Non-standard numerical scheme for singularly perturbed parabolic partial differential equation with long time lag arising in control theory

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Abstract

For the numerical solution of singularly perturbed second-order parabolic partial differential equation of one dimensional convection-diffusion type with long time delays arising in control theory, a novel class of fitted operator finite difference methods is constructed using non-standard finite difference methods. Since the two parameters; time lag and perturbation parameters are sources for the simultaneous occurrence of time-consuming and high speed phenomena of the physical systems that depends on the present and past history, our study here is to capture the effect of the two parameters on the boundary layer. The spatial derivative is suitably replaced by a difference operator followed by the time derivative is replaced by the Crank-Nicolson based scheme. A second-order parameter-uniform error bounds are established to provide numerical results.

KEYWORDS

singular perturbation; non-standard finite difference method; large time delay; parabolic partial differential equation; boundary layer.

[*]Both the authors have made equal and substantive contributions to the article and assume full responsibility for its content. Both the authors read and approved the final manuscript.

1 | INTRODUCTION

Any system involving the return to the input of a part of the output of a system control will almost always involve time delays. In contrast to non-delay parabolic partial differential equations which relate an unknown function to its derivatives evaluated at the same time, parabolic partial differential equations with delay in time arise when the rate of change of a time dependent process is determined not only by its current state but also by a certain past state in its mathematical modeling. A realistic model with time delay partial differential equations has significantly more complicated dynamics than a model without time delay partial differential equations, because a time delay can cause a stable equilibrium to become unstable. Such type of equation arises frequently in the mathematical modelling of science and engineering [17, 28, 20, 5, 23, 32, 15, 18, 19, 29, 30, 33, 14, 2, 3, 4, 8]. The occurrence of boundary layer in singular perturbation problem was originated in nineteenth century [31]. The solution to such problems undergoes abrupt changes in narrow regions of the domain due to the multiscale character of the associated perturbation parameter(s) [26, 7, 22]. In general, there are two type of difficulty associated with singularly perturbed time delay partial differential equations; one is due to the presence of the perturbation parameter multiplied to the highest order derivative term and another one is due to the presence of retarded argument. The two parameters makes complexity of the system modeled by singularly perturbed time delay partial differential equations that renders it unlikely to obtain an analytical solution and numerical solution of the problem would seem more practical. However, for the numerical solution of such problems, these layers are connected with additional difficulties; besides instabilities of certain discretization methods, high computational costs and insufficient resolution are essentially due to the existence of layers [8, 7]. Solutions of time delay differential equations are of immense interest, equally in applications and theory. An important example arises in the numerical study of the overall control system

$$\frac{\partial u(x,t)}{\partial t} =\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2}+v[g\left(u(x,t-\tau)\right)]\left(\frac{\partial u(x,t)}{\partial x}\right)+c[f\left(u(x,t-\tau)\right)-u(x,t)]$$

defined on a one dimensional spatial domain $0<x<1$, where v is the instantaneous material strip velocity depending on a prescribed spatial average of the time-delayed temperature distribution $u(x,t-\tau)$, and f represents a distributed temperature source function depending on $u(x,t-\tau)$ [33]. A wide range of delay parabolic partial differential Equations models can be found in Wu [33] and Murray [23]. Much attention has been paid to delay parabolic differential equations and their numerical approximations. Among the first rigorous numerical treatments of singularly perturbed parabolic delay differential equations with delay is the pioneering work of Ansari *et al*. [1] in which second order singularly perturbed delay parabolic differential equations are approximated by finite differences on piecewise Shishkin meshes. In the papers, [11, 12, 13, 6, 27] the authors considered numerical study of singularly perturbed parabolic convection-diffusion equation with time delay. Recently, Gowrisankar and Natesan [9], Kumar and Kumari [16] and Podila and Kumar [24] proposed a robust numerical approach for solving convection-diffusion singularly perturbed parabolic partial differential equations with large delay in time.

Most of the previous works for numerical solution of singularly perturbed delay parabolic partial differential equations of convection-diffusion type have been studied on the ε-uniform convergence of solutions based on fitted mesh and a few interest has been paid to construction fitted operator finite difference method of solutions. As a result, when the perturbation parameter ε becomes very small, it is critical to improve appropriate numerical techniques to cope with the oscillatory character of the solutions, whose accuracy is independent of the parameter value ε. In this work, a denominator function has been introduced in the nonstandard method, in such a way that it suits the central difference and thus obtained are very stable for all the finite values of step-sizes. The goal of this study is to implement such a numerical method and provide parameter uniform error estimates for singularly perturbed convection-diffusion parabolic systems with large time lag.

Notation For any given function $g(x,t) \in C^{(k)}(\Omega_x \times \Omega_t)$ (k is a non-negative integer), $\|.\|_\infty$ is standard supremum norm over the domain $\Omega_x \times \Omega_t$ given by $\|.\|_\infty = \sup_{x\in\Omega_x, t\in\Omega_t}|g(x,t)|$. Through out this paper C (in some case indexed) is denoted for positive constant independent of perturbation parameter ε.

2 | PROBLEM FORMULATION

Let $\Omega_x = (0,1)$, $D = \Omega_x \times (0,T]$, and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where Γ_l and Γ_r are the left and the right side of the rectangular domain D corresponding to $x = 0$ and $x = 1$, respectively and $\Gamma_b = [0,1] \times [-\tau, 0]$. Here in this paper, we consider the following class of second-order singularly perturbed time delayed one-dimensional parabolic convection-diffusion problem:

$$\pounds_{\varepsilon,x} u(x,t) \equiv \frac{\partial u(x,t)}{\partial t} - \varepsilon u_{xx}(x,t) + a(x,t)u_x(x,t) + b(x,t)u(x,t) = -c(x,t)u(x,t-\tau) + f(x,t), (x,t) \in D, \tag{2.1}$$

initial condition

$$u(x,t) = \phi_b(x,t),\ (x,t) \in \Gamma b, \tag{2.2}$$

and subject to the boundary condition

$$u(0,t) = \phi_l(t),\ \Gamma_l = \{(0,t) : 0 \le t \le T\}, \tag{2.3}$$

$$u(1,t) = \phi_r(t),\ \Gamma_r = \{(1,t) : 0 \le t \le T\}. \tag{2.4}$$

$0 < \varepsilon \ll 1$ is a singular perturbation parameter and $\tau > 0$ represents the delay parameter and the functions $a(x,t), b(x,t), c(x,t), f(x,t)$ on $\overline{D}$ and $\phi_b(x,t)$, $\phi_l(t)$, $\phi_r(t)$ on Γ are sufficiently smooth, bounded functions and independent of ε. For small values of perturbation parameter($\varepsilon \to 0$) the solution of the problem typically exhibits layer behavior depending on the sign of the convection term. When $a(x,t) \ge \alpha > 0, b(x,t) \ge \beta > 0$, $c(x,t) \ge \vartheta > 0, (x,t) \in \overline{D}$, the solutions of (2.1)-(2.4) exhibits boundary layer along $x = 1$($i.e$, in the neighborhood of Γ_r). In this study, our aim is to obtain and examine the approximate solution to observe the effects of the delay parameter τ and perturbation parameter ε on the boundary layer.

