) = 3\lambda\left(1 + \tan\left((k - x + \lambda t)\frac{\sqrt{18b\lambda}}{12b}\right)^2\right) \tag{59}$$

In Figure 1 we consider (59) with $\lambda = -1, k = 1$ and $b = 1$.

The solution for $n = 2$ is given by

$$h(w) = \frac{-24b\lambda e^{-\frac{-2b\lambda}{2b}(k-w)}}{24b\lambda + e^{\frac{\sqrt{-2b\lambda}}{b}(k-w)}} \tag{60}$$

and

$$h(w) = \frac{-24b\lambda e^{-\frac{-2b\lambda}{2b}(k-w)}}{24b\lambda + e^{\frac{\sqrt{-2b\lambda}}{b}(w-k)}} \tag{61}$$

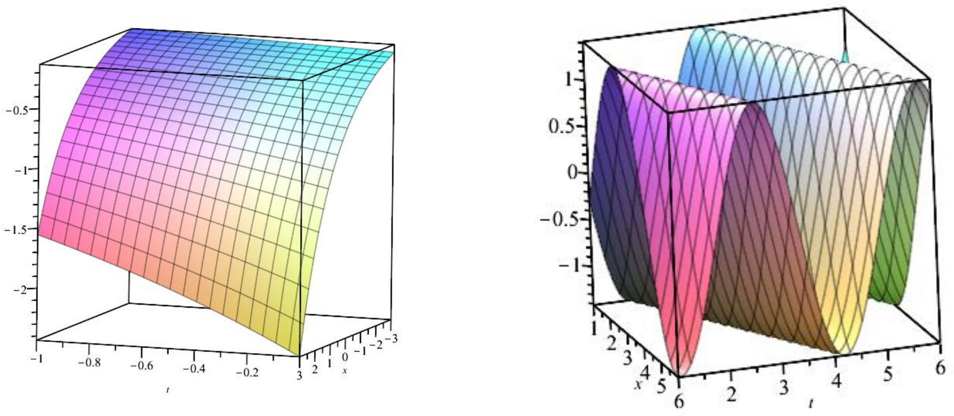

FIGURE 2 Exact solutions of (62) and (63)

where $k \in \mathbb{R}$. This give us the following solution for Equation (1)

$$u(t,x)\frac{-24b\lambda e^{-\frac{-2b\lambda}{2b}(k-x+\lambda t)}}{24b\lambda + e^{\frac{\sqrt{-2b\lambda}}{b}(k-x+\lambda t)}} \tag{62}$$

and

$$u(t,x)\frac{-24b\lambda e^{-\frac{-2b\lambda}{2b}(k-x+\lambda t)}}{24b\lambda + e^{\frac{\sqrt{-2b\lambda}}{b}(k-x+\lambda t)}} \tag{63}$$

In Figure 2 we consider solution (62) with $\lambda = -1$, $k = 1$ and $b = 3$ and solution (63) with $\lambda = -2$, $k = 1$ and $b = 1$ respectively.

Case 3b. By using the generator V_3, we obtain

$$w = xt^{-\frac{1}{3}}, \quad u = h(w)t^{-\frac{2}{3n}} \tag{64}$$

and the ordinary reduced equation

$$-\frac{1}{3}wh'' + nh^{n-1}(h')^2 - \beta h^{iv)} + h^n h'' - \frac{2+n}{3n}h' = 0 \tag{65}$$

which is reduced to

$$-\frac{1}{3}wh' + \frac{1}{3}h - \beta h''' + h^n h' - \frac{2+n}{3n} = 0 \tag{66}$$

Case 4. By using the generator $V_1 + \lambda_2 V_2 + \lambda_4 V_4$, we obtain

$$w = x - \lambda_2 t, \quad u = h(w)e^{\lambda_4 t} \tag{67}$$

and the ordinary differential equation

$$\lambda_4 h' - \lambda_2 h'' - \beta h^{iv)} + h'' - \alpha h = 0. \tag{68}$$

∘ We have the following solution of (68) in the general case of α, β, λ_2 and λ_4

$$h(w) = \sum_{k=1}^{4} C_k e^{RootOf(\beta Z^4 + (\lambda_2 - 1)Z^2 - \lambda_4 Z + \alpha,\, index=k)w} \tag{69}$$

where $C_k \in \mathbb{R}$, $k = 1 \dots 4$ and the corresponding solution of (1)

$$u(t,x) = \sum_{k=1}^{4} C_k e^{RootOf(\beta Z^4 + (\lambda_2 - 1)Z^2 - \lambda_4 Z + \alpha,\, index=k)(x - \lambda_2 t)} \tag{70}$$