3 | BOUNDS FOR THE SOLUTION OF THE CONTINUOUS PROBLEM

The existence and uniqueness of the solution for the model problem (2.1)-(2.4) can be guaranteed by the sufficient smoothness of $\phi_l(t), \phi_b(x,t)$ and $\phi_r(t)$ along with appropriate compatibility conditions at the corner points $(0,0), (1,0), (0,-\tau)$ and $(1,-\tau)$, and delay terms [25]:

$$\begin{cases} \phi_b(0,0) = \phi_l(0), \\ \phi_b(1,0) = \phi_r(0), \end{cases} \tag{3.1}$$

and

$$\begin{cases} \frac{\partial\phi_l(0)}{\partial t} - \varepsilon\frac{\partial^2\phi_b(0,0)}{\partial x^2} + a(0,0)\frac{\partial\phi_b(0,0)}{\partial x} + b(0,0)\phi_b(0,0) = -c(0,0)\phi_b(0,-\tau) + f(0,0), \\ \frac{\partial\phi_r(0)}{\partial t} - \varepsilon\frac{\partial^2\phi_b(1,0)}{\partial x^2} + a(1,0)\frac{\partial\phi_b(1,0)}{\partial x} + b(1,0)\phi_b(1,0) = -c(1,0)\phi_b(0,-\tau) + f(1,0). \end{cases} \tag{3.2}$$

Lemma 3.1 (Continuous maximum principle) Let $\Psi(x,t) \in C^2(D) \cap C^0(\bar{D})$, with $\pounds_{\varepsilon,x}\Psi(x,t) \geq 0$ in D and $\Psi(x,t) \geq 0$ for all $(x,t) \in \Gamma$. Then we have $\Psi(x,t) \geq 0$ for all $(x,t) \in \bar{D}$.

Proof Suppose there exists $(x^*,t^*) \in \bar{D}$ be such that $\Psi(x^*,t^*) = \min_{(x,t)\in\bar{D}} \Psi(x,t)$ and suppose that $\Psi(x,t) < 0$ which implies $(x^*,t^*) \notin \Gamma$ as $\Psi(x,t) \geq 0$ on Γ. Then, we have $\Psi_x(x^*,t^*) = \Psi_t(x^*,t^*) = 0$ and $\Psi_{xx}(x^*,t^*) \geq 0$ and thus $\pounds_{\varepsilon,x}\Psi(x^*,t^*) < 0$ which contradicts the given hypothesis and hence $\Psi(x,t) \geq 0$ for all $(x,t) \in \bar{D}$.

In a case when boundary layer will occur in the neighborhood of Γ_r then by using compatibility conditions in (3.1) and(3.2) we can say that there exist a constant C independent of ε such that for all $(x,t) \in \overline{D}$ we have the following Lemma.

Lemma 3.2 The solution $u(x,t)$ of the continuous problem (2.1)-(2.4) satisfy the following estimate:

$$|u(x,t) - \phi_b(x,0)| \leq Ct. \tag{3.3}$$

Proof For the proof reader can refer to Das and Natesan [6].

Lemma 3.3 The bound on the solution $u(x,t)$ of the continuous problem (2.1)-(2.4) satisfy the following bound:

$$|u(x,t)| \leq C, (x,t) \in \bar{D}.$$

Proof From Lemma 3.2, we have

$$\begin{cases} |u(x,t)| - |\phi_b(x,0)| \leq |u(x,t) - \phi_b(x,0)| \leq Ct. \\ \Rightarrow u(x,t) \leq Ct + \phi_b(x,0), \forall (x,t) \in \bar{D} \end{cases}$$

Since $t \in (0,T]$, so it is bounded and $\phi_b(x,0) \in C^2(\bar{D})$. Therefore, $Ct+\phi_b(x,0)$ is bounded by some constant C and hence $|u(x,t)| \leq C, (x,t) \in \bar{D}$.

Lemma 3.4 (Uniform stability estimate for continuous problem) The uniform stability bound on the solution $u(x,t)$ of the continuous problems (2.1)-(2.2) satisfy:

$$\|u\| \leq \beta^{-1}\|\pounds_{\varepsilon,x}u\| + \max(|\phi_b|, (|\phi_l| + |\phi_r|)).$$

Proof For the barrier functions $\Psi^{\pm}(x,t) = \beta^{-1}\|\mathcal{L}_{\varepsilon,x}u\| + \max(|\phi_b|, (|\phi_l| + |\phi_r|)) \pm u(x,t), (x,t) \in \bar{D}$ we have

$$\Psi^{\pm}(0,t) = \beta^{-1}\|\mathcal{L}_{\varepsilon,x}u\| + \max(\phi_b, \max(\phi_l, \phi_r)) \pm u(0,t) \geq 0,$$

$$\Psi^{\pm}(1,t) = \beta^{-1}\|\mathcal{L}_{\varepsilon,x}u\| + \max(|\phi_b|, (|\phi_l| + |\phi_r|)) \pm u(1,t) \geq 0,$$

$$\begin{aligned}\mathcal{L}_{\varepsilon,x}\Psi^{\pm} =& b\left[\beta^{-1}\|\mathcal{L}_{\varepsilon,x}u\| + \max(|\phi_b|, (|\phi_l| + |\phi_r|))\right] \pm \mathcal{L}_{\varepsilon,x}u(x,t)\\ &\geq \|\mathcal{L}_{\varepsilon,x}u\| + \beta\max(|\phi_b|, (|\phi_l| + |\phi_r|)) \pm \mathcal{L}_{\varepsilon,x}u(x,t) \geq \|\mathcal{L}_{\varepsilon,x}u\| \pm \|\mathcal{L}_{\varepsilon,x}u(x,t)\| \geq 0.\end{aligned}$$

Thus, by applying the maximum principle we obtain the required result.

Lemma 3.5 Let $u(x,t)$ be the solution of the continuous problem (2.1)-(2.4) for the case when boundary layer occur in the neighborhood of Γ_r then the derivatives of $u(x,t)$ satisfy the following bound:

$$\left|\frac{\partial^i u}{\partial x^i}\right| \leq C\left(1 + \varepsilon^{-i} exp\left(-\alpha(1-x)/\varepsilon\right)\right), \forall(x,t) \in \bar{D}$$

where i non-negative integers such that $0 \leq i \leq 4$..

Proof For the proof reader can refer to Das and Natesan [6].

Lemma 3.6 [6] Suppose Lemmas 3.1 and 3.3 hold and by using mean value theorem the bound on the derivatives of $u(x,t)$ with respect to $'t'$ is given by

$$\left|\frac{\partial^j u}{\partial t^j}\right| \leq C, \forall(x,t) \in \bar{D}, j = 0,1,2,3.$$

4 | NUMERICAL SCHEME FORMULATION

4.1 | The time semidiscretization

On the time domain $[0,T]$ we introduce the equidistant meshes with uniform step size Δt such that

$$\bar{\Omega}_t^M = \{t_n = n\Delta t, n = 0,1,...,M, \Delta t = T/M\},$$

where $M = T/\Delta t$ is the total number of mesh elements in the domain $[0,T]$. For the delay term $u(x,t-\tau)$ we first divide the given interval $[-\tau,0]$ into s equal parts with spacing $\Delta t = \tau/s$ for some positive integer k and use the same spacing for the interval $[0,T]$ and so the mesh in the interval $[-\tau,T]$ is defined as

$$\bar{\Omega}_t^s = \{t_n = n\Delta t, n = 0,1,...,s, t_s = \tau, \Delta t = \tau/s, -s \leq n \leq M\}.$$

Thus the uniform meshes $\bar{\Omega}_t^s$ and $\bar{\Omega}_t^M$ with step size Δt, with s and M mesh elements are used on $[-\tau,0]$ and $[0,T]$, respectively. Here, we propose a numerical scheme to solve Equations (2.1)-(2.4), which consists of the

Crank-Nicolson method for the time derivative. This gives the following system of semi-discretize problem,

$$\begin{cases} \dfrac{U^{n+1}(x)-U^{n}(x)}{\Delta t}-\varepsilon\left(U_{xx}\right)^{n+1/2}(x)+a^{n+1/2}(x)\left(U_{x}\right)^{n+1/2}(x)+ \\ b^{n+1/2}(x)\,U^{n+1/2}(x)=-c^{n+1/2}(x)U^{n+1/2-s}(x)+f^{n+1/2}(x), \\ U^{n+1}(0)=\phi_{l}\left(t_{n+1}\right),0\leq n\leq M, \\ U^{n+1}(1)=\phi_{r}\left(t_{n+1}\right),0\leq n\leq M, \\ U^{n+1}(x)=\phi_{b}\left(x,t_{n+1}\right),x\in\Omega_{x},-(s+1)\leq n\leq -1, \end{cases} \tag{4.1}$$

where $U^{n+1}(x)$ is the approximate solution of $u(x,t_{n+1})$ at $(n+1)th$ time level. The above equation (4.1) can be rewritten in operator form as

$$\begin{cases} \hat{\pounds}^{M}_{\varepsilon,x}U^{n+1}(x)\equiv -\dfrac{\varepsilon}{2}\left(U_{xx}\right)^{n+1}(x)+\dfrac{a^{n+1/2}(x)}{2}\left(U_{x}\right)^{n+1}(x)+\dfrac{1}{2}\left(\dfrac{2}{\Delta t}+b^{n+1/2}(x)\right)U^{n+1}(x) \\ \qquad\qquad =\hat{H}^{n}(x) \\ U^{n+1}(0)=\phi_{l}\left(t_{n+1}\right),0\leq n\leq M, \\ U^{n+1}(1)=\phi_{r}\left(t_{n+1}\right),0\leq n\leq M, \\ U^{n+1}(x)=\phi_{b}\left(x,t_{n+1}\right),x\in\Omega_{x},-(s+1)\leq n\leq -1, \end{cases} \tag{4.2}$$

where

$$\hat{H}^{n}(x)=\begin{cases} \dfrac{\varepsilon}{2}\left(U_{xx}\right)^{n}(x)-\dfrac{a^{n+1/2}(x)}{2}\left(U_{x}\right)^{n}(x)-\dfrac{1}{2}\left(\dfrac{-2}{\Delta t}+b^{n+1/2}(x)\right)U^{n}(x)- \\ c^{n+1/2}(x)\phi_{b}^{n+1/2}(x)+f^{n+1/2}(x),\text{if }\ t_{n}<s, \\ \dfrac{\varepsilon}{2}\left(U_{xx}\right)^{n}(x)-\dfrac{a^{n+1/2}(x)}{2}\left(U_{x}\right)^{n}(x)-\dfrac{1}{2}\left(\dfrac{-2}{\Delta t}+b^{n+1/2}(x)\right)U^{n}(x)- \\ c^{n+1/2}(x)U^{n+1/2-s}(x)+f^{n+1/2}(x),\text{if }\ t_{n}\geq s. \end{cases}$$

The semidiscrete difference operator $\pounds^{M}_{\varepsilon,x}U^{n+1}(x)$ in Equation (4.2) satisfies the maximum principle as follows.

Lemma 4.1 (Semi-discrete maximum principle) Let $\Upsilon^{n+1}(x)$ be a smooth function such that $\Upsilon^{n+1}(0)\geq 0$ and $\Upsilon^{n+1}(1)\geq 0$. Then $\pounds^{M}_{\varepsilon,x}\Upsilon^{n+1}(x)\geq 0$ for all $x\in D$, implies that $\Upsilon^{n+1}(x)\geq 0$ for all $x\in\bar{D}$.

Proof Let $(x^{*},t_{n+1})\in\{(x,t_{n+1}):x\in\bar{D}\}$ be such that $\Upsilon^{n+1}(x^{*})=\min_{(x)\in\bar{D}}\Upsilon^{n+1}(x)$ and suppose $\Upsilon^{n+1}(x)<0$. It is clear that $(x^{*},t_{n+1})\notin\{(0,t_{n+1}),(1,t_{n+1})\}$ as $\Upsilon^{n+1}(x)\geq 0$ on $\{0,1\}$. Then, we have $\Upsilon^{n+1}_{x}(x^{*})=0$ and $\Upsilon_{xx}(x^{*})\geq 0$ and thus

$$\begin{aligned} \pounds^{M}_{\varepsilon,x}\Upsilon^{n+1}_{x}(x^{*})&=-\frac{\varepsilon}{2}\left(\Psi_{xx}\right)^{n+1}(x^{*})+\frac{a^{n+1/2}(x^{*})}{2}\left(\Upsilon_{x}\right)^{n+1}(x^{*})+\frac{R^{n+1/2}(x^{*})}{2}\Upsilon^{n+1}(x^{*}) \\ &\leq\frac{R^{n+1/2}(x^{*})}{2}\Upsilon^{n+1}(x^{*})<0 \end{aligned}$$

which contradicts our supposition and $\Upsilon^{n+1}(x^{*})\geq 0$, which implies $\Upsilon^{n+1}(x)\geq 0$ for all $(x)\in\bar{D}$.

The local truncation error e_{n+1} of the temporal semi-discretization (4.2) is given by $U^n(x) - u(x, t_n)$ where $u(x, t_n)$ and $U^n(x)$ are the exact and approximate solution of the problem in (2.1)-(2.4). Now we follow the following lemma for the error estimate e_{n+1}.

Lemma 4.2 (Local error estimate) Suppose that Lemmas 3.5 and 3.6 hold. Then the local error estimate associated to the semi-discretized problem (4.2) is given by

$$\|e_{n+1}\|_{\infty} \leq C\,(\Delta t)^3$$

Proof The proof can be done by using the Taylor's series expansion up to $O\left((\Delta t)^3\right)$ such that $u(x, t_{n+1/2}) = u(x, t_n + \Delta t/2)$, $u(x, t_n) = u(x, t_n - \Delta t/2)$ and applying the maximum principle given at Lemma (4.1). For more detail the reader referred to Kumar *et al*. [16].

The contribution of each time step to the global error of the time semi-discretization is measured by local truncation error e^{n+1} given by $E_n = \sum_{k=1}^{n} e_k$. Then E^n satisfy the following Lemma.

Lemma 4.3 (Global error estimate.) Under the hypothesis of Lemma 4.2, global error estimate in the temporal direction is given by

$$\|E_n\|_{\infty} \leq C\,(\Delta t)^2$$

where E_n is the global error in the temporal direction at $(n+1)\,th$ time level.