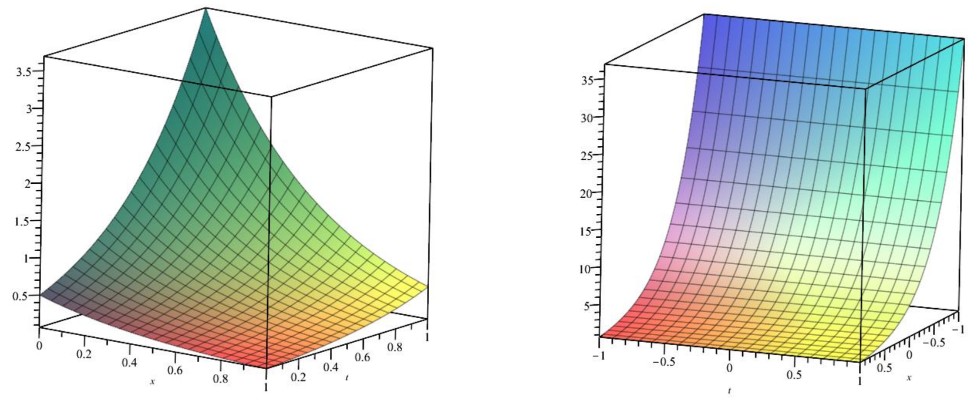

FIGURE 3 Exact solution (85) and (86)

where $C_k \in \mathbb{R}$, $k = 1 \ldots 4$.

∘ In the case $\alpha + 16\beta + 4\lambda_2 - 4 + 2\lambda_4 = 0$, we obtain the following solutions of (68)

$$h(w) = \frac{c_1 sech(w)^2}{(1 + tanh(w))^2} \tag{71}$$

$$h(w) = \frac{c_2 csch(w)^2}{(1 + coth(w))^2} \tag{72}$$

and these give us the following solutions for Equation (1) respectively

$$u(t, x) = \frac{c_1 sech(x - \lambda_2 t)^2}{(1 + tanh(x - \lambda_2 t))^2} \tag{73}$$

$$u(t, x) = \frac{c_2 csch(x - \lambda_2 t)^2}{(1 + coth(x - \lambda_2 t))^2}. \tag{74}$$

where $c_1, c_2 \in \mathbb{R}$. In Figure 3 we consider solution (73) and (74) respectively.

∘ In the case $\lambda_4 = 0$, we obtain the following general solution of (68)

$$\begin{aligned} h(w) = {} & C_1 e^{-\frac{\sqrt{-2\beta(\lambda_2-1+\sqrt{-4\alpha\beta+\lambda_2^2-2\lambda_2+1})w}}{2b}} + C_2 e^{\frac{\sqrt{-2\beta(\lambda_2-1+\sqrt{-4\alpha\beta+\lambda_2^2-2\lambda_2+1})w}}{2b}} \\ & + C_3 e^{-\frac{\sqrt{-2\beta(-\lambda_2+1+\sqrt{-4\alpha\beta+\lambda_2^2-2\lambda_2+1})w}}{2b}} + C_4 e^{\frac{\sqrt{-2\beta(-\lambda_2+1+\sqrt{-4\alpha\beta+\lambda_2^2-2\lambda_2+1})w}}{2b}} \end{aligned} \tag{75}$$

where $c_i \in \mathbb{R}$, $i = 1 \ldots 3$.