Proof Using local error estimates given in Lemma 4.2, the global error estimate at the $(n+1)\,th$ time step is given by

$$\begin{aligned}
\|E_n\|_{\infty} &= \left\|\sum_{k=1}^{n} e_k\right\|, n \leq \frac{T}{\Delta t} \\
&\leq \|e_1\| + \|e_2\| + ... + \|e_n\| \\
&\leq C_0\,((n)\Delta t)^2\,(\Delta t) \\
&\leq C_0 T\,(\Delta t)^2\,, \text{since} \quad n\,(\Delta t) \leq T \\
&\leq C\,(\Delta t)^2\,, C = C_0 T,
\end{aligned}$$

where C is constant independent of ε and Δt.

The bounds on the derivatives in x direction and asymptotic behavior with respect to ε of the solution $U^{n+1}(x)$ of the problems in (4.2) is given by the following Lemma.

Lemma 4.4 [9] The solution $U^n(x)$ of semi-discretized problem (4.2) and its derivatives satisfies

$$\left|\frac{d^i U^n(x)}{dx^i}\right| \leq C\left(1 + \varepsilon^{-i} exp\left(-\alpha\,(1-x)\,/\varepsilon\right)\right), \forall (x) \in \bar{D}, 0 \leq i \leq 4.$$

4.2 | Spatial discretization

Consider the semi-discretized problem corresponding to the Equation (4.1):

$$-\frac{\varepsilon}{2}\left(U_{xx}\right)^{n+1}(x)+\frac{a^{n+1/2}(x)}{2}\left(U_x\right)^{n+1}(x)+\frac{1}{2}\left(\frac{2}{\Delta t}+b^{n+1/2}(x)\right)U^{n+1}(x)=\hat{H}^n(x). \tag{4.3}$$

Using the homogeneous problems corresponding to (4.3) with constant coefficients gives

$$-\varepsilon\left(U_{xx}\right)^{n+1}(x)+\hat{\alpha}\left(U_x\right)^{n+1}(x)+\hat{r}^*U^{n+1}(x)=0 \tag{4.4}$$

where $\frac{1}{2}\left(\frac{2}{\Delta t}+b^{n+1/2}(x)\right)\geq\hat{r}^*>0$. From Equation (4.4) we have two linear independent solutions $exp\left(\hat{\lambda}_1 x\right)$ and $exp\left(\hat{\lambda}_2 x\right)$ such that

$$\hat{\lambda}_{1,2}=\frac{-\hat{\alpha}\pm\sqrt{\hat{\alpha}^2+4\varepsilon\hat{r}^*}}{-2\varepsilon}. \tag{4.5}$$

Now we partitioned spatial domain $[0,1]$ into N number of mesh elements with a uniform meshes of equal length of h. This gives the spatial mesh

$$\Omega_x^N=\{x_m=mh, m=1,2,...,N, x_0=0, x_N=1, h=1/N\}$$

where x_m is nodal points. Let us denote the approximate solution to $u(x,t_n)$ at the grid point x_m by $U_m=c_1 exp\left(\hat{\lambda}_1 x_m\right)+c_2 exp\left(\hat{\lambda}_2 x_m\right)$. Using the method in [21] we have

$$\begin{vmatrix} U_{m-1} & exp\left(\hat{\lambda}_1 x_{m-1}\right) & exp\left(\hat{\lambda}_2 x_{m-1}\right) \\ U_m & exp\left(\hat{\lambda}_1 x_m\right) & exp\left(\hat{\lambda}_2 x_m\right) \\ U_{m+1} & exp\left(\hat{\lambda}_1 x_{m+1}\right) & exp\left(\hat{\lambda}_2 x_{m+1}\right) \end{vmatrix}=0.$$

Evaluation of the determinant gives:

$$exp\left(\frac{\hat{\alpha}h}{\varepsilon}\right)U_{m-1}-2\cosh\left(\frac{h\sqrt{\hat{\alpha}^2+4\varepsilon\hat{r}^*}}{2\varepsilon}\right)U_m+exp\left(\frac{-\hat{\alpha}h}{\varepsilon}\right)U_{m+1}=0, \tag{4.6}$$

which is an exact difference scheme for (4.4). With some manipulations (4.6) yields the following scheme for the non homogeneous problem corresponding to the problem (4.4)

$$-\varepsilon\frac{U_{m-1}^{n+1}-2U_m^{n+1}+U_{m+1}^{n+1}}{\frac{2h\varepsilon}{\hat{\alpha}}\tanh\left(\frac{\hat{\alpha}h}{2\varepsilon}\right)}+\hat{\alpha}\frac{U_{m+1}^{n+1}-U_{m-1}^{n+1}}{2h}=\hat{H}_m^{n+1}. \tag{4.7}$$

According to Mickens [21] we introduce a denominator function that constitutes a general property of the schemes (4.7). Motivated by (4.7), the non-standard finite difference scheme for the variable coefficient problem is given by

$$-\frac{\varepsilon}{2}\frac{\delta_x^2 U_m^{n+1}}{\hat{\gamma}^2}+\frac{a_m^{n+1/2}}{2}D_x^0 U_m^{n+1}+\frac{1}{2}\left(\frac{2}{\Delta t}+b_m^{n+1/2}\right)U_m^{n+1}=\hat{H}_m^n, \tag{4.8}$$

where

$$\hat{H}^n(x)=\begin{cases}\frac{\varepsilon}{2}\frac{\delta_x^2 U_m^{n+1}}{\hat{\gamma}^2}-\frac{a_m^{n+1/2}}{2}D_x^0 U_m^{n+1}-\frac{1}{2}\left(\frac{-2}{\Delta t}+b_m^{n+1/2}\right)U_m^n-c_m^{n+1/2}\phi_b^{n+1/2}(x_m)+\\ f_m^{n+1/2}, \text{if } \ t_n<s,\\ \frac{\varepsilon}{2}\frac{\delta_x^2 U_m^{n+1}}{\hat{\gamma}^2}-\frac{a_m^{n+1/2}}{2}D_x^0 U_m^{n+1}-\frac{1}{2}\left(\frac{-2}{\Delta t}+b_m^{n+1/2}\right)U_m^n-c_m^{n+1/2}U_m^{n+1/2-s}+\\ f_m^{n+1/2}, \text{if } \ t_n\geq s,\end{cases}$$

with

$$\delta_x^2 U_m^n=U_{m-1}^n-2U_m^n+U_{m+1}^n, \hat{\gamma}^2=\frac{2h\varepsilon}{a_m^n}\tanh\left(\frac{a_m^{n+1/2}h}{2\varepsilon}\right), D_x^0 U_m^n=\frac{U_{m+1}^n-U_{m-1}^n}{2h}.$$

Equation (4.8) can be rewritten as:

$$\begin{cases}\hat{\pounds}_{\varepsilon,m}^{N,M}U_m^{n+1}=\hat{H}_m^{n+1},\\ U^{n+1}(0)=\phi_l(t_{n+1}), 0\leq n\leq M,\\ U^{n+1}(1)=\phi_r(t_{n+1}), 0\leq n\leq M,\\ U^{n+1}(x_m)=\phi_b(x_m,t_{n+1}), -(s+1)\leq n\leq -1, x_m\in\bar{\Omega}^N,\end{cases} \tag{4.9}$$

where

$$\hat{\pounds}_{\varepsilon,m}^{N,M}U_m^{n+1}=-\frac{\varepsilon}{2}\frac{\delta_x^2 U_m^{n+1}}{\hat{\gamma}^2}+\frac{a_m^{n+1/2}}{2}D_x^0 U_m^{n+1}+\frac{1}{2}\left(\frac{2}{\Delta t}+b_m^n\right)U_m^{n+1}.$$

Lemma 4.5 (Discrete maximum principle) Let $\Psi^{n+1}(x_m)$ be a mesh function such that $\Psi^{n+1}(x_0)\geq 0$ and $\Psi^{n+1}(x_N)\geq 0$. Then $\pounds_{\varepsilon,m}^{N,M}\Psi^{n+1}(x_m)\geq 0$ for $1\leq m\leq N-1$, implies that $\Psi^{n+1}(x_m)\geq 0$ for $0\leq m\leq N$.

Proof Let $k^*\in\{0,1,...,N\}$ be such that $\Psi_{k^*}^{n+1}=\min_{1\leq m\leq N}\Psi_{k^*}^{n+1}$ and suppose $\Psi_{k^*}^{n+1}<0$. It is clear that $k^*\notin\{0,N\}$. Also we have $\Psi_{k^*+1}^{n+1}-\Psi_{k^*}^{n+1}\geq 0$ and $\Psi_{k^*+1}^{n+1}-\Psi_{k^*-1}^{n+1}\leq 0$. Now from (4.9) we have

$$\begin{aligned}\pounds_{\varepsilon,m}^{N,M}\Psi_{k^*}^{n+1}&=-\frac{\varepsilon}{2}\frac{\delta_x^2\Psi_{k^*}^{n+1}}{\gamma^2}+\frac{a_{k^*}^{n+1/2}}{2}D_x^0\Psi_{k*}^{n+1}+\frac{R_{k*}^{n+1/2}}{2}\Psi_{k*}^{n+1}\\ &=-\frac{\varepsilon}{2}\frac{\Psi_{k^*-1}^{n+1}-2\Psi_{k^*}^{n+1}+\Psi_{k^*+1}^{n+1}}{\gamma^2}+\frac{a_{k^*}^{n+1/2}}{2}\frac{\Psi_{k^*+1}^{n+1}-\Psi_{k^*-1}^{n+1}}{2h}+\frac{R_{k*}^{n+1/2}}{2}\Psi_{k*}^{n+1}\\ &=-\frac{\varepsilon}{2}\frac{\left(\Psi_{k^*-1}^{n+1}-\Psi_{k^*}^{n+1}\right)+\left(\Psi_{k^*+1}^{n+1}-\Psi_{k^*}^{n+1}\right)}{\gamma^2}+\frac{a_{k^*}^{n+1/2}}{2}\frac{\Psi_{k^*+1}^{n+1}-\Psi_{k^*-1}^{n+1}}{2h}+\frac{R_{k*}^{n+1/2}}{2}\Psi_{k*}^{n+1}\\ &<0,\end{aligned}$$

which contradicts the given hypothesis $\pounds_{\varepsilon,m}^{N,M}\Psi^{n+1}(x_m) \geq 0$ and our supposition $\Psi_{k^*}^{n+1} < 0$. For $k^* = 0, 1, ..., N$, which gives $\Psi_{k^*}^{n+1} \geq 0$, and hence $\Psi^{n+1}(x_m) \geq 0$, for all $m = 0, 1, ..., N$.

Lemma 4.6 (Uniform stability estimate) The solution U_m^{n+1} of the discrete scheme in (4.9) satisfy the bound

$$\left|U_m^{n+1}\right| \leq \frac{\max\left|\pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\right|}{r^*} + \max\left\{\left|\phi_l(t_{n+1})\right|, \left|\phi_r(t_{n+1})\right|\right\},$$

Proof We define two barrier functions as

$$\wp_{m,n+1}^{\pm} = \frac{\max\left|\pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\right|}{r^*} + \max\left\{\left|\phi_l(t_{n+1})\right|, \left|\phi_r(t_{n+1})\right|\right\} \pm U_m^{n+1}.$$

Then on the boundary points, we obtain

$$\wp_{0,n+1}^{\pm} = \frac{\max\left|\pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\right|}{r^*} + \max\left\{\left|\phi_l(t_{n+1})\right|, \left|\phi_r(t_{n+1})\right|\right\} \pm U_0^{n+1} = \max\left|\pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\right| \geq 0,$$
$$\wp_{1,n+1}^{\pm} = \frac{\max\left|\pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\right|}{r^*} + \max\left\{\left|\phi_l(t_{n+1})\right|, \left|\phi_r(t_{n+1})\right|\right\} \pm U_N^{n+1} = \max\left|\pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\right| \geq 0.$$

Now for $0 < m < N$, we have

$$\begin{aligned}\pounds_{\varepsilon,m}^{N,M}\wp_{m,n+1}^{\pm} &= -\frac{\varepsilon}{2}\frac{\delta_x^2\wp_{m,n+1}^{\pm}}{\gamma^2} + \frac{a_m^{n+1/2}}{2}D_x^0\wp_{m,n+1}^{\pm} + \frac{R_m^{n+1/2}}{2}\wp_{m,n+1}^{\pm},\\ &= \frac{R_m^{n+1/2}}{2}\left(\frac{\max\left|\pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\right|}{r^*} + \max\left\{\left|\phi_l(t_{n+1})\right|, \left|\phi_r(t_{n+1})\right|\right\}\right) \pm \pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\\ &= \frac{R_m^{n+1/2}}{2}\left(\frac{\max\left|\pounds_{\varepsilon,m}^{N,M}U_m^{n+1}\right|}{r^*} + \max\left\{\left|\phi_l(t_{n+1})\right|, \left|\phi_r(t_{n+1})\right|\right\}\right) \pm H_m^{n+1}\\ &\geq 0, \quad \text{since} \quad R_m^{n+1/2} \geq r^*.\end{aligned}$$

Using discrete maximum principle given in Lemma 4.5 yields $\wp_{m,n+1}^{\pm} \geq 0, m = 0, 1, 2, ..., N$.

Lemma 4.7 For all $k \in Z^+$ on a fixed number of mesh numbers N, and $\varepsilon \longrightarrow 0$, we have

$$\lim_{\varepsilon\longrightarrow 0}\max_{1\leq m\leq N-1}\frac{exp\left(-\alpha x_m/\varepsilon\right)}{\varepsilon^k} = 0 \quad \text{and}$$
$$\lim_{\varepsilon\longrightarrow 0}\max_{1\leq m\leq N-1}\frac{exp\left(-\alpha\left(1-x_m\right)/\varepsilon\right)}{\varepsilon^k} = 0$$

where $x_m = mh, \forall m = 1, 2, ...N-1$.

Proof The proof is given in [10].

5 | CONVERGENCE ANALYSIS OF THE METHOD

Next, we consider the semidiscrete problem in Equation (4.2) and the discrete scheme in (4.9) to find the truncation error of the spatial direction discretization.