This give us the following solution for Equation (1)

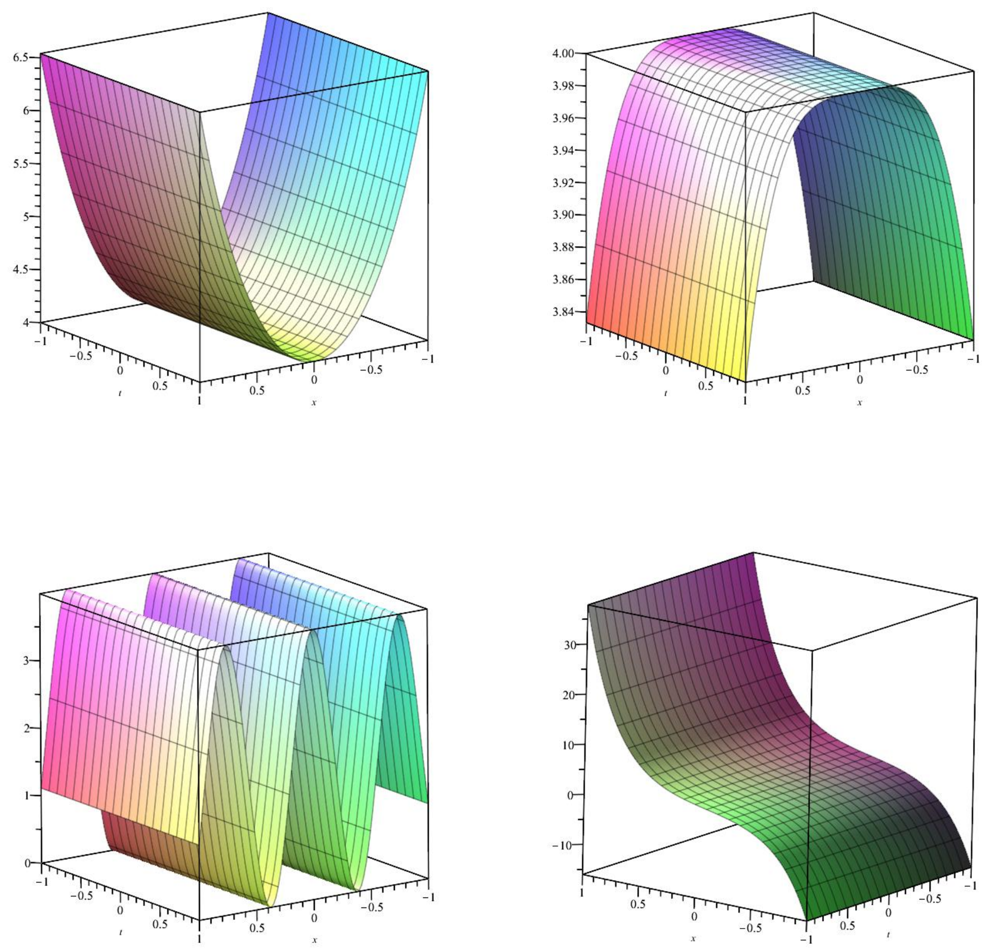

FIGURE 4 Exact solution of Equation (1) given by equation (76).

$$
\begin{aligned}
u(t,x) = {} & C_1 e^{-\frac{\sqrt{-2\beta(\lambda_2-1+\sqrt{-4\alpha\beta+\lambda_2^2-2\lambda_2+1})(x-\lambda_2 t)}}{2b}} + C_2 e^{\frac{\sqrt{-2\beta(\lambda_2-1+\sqrt{-4\alpha\beta+\lambda_2^2-2\lambda_2+1})(x-\lambda_2 t)}}{2b}} \\
& + C_3 e^{-\frac{\sqrt{-2\beta(-\lambda_2+1+\sqrt{-4\alpha\beta+\lambda_2^2-2\lambda_2+1})(x-\lambda_2 t)}}{2b}} + C_4 e^{\frac{\sqrt{-2\beta(-\lambda_2+1+\sqrt{-4\alpha\beta+\lambda_2^2-2\lambda_2+1})(x-\lambda_2 t)}}{2b}}
\end{aligned}
\tag{76}
$$

In Figure 4 we consider solution (76) in the cases
$C_i = 1, i = 1 \ldots 4, \lambda_2 = -1, \beta = 1, \alpha = -1,$
$C_i = 1, i = 1 \ldots 4, \lambda_2 = 1, \beta = 2, \alpha = 3,$
$C_i = 1, i = 1 \ldots 4, \lambda_2 = 70, \beta = 1, \alpha = 1,$
$C_1 = C_2 = 1, C_3 = -0.5, C_4 = 1, \lambda_2 = -50, \beta = 4, \alpha = -4,$
respectively.

∘ In the case $\lambda_4 = 0$ we obtain the following solution of (68)

$$h(w) = c_1 \sinh(c_2 w) + c_3 \cosh(c_2) w \tag{77}$$

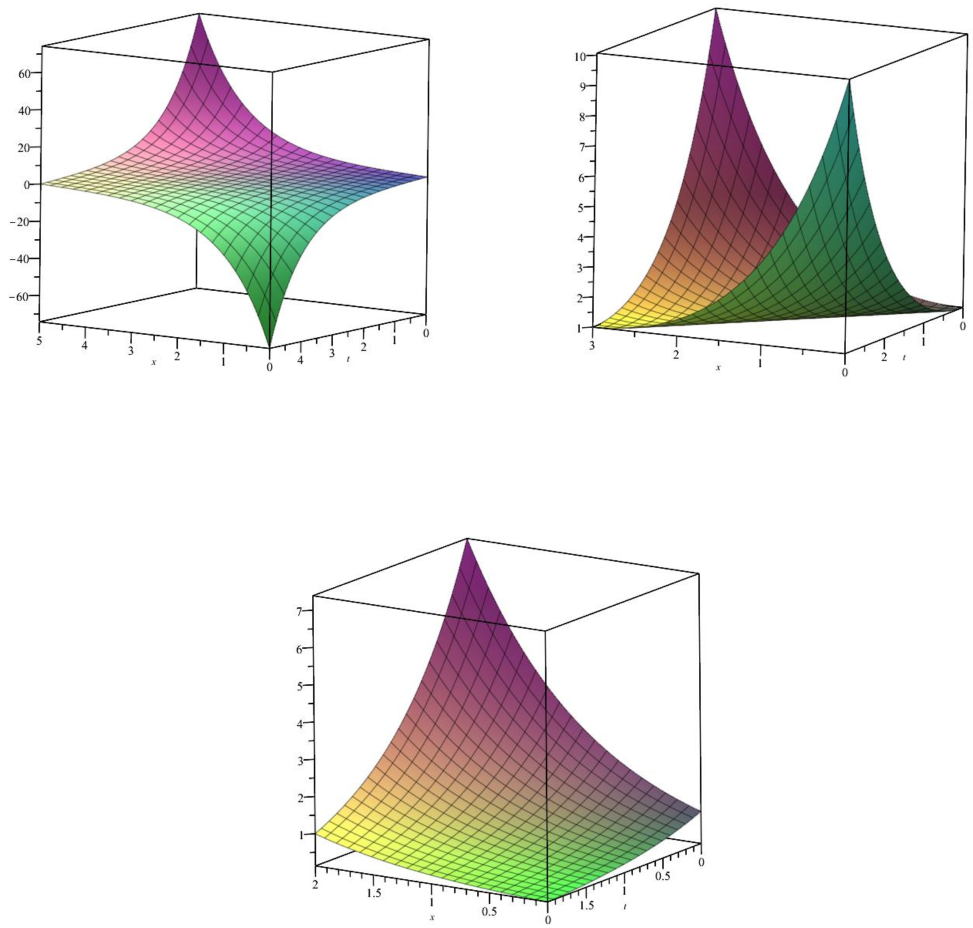

FIGURE 5 Exact solution of Equation (1) given by equation (78).

where $c_2^2\lambda_2 + \beta c_2^4 - c_2^2 + \alpha = 0$, with the following solution for Equation (1)

$$u(t,x) = c_1 \sinh(c_2(x-\lambda_2 t)) + c_3 \cosh(c_2)(x-\lambda_2 t). \tag{78}$$

In Figure 5 we consider the solution (78) in the case $c_1 = c_2 = 1, c_3 = 0, \lambda_2 = 1, \beta = 1$, the case $c_1 = 0, c_2 = 1, c_3 = 1, \lambda_2 = 1, \beta = 1$ and the case $c_i = 1, i = 1 \dots 3, \lambda_2 = 1, \beta = 1$ respectively.