Theorem 5.1 (Error estimate in the spatial direction) Let $U^{n+1}(x_m)$ be the solution of continuous solution (4.2) after temporal discretization and U_m^{n+1} be the approximate solutions of (4.9) after the full discretization. Then, the numerical solution U_m^{n+1} of the problem in (4.9) satisfies the error bound

$$\left|\pounds_{\varepsilon,m}^{N,M}\left(U^{n+1}(x_m)-U_m^{n+1}\right)\right| \le Ch^2$$

Proof Consider the error bound in the spatial direction

$$\begin{aligned}
&\left|\pounds_{\varepsilon,m}^{N,M}\left(U^{n+1}(x_m)-U_m^{n+1}\right)\right| = \\
&\left|-\frac{\varepsilon}{2}\left(U_{xx}\right)^{n+1}(x_m)+\frac{a^{n+1/2}(x_m)}{2}\left(U_x\right)^{n+1}(x_m)-\left\{-\frac{\varepsilon}{2}\frac{\delta_x^2 U_m^{n+1}}{\gamma^2}+\frac{a_m^{n+1/2}}{2}D_x^0 U_m^{n+1}\right\}\right| \\
&\left|-\frac{\varepsilon}{2}\left(\left(U_{xx}\right)^{n+1}(x_m)-\frac{\delta_x^2 U_m^{n+1}}{\gamma^2}\right)+\frac{a^{n+1/2}(x_m)}{2}\left(\left(U_x\right)^{n+1}(x_m)-D_x^0 U_m^{n+1}\right)\right| \\
&=\left|-\frac{\varepsilon}{2}\left(1-\frac{h^2}{\gamma^2}\right)\left(U_{xx}\right)^{n+1}(x_m)-\frac{h^2}{12}a^{n+1/2}(x_m)\left(U_{xxx}\right)^{n+1}(x_m)-\frac{\varepsilon h^4}{24\gamma^2}\left(U_{xxxx}\right)^{n+1}(x_m)\right| \\
&\le h^2\left(\left|-\frac{1}{12}a^{n+1/2}(x_m)\left(U_{xxx}\right)^{n+1}(x_m)\right|+\left|-\frac{\varepsilon h^2}{24\gamma^2}\left(U_{xxxx}\right)^{n+1}(x_m)\right|\right) \\
&\le Ch^2,\ \text{where } \gamma^2=\frac{2h\varepsilon}{a_m^n}tanh\left(\frac{a_m^n h}{2\varepsilon}\right)\approx h^2-\frac{h^4}{24}\frac{\alpha^2}{\varepsilon^2}
\end{aligned}$$

Applying the bound given in Lemma (3.5) and Lemma (4.7) gives

$$\left|\pounds_{\varepsilon,m}^{N,M}\left(U^{n+1}(x_m)-U_m^{n+1}\right)\right| \le Ch^2 = CN^2$$

Theorem 5.2 (Error estimate in the fully discrete scheme) Let u be the solution of the problem (2.1)-(2.4) and U be the numerical solution of (4.9). For the fully discrete scheme, the following parameter uniform error estimate holds:

$$\sup_{0\le\varepsilon<1}|u-U| \le C\left(h^2+(\Delta t)^2\right).$$

Proof Immediate result follows from the combination of temporal error bound (Lemma 4.3) and spatial error bound (Theorem 5.1).

6 | NUMERICAL RESULTS

In this section, we present test examples with left and right end boundary layers obtained by the proposed method to support the theoretical discussion. We utilize the double mesh technique to calculate the maximum point wise error and rate of convergence because the exact solution to the issues is unknown. The maximum pointwise errors $E_\varepsilon^{N,\Delta t}$ and the corresponding order of convergence $p_\varepsilon^{N,\Delta t}$ are computed as

$$E_\varepsilon^{N,\Delta t} = \max_{m,n}\left|U_{m,n}^{N,\Delta t} - U_{m,n}^{4N,\frac{\Delta t}{2}}\right|, \qquad p_\varepsilon^{N,\Delta t} = log_2\left(\frac{E_\varepsilon^{N,\Delta t}}{E_\varepsilon^{4N,\frac{\Delta t}{2}}}\right)$$

and from these values we obtain the ε-uniform error $E^{N,\Delta t}$ and the corresponding ε-uniform order of convergence $p^{N,\Delta t}$ by:

$$E^{N,\Delta t} = \max_\varepsilon E_\varepsilon^{N,\Delta t} \text{ and } p^{N,\Delta t} = log_2\left(\frac{E^{N,\Delta t}}{E^{4N,\frac{\Delta t}{2}}}\right).$$

where $U_{m,n}^{N,\Delta t}$ is the numerical solutions obtained by using N, M mesh intervals in space and time direction, respectively. To compute $U_{m,n}^{4N,\frac{\Delta t}{2}}$ we use $4N$ and $2M$ mesh intervals in spatial and temporal direction, respectively.

Example 6.1 Consider

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{(5-x^2)}{3}\frac{\partial u}{\partial x} + tu(x,t) = -u(x,t-\tau) + t^3 x\left(1-x\right) sin\left(\pi x\right), (x,t) \in (0,1)\times(0,2],$$

with

$$\begin{cases} u(0,t) = 0, u(1,t) = 0, t \in (0,2]\,, \\ u(x,t) = 0, (x,t) \in [0,1]\times[-\tau,0]\,. \end{cases}$$

Example 6.2 Consider

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon\dfrac{\partial^2 u}{\partial x^2} + (2-x^2)\dfrac{\partial u}{\partial x} + (x+1)(t+1)u(x,t) = -u(x,t-\tau) \\ \qquad\qquad\qquad + 10t^2 exp(-t)x(1-x), (x,t) \in (0,1)\times(0,2], \end{cases}$$

with

$$\begin{cases} u(0,t) = 0, u(1,t) = 0, t \in (0,2]\,, \\ u(x,t) = 0, (x,t) \in [0,1]\times[-\tau,0]\,. \end{cases}$$

We have illustrated the maximum point wise errors $E_\varepsilon^{N,\Delta t}$ and the corresponding numerical rates of convergence $p_\varepsilon^{N,\Delta t}$ calculated by numerical scheme (4.9) for Example 6.1 and Example 6.2 in Table 1 and Table 3, respectively. The numerical results presented in Table 1 and Table 3 shows the fact that the proposed numerical method is accurate of order $O\left(h^2 + (\Delta t)^2\right)$ as predicted by the theory. From the tables Table 1,Table 2,Table 3 and Table 4 one can clearly observe the ε-uniform convergence of the proposed scheme (4.9). Figures 1 and 2 clearly indicate that the boundary layer is located at the right side of the rectangular domain. The two figures (Figure 1 and Figure 2) shows the effect of perturbation parameter and that of

TABLE 1 Maximum pointwise errors ($E_\varepsilon^{N,\Delta t}$) and the corresponding rate of convergence ($p_\varepsilon^{N,\Delta t}$) of the scheme (4.9) for Example 6.1.