∘ In the case $\lambda_4 = 0$, we obtain the following solution of (68)

$$h(w) = c_1 \sin(c_2 w) + c_3 \cos(c_2 w) \tag{79}$$

where $c_i \in \mathbb{R}$, $i = 1 \dots 3$, with the following solution for Equation (1)

$$u(t,x) = c_1 \sin(c_2(x-\lambda_2 t)) + c_3 \cos(c_2(x-\lambda_2 t)) \tag{80}$$

In Figure 6 we consider the solution (80) in the case $c_1 = c_2 = 1, c_3 = 1, \lambda_2 = 10, \beta = 1$, the case $c_1 = 0, c_2 = c_3 = 1, \lambda_2 = 0.5, \beta = 2$ and the case $c_1 = 100, c_2 = 1, c_3 = 0, \lambda_2 = 0, \beta = 1$ respectively.

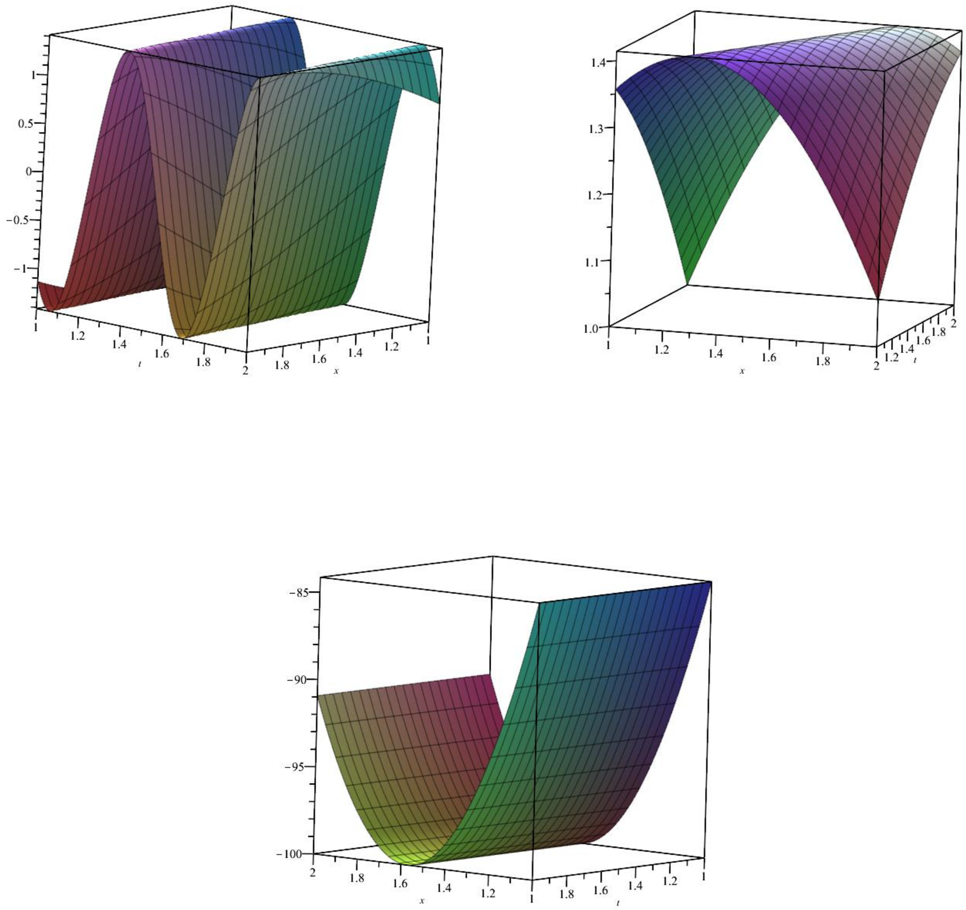

FIGURE 6 Exact solution of Equation (1) given by equation (80).