$\varepsilon\downarrow$	N=8	N=32	N=128	N=512	N=2048
	M=32	M=64	M=128	M=256	M=512
2^{-10}	1.6006e-02	4.5339e-03	1.0927e-03	1.8560e-04	4.4588e-05
	1.8198	2.0529	2.5576	2.0575	–
2^{-12}	1.6006e-02	4.5339e-03	1.1003e-03	2.6609e-04	5.1185e-05
	1.8198	2.0429	2.0479	2.3781	–
2^{-14}	1.6006e-02	4.5339e-03	1.1003e-03	2.6678e-04	7.1020e-05
	1.8198	2.0429	2.0442	1.9094	–
2^{-16}	1.6006e-02	4.5339e-03	1.1003e-03	2.6678e-04	7.1192e-05
	1.8198	2.0429	2.0442	1.9059	–
2^{-18}	1.6006e-02	4.5339e-03	1.1003e-03	2.6678e-04	7.1192e-05
	1.8198	2.0429	2.0442	1.9059	–
2^{-20}	1.6006e-02	4.5339e-03	1.1003e-03	2.6678e-04	7.1192e-05
	1.8198	2.0429	2.0442	1.9059	–
2^{-22}	1.6006e-02	4.5339e-03	1.1003e-03	2.6678e-04	7.1192e-05
	1.8198	2.0429	2.0442	1.9059	–
2^{-30}	1.6006e-02	4.5339e-03	1.1003e-03	2.6678e-04	7.1192e-05
	1.8198	2.0429	2.0442	1.9059	–
$E^{N,\Delta t}$	1.6006e-02	4.5339e-03	1.1003e-03	2.6678e-04	7.1192e-05
$p^{N,\Delta t}$	1.8198	2.0429	2.0442	1.9059	-

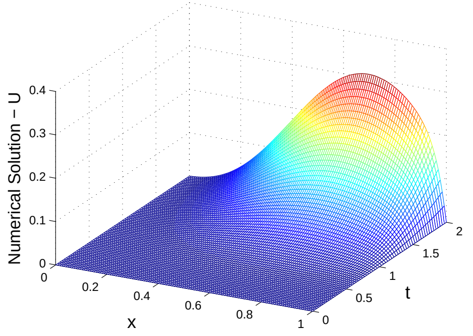
retarded argument on the steepness of layer of the solution. In order to reveal the numerical order of convergence, we have plotted the maximum pointwise errors of Example 6.1 and Example 6.2 in Figure 3 (a) and Figure 3(b), respectively in the log-log scale for which again confirms the effectiveness of the proposed method and also it gives close to second-order. Moreover, we note that in this paper computations associated with the examples discussed above were performed using MATLAB R2013A software package.

7 | CONCLUSION

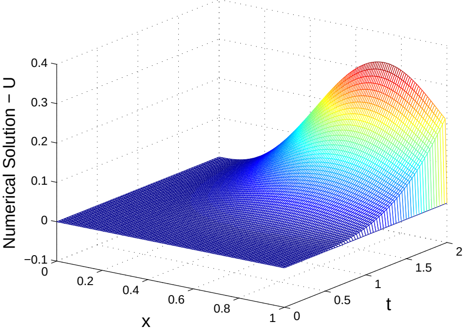
In this paper, singularly perturbed parabolic convection-diffusion problem with large retarded argument is considered. Because of the perturbation parameter and the retarded argument, the solution of the investigated problem exhibits boundary layer behavior on the right side of the spatial domain, which is dependent on the sign of the convection term coefficients. To stabilize the breakdown solution of the problem in the boundary regions we employ the nonstandard finite difference method for the space discretization and the Crank-Nicolson

TABLE 2 Maximum pointwise errors ($E_\varepsilon^{N,\Delta t}$) of the scheme (4.9) for Example 6.1.

$\varepsilon\downarrow$	Number of mesh intervals N=M					
	16	32	64	128	256	512
1	4.4487e-04	1.8416e-04	8.0850e-05	3.7441e-05	1.7955e-05	8.7830e-06
2^{-2}	1.1336e-03	5.0859e-04	2.4745e-04	1.2449e-04	6.2818e-05	3.1601e-05
2^{-4}	2.6338e-03	8.5968e-04	2.9989e-04	1.4663e-04	8.5631e-05	4.6177e-05
2^{-6}	6.1203e-03	2.5280e-03	8.0020e-04	2.3234e-04	7.2644e-05	4.1308e-05
2^{-8}	6.9800e-03	3.9397e-03	1.8667e-03	6.9905e-04	2.1082e-04	5.9430e-05
2^{-10}	6.9802e-03	3.9669e-03	2.1290e-03	1.0927e-03	4.9067e-04	1.7922e-04
2^{-12}	6.9802e-03	3.9669e-03	2.1290e-03	1.1003e-03	5.5950e-04	2.8014e-04
2^{-14}	6.9802e-03	3.9669e-03	2.1290e-03	1.1003e-03	5.5951e-04	2.8210e-04
2^{-16}	6.9802e-03	3.9669e-03	2.1290e-03	1.1003e-03	5.5951e-04	2.8210e-04
2^{-20}	6.9802e-03	3.9669e-03	2.1290e-03	1.1003e-03	5.5951e-04	2.8210e-04
2^{-24}	6.9802e-03	3.9669e-03	2.1290e-03	1.1003e-03	5.5951e-04	2.8210e-04
2^{-28}	6.9802e-03	3.9669e-03	2.1290e-03	1.1003e-03	5.5951e-04	2.8210e-04
2^{-30}	6.9802e-03	3.9669e-03	2.1290e-03	1.1003e-03	5.5951e-04	2.8210e-04
$E^{N,\Delta t}$	6.9802e-03	3.9669e-03	2.1290e-03	1.1003e-03	5.5951e-04	2.8210e-04
$p^{N,\Delta t}$	0.81526	0.89784	0.95228	0.97566	9.8796	-



(a)



(b)

FIGURE 1 Surface plot of the numerical solution for Example 6.1 with $N = 128, M = 64$, a $\varepsilon = 2^{-4}$,b $\varepsilon = 2^{-18}$.

method for the time discretization both on a uniform mesh. Thus, the proposed nonstandard finite difference method converges properly, is unconditionally stable and is robust with respect to the singular perturbation

TABLE 3 Maximum pointwise errors $E_{\varepsilon}^{N,\Delta t}$ and the corresponding rate of convergence $p_{\varepsilon}^{N,\Delta t}$ of the scheme (4.9) for Example 6.2.

$\varepsilon\downarrow$	N=8 M=32	N=32 M=64	N=128 M=128	N=512 M=256	N=2048 M=512
2^{-10}	8.3727e-03	2.5378e-03	5.7067e-04	1.1055e-04	3.8159e-05
	1.7221	2.1528	2.3680	1.5346	–
2^{-12}	8.3727e-03	2.5378e-03	5.8446e-04	1.4370e-04	2.9594e-05
	1.7221	2.1184	2.0240	2.2797	–
2^{-14}	8.3727e-03	2.5378e-03	5.8446e-04	1.4379e-04	3.7693e-05
	1.7221	2.1184	2.0231	1.9316	–
2^{-16}	8.3727e-03	2.5378e-03	5.8446e-04	1.4379e-04	3.7717e-05
	1.7221	2.1184	2.0231	1.9307	–
2^{-18}	8.3727e-03	2.5378e-03	5.8446e-04	1.4379e-04	3.7717e-05
	1.7221	2.1184	2.0231	1.9307	–
2^{-20}	8.3727e-03	2.5378e-03	5.8446e-04	1.4379e-04	3.7717e-05
	1.7221	2.1184	2.0231	1.9307	–
2^{-22}	8.3727e-03	2.5378e-03	5.8446e-04	1.4379e-04	3.7717e-05
	1.7221	2.1184	2.0231	1.9307	–
2^{-24}	8.3727e-03	2.5378e-03	5.8446e-04	1.4379e-04	3.7717e-05
	1.7221	2.1184	2.0231	1.9307	–
2^{-30}	8.3727e-03	2.5378e-03	5.8446e-04	1.4379e-04	3.7717e-05
	1.7221	2.1184	2.0231	1.9307	–
$E^{N,\Delta t}$	8.3727e-03	2.5378e-03	5.8446e-04	1.4379e-04	3.7717e-05
$p^{N,\Delta t}$	1.7221	2.1184	2.0231	1.9307	-

parameter (see Table 5). Analytically we have proved that the nonstandard finite difference method provides second order ε-uniform convergent results, two numerical experiments are carried out to validate the analytical findings.