∘ In the case $\alpha = \lambda_4 = 0$ we obtain the following solutions according to if $b(\lambda_2 - 1)$ is a positive, negative o zero real number, respectively

$$h(w) = k_1 \sin\left(\frac{\sqrt{\lambda_2 - 1}w}{\sqrt{b}}\right) + k_2 \cos\left(\frac{\sqrt{\lambda_2 - 1}w}{\sqrt{b}}\right) \tag{81}$$

$$h(w) = k_1 e^{\left(\frac{\sqrt{\lambda_2 - 1}iw}{\sqrt{b}}\right)} + k_2 e^{\left(\frac{\sqrt{\lambda_2 - 1}iw}{\sqrt{b}}\right)} \tag{82}$$

$$h(w) = k_1 w + k2, \tag{83}$$

and these give us the following solutions for Equation (1) respectively

$$u(t, x) = k_1 \sin\left(\frac{\sqrt{\lambda_2 - 1}(x - \lambda_2 t)}{\sqrt{b}}\right) + k_2 \cos\left(\frac{\sqrt{\lambda_2 - 1}(x - \lambda_2 t)}{\sqrt{b}}\right) \tag{84}$$

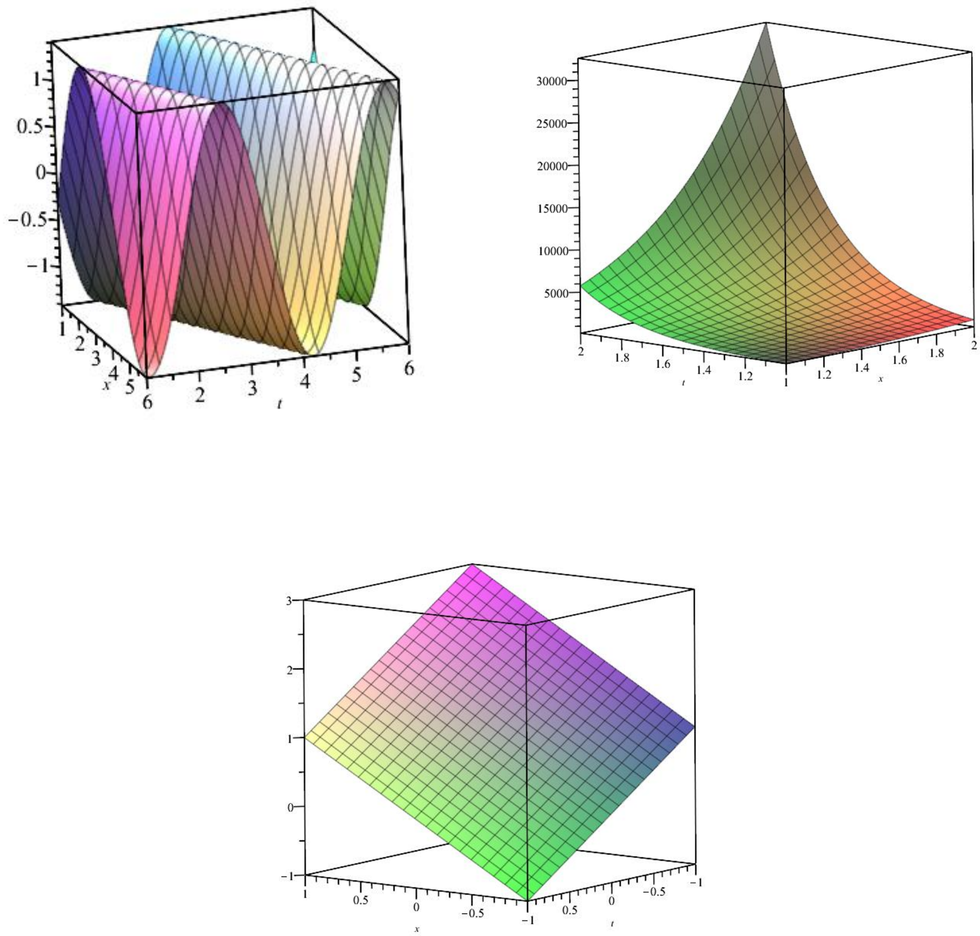

FIGURE 7 Exact solution (84), (85) and (86)