8 | ACKNOWLEDGMENTS

The authors wish to appreciate the anonymous referees for their constructive suggestions.

TABLE 4 Maximum pointwise errors ($E_\varepsilon^{N,\Delta t}$) of the scheme (4.9) for Example 6.2.

$\varepsilon\downarrow$	N=16 M=20	N=32 M=40	N=64 M=80	N=128 M=160	N=256 M=320	N=512 M=640
1	3.8040e-04	1.4725e-04	6.2939e-05	2.8834e-05	1.3764e-05	6.7191e-06
2^{-2}	3.8040e-04	3.6866e-04	1.7495e-04	8.5493e-05	4.2296e-05	2.1041e-05
2^{-4}	1.9081e-03	5.5763e-04	1.8604e-04	1.0896e-04	5.8832e-05	3.0523e-05
2^{-6}	3.7751e-03	1.6115e-03	5.1191e-04	1.4229e-04	4.4252e-05	2.8661e-05
2^{-8}	4.2718e-03	2.2877e-03	1.0477e-03	4.2282e-04	1.3082e-04	3.5920e-05
2^{-10}	4.2723e-03	2.3338e-03	1.2139e-03	6.0468e-04	2.6934e-04	1.0703e-04
2^{-12}	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1196e-04	1.5320e-04
2^{-14}	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1204e-04	1.5671e-04
2^{-16}	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1204e-04	1.5671e-04
2^{-20}	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1204e-04	1.5671e-04
2^{-24}	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1204e-04	1.5671e-04
2^{-28}	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1204e-04	1.5671e-04
2^{-30}	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1204e-04	1.5671e-04
$E^{N,\Delta t}$	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1204e-04	1.5671e-04
$p^{N,\Delta t}$	0.87233	0.94279	0.97311	0.98697	0.99363	-

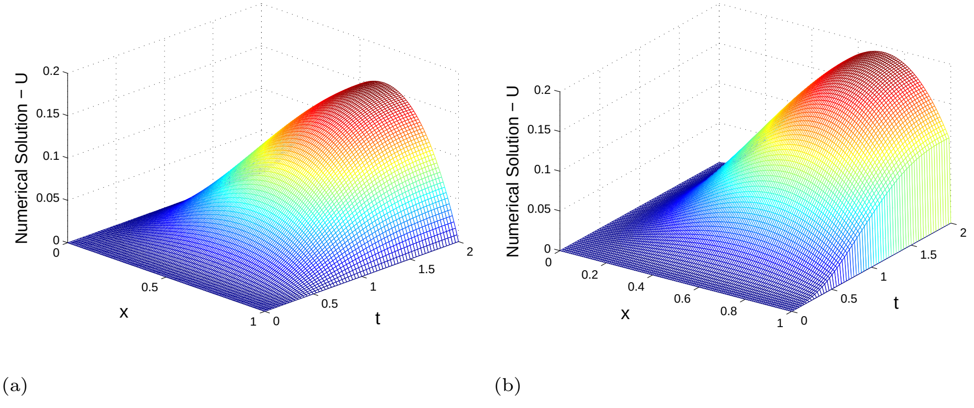
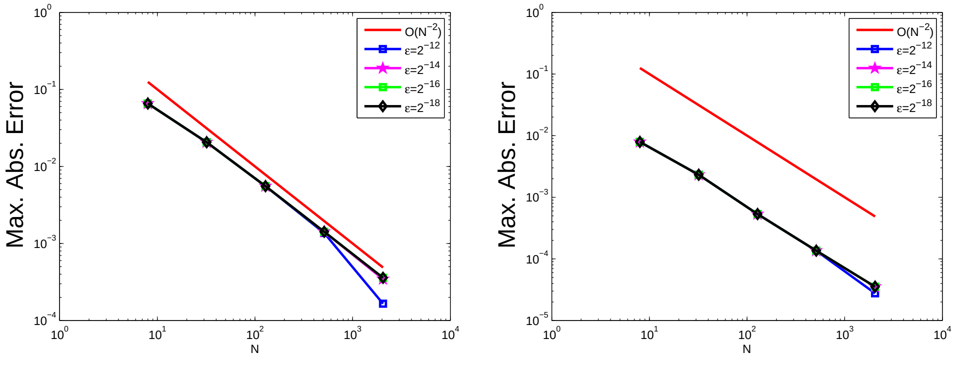


FIGURE 2 Surface plot of the numerical solution for Example 6.2 with $N = 120, M = 64$, a $\varepsilon = 2^{-4}$,b $\varepsilon = 2^{-18}$.

TABLE 5 Comparison of uniform error ($E^{N,\Delta t}$) and the corresponding uniform rate of convergence ($p^{N,\Delta t}$) for Example 6.2.

Methods		N=16 M=20	N=32 M=40	N=64 M=80	N=128 M=160	N=256 M=320
Proposed method	$E^{N,\Delta t}$	4.2723e-03	2.3338e-03	1.2141e-03	6.1847e-04	3.1204e-04
	$p^{N,\Delta t}$	0.87233	0.94279	0.97311	0.98697	-
Method in [24]	$E^{N,\Delta t}$	1.4212e-02	7.8114e-03	4.1163e-03	2.1158e-03	1.0729e-03
	$p^{N,\Delta t}$	0.8635	0.9242	0.9601	0.9797	-
Method in [9]	$E^{N,\Delta t}$	1.6119e-02	9.9504e-03	5.8541e-03	3.3439e-03	1.8650e-03
	$p^{N,\Delta t}$	0.6960	0.7653	0.8079	0.8424	-
		N=M=16	N=32	N=64	N=128	N=256
Proposed method	$E^{N,\Delta t}$	3.9940e-03	2.1965e-03	1.1459e-03	5.8446e-04	2.9506e-04
	$p^{N,\Delta t}$	0.86263	0.93873	0.97130	0.98610	-
Method in [16]	$E^{N,\Delta t}$	3.06e-02	1.72e-02	9.00e-03	4.58E-03	2.30e-03
	$p^{N,\Delta t}$	0.8311	0.9344	0.9746	0.9937	-



(a) (b)

FIGURE 3 Log-Log plot of the maximum error for Example 6.1 on left ([a]) and Example 6.2 on right([b]).

9 | CONFLICT OF INTEREST

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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