$$u(t,x) = k_1 e^{\left(\frac{\sqrt{\lambda_2 - 1}\, i(x - \lambda_2 t)}{\sqrt{b}}\right)} + k_2 e^{\left(\frac{\sqrt{\lambda_2 - 1}\, i(x - \lambda_2 t)}{\sqrt{b}}\right)} \tag{85}$$

$$u(t,x) = k_1(x - t) + k2, \tag{86}$$

In Figure 7 we consider solution (84) with $k_1 = k_2 = 1, \lambda_2 = 2, b = 1$, solution (85) with $k_1 = 1, k_2 = 1, \lambda_2 = -2, b = 1$ and solution (86) with $k_1 = 1, k_2 = 1, \lambda_2 = 1$ respectively.

Case 5a. By using the generator V_3, we have

$$u = h(w) - \tfrac{2}{3} ln(t) \tag{87}$$

where $w = xt^{-\frac{1}{3}}$ and we have the ordinary reduced equation given by

$$-\frac{1}{3} w h'' + c_1 h'' e^h + c_1 (h')^2 e^h - \alpha c_2 e^{2h} - \beta h^{iv)} - \frac{1}{3} h' = 0. \tag{88}$$

Case 5b. From the generator $\lambda V_1 + V_2$, one has

$$u = h(w) \tag{89}$$

where $w = x - \lambda t$. By substituting (89) into (1), we have the ordinary differential equation given by

$$-\lambda h'' - \beta h^{iv)} + c_1 h'' e^h + c_1 (h')^2 e^h - \alpha c_2 e^{2h} = 0. \tag{90}$$

For $c_2 = 0$ we have the following solutions of (90) where λ is positive, negative or $\lambda = 0$ respectively

$$h(w) = k_1 \sin\left(w\sqrt{\frac{\lambda}{\beta}}\right) + k_2 \cos\left(w\sqrt{\frac{\lambda}{\beta}}\right) \tag{91}$$

$$h(w) = k_1 e^{iw\sqrt{\frac{\lambda}{\beta}}} + k_2 e^{-iw\sqrt{\frac{\lambda}{\beta}}} \tag{92}$$

$$h(w) = k_1 + k_2 w \tag{93}$$

and we get the following solutions of Equation (1)

$$u(t,x) = k_1 \sin\left((x-\lambda t)\sqrt{\frac{\lambda}{\beta}}\right) + k_2 \cos\left((x-\lambda t)\sqrt{\frac{\lambda}{\beta}}\right) \tag{94}$$

$$u(t,x) = k_1 e^{i(x-\lambda t)\sqrt{\frac{\lambda}{\beta}}} + k_2 e^{-i(x-\lambda t)\sqrt{\frac{\lambda}{\beta}}} \tag{95}$$

$$u(t,x) = k_1 + k_2 x. \tag{96}$$

Case 6. From the generator $\lambda V_1 + V_2$, we have $u = h(w)$, $w = x - \lambda t$ and we have the reduced differential equation given by

$$-\lambda h'' - \beta h^{iv)} + g''(h')^2 + g' h'' - \alpha f = 0. \tag{97}$$

and integrating twice respect w we obtain

$$-\lambda h - \beta h'' + g - \alpha \int \left(\int f dw \right) dw = 0. \tag{98}$$

For f, g identity functions or constant functions, we have similar solutions to (94), (95) and (96).

CONCLUDING REMARKS

We have studied the generalized Ostrovsky equation (1) with real dispersion coefficients, from the point of view of symmetry analysis. First of all, we have derived conservation laws for the subjacent equation through multiplier approach conservation theorem and we and we have recourse to the invariance and multiplier perspective by using the Euler-Lagrange operator. We have obtained several local conservation laws for Equation (1) which preserve densities, fluxes and energy. Secondly, we have calculated Lie point symmetries of the equation and subsequently we have performed symmetry reductions. Also, we have obtained travelling wave solutions of significance importance by using the vector fields. Finally, we have obtained Lie symmetry groups generated by means of the vector field and new solutions for Equation (1) associated to them.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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$\square$