The wave speed for a time-periodic bistable 3-species lattice competition system

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Abstract

In this paper, we consider propagation direction, which can be used to predict which species will occupy the habitat or win the competition eventually, of bistable wave for a 3-species time-periodic lattice competition system with bistable nonlinearity, aiming to address an open problem proposed in [J.-S. Guo et al, The sign of traveling wave speed in bistable dynamics, *Discret. Contin. Dyn. Syst.*, 40 (2020), 3451]. As a first step, by transforming the competition system to a cooperative one, we study the asymptotic behavior for the bistable wave profile and then prove the uniqueness of the bistable wave speed. Secondly, we utilize comparison principle and build up two couples of upper and lower solutions to judge the sign of the bistable wave speed which provides partially the answer to the open problem. As an application, we reduce the time-periodic system to a space-time homogeneous system, we obtain the corresponding criteria and carry out numerical simulations to illustrate the availability of our results. Moreover, an interesting phenomenon we found is that the two weak competitors can wipe out the strong competitor under some circumstances.

Keywords and Phrases: Propagation direction, bistable wave, lattice system
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1 Introduction

This paper is devoted to the propagation direction, which is determined by the sign of wave speed, of traveling wave solutions (TWS) for the following bistable lattice system

$$\begin{cases} u_j'(t) = d_1(t)\mathcal{D}_2[u_j](t) + u_j(t)(r_1(t) - a_{11}(t)u_j(t) - a_{12}(t)v_j(t)), \\ v_j'(t) = d_2(t)\mathcal{D}_2[v_j](t) + v_j(t)(r_2(t) - b_{11}(t)v_j(t) - b_{12}(t)u_j(t) - b_{13}(t)w_j(t)), \\ w_j'(t) = d_3(t)\mathcal{D}_2[w_j](t) + w_j(t)(r_3(t) - c_{11}(t)w_j(t) - c_{12}(t)v_j(t)),\ j \in \mathbb{Z}, t > 0. \end{cases} \tag{1.1}$$

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In model (1.1) and in the sense of biology, one can interpret $u_j(t), v_j(t)$ and $w_j(t)$ as the population densities of three species at position j and time t respectively, $d_i(t)$ as the diffusivity coefficient and $r_i(t)$ as the growth rate of the species. Here, the coefficients $a_{1i}(t), c_{1i}(t), i = 1, 2$ and $d_k(t), b_{1k}(t), k = 1, 2, 3$ are assumed to be positive T-periodic functions with T being a positive number. Biologically speaking, $a_{1i}(t), b_{1k}(t), c_{1i}(t)$ are the intra-specific competitive coefficients as $i = k = 1$, while $i = 2$ or $k = 2, 3$, they represent the inter-specific competitive coefficients. The term $\mathcal{D}_2[s_j](t)$ appeared in (1.1) is the second order central difference and is defined as $\mathcal{D}_2[s_j](t) := s(t, j+1) + s(t, j-1) - 2s(t, j)$ for $s = u, v, w$. Obviously, system (1.1) is a competitive system, it models such a relationship between three species: v competes with u and w for common resources, while there is no competition between u and w. The biological interpretation is that species u and w have different preferences for food resources, while species v has the same food preferences as u and w.

As we all know, nature is a constantly changing and relatively stable system, in which competition for survival between species is a common phenomenon in nature. Therefore, to study the dynamic behavior between different species, it is necessary to study the phenomenon of competition between species and establish a reasonable model. Lotka-Volterra competitive diffusion system is one of the classical biological models to describe inter- and intra-specific interactions. When the environment is assumed to be homogenous, the general form of 3-species Lotka-Volterra competition diffusion model in the above biological context is as follows:

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - a_1 v), \\ v_t = d_2 v_{xx} + r_2 v(1 - v - a_2 u - a_3 w), \\ w_t = d_3 w_{xx} + r_3 w(1 - w - a_4 v), t \in \mathbb{R}^+, x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $d_k, r_k, k = 1, 2, 3$ and $a_l, l = 1, 2, 3, 4$ are positive constants. As a matter of fact, system (1.2) can be regarded as an extension of the classic 2-species Lotka-Volterra system which has been studied extensively in past decades, see for examples [1,12,21–23,29,32] and more references therein. Due to the benefit from the classic Lotka-Volterra system in application of ecology, more and more works also have been devoted to the system (1.2). For instance, we refer the readers to [14, 26] for the selection mechanism of minimum wave speed in the monostable model; [4] for the stability of monotone traveling wave solutions; [5] for the exact traveling wave solutions of (1.2) with nontrivial three components; [25] for the uniqueness of traveling wavefronts; [13, 33] for the sign of wave speed in the bistable model. Related to the present paper, we particularly mention that Guo et al [13] studied two different cases for system (1.2): (1) the case where two species are weakly competitive and one species is strongly competitive, (2) the case where all three species are very strong competitors. They obtained some new observations in contrast with the 2-species Lotka-Volterra model. In addition to system (1.2), we further refer the readers to [11,15,17,28,31] for discrete three-species competition system; [8] and [19] for three-component competition system with nonlocal dispersal; [18, 24] for competitive-cooperative Lotka-Volterra system of three species.

In their recent paper, besides the model (1.2), Guo et al [13] also proposed a discrete version of (1.2) as below

$$\begin{cases} u_j'(t) = d_1\mathcal{D}_2[u_j](t) + r_1[u_j(1-u_j-b_2v_j)](t), \\ v_j'(t) = d_2\mathcal{D}_2[v_j](t) + r_2[v_j(1-b_1u_j-v_j-b_3w_j)](t), \\ w_j'(t) = d_3\mathcal{D}_2[w_j](t) + r_3[w_j(1-b_2v_j-w_j)](t), t\in\mathbb{R}^+, j\in\mathbb{Z}, \end{cases} \tag{1.3}$$

in which, the parameters d_k, r_k and $b_k, k = 1, 2, 3$ are positive numbers and can be interpreted as the ones in system (1.2). In (1.3), although the sign of wave speed of (1.2) has been addressed for certain special cases, it is still largely left open for the discrete case (1.3). One of the reasons is that their method used on system (1.2) relies on the integration of the corresponding wave profile system, so it seems that such a method can't be applied to system (1.3) directly due to the central difference involved in (1.3). Another might be that the combination of patchy environments and periodicity can make the corresponding analysis more difficult. In this paper, we try to make some progress in this direction and this is our main motivation. Our strategy is to use the upper-lower solution method to investigate the sign of the bistable wave speed of (1.1). As a matter of fact, this method has been proved to be valid in this subject for several diffusion systems, see for instances [22, 26, 29].

In recent years, an increasing number of scholars are attracted to traveling wave solutions that have advantages in describing the development, migration and invasion of biological populations. In particular, the sign of wave speed of traveling wave solution can be used to explain the outcome of competition between different species which makes it a meaningful topic. In this paper, we will study the propagation direction of traveling wave solutions for (1.1) which is a lattice competition system. To the best of our knowledge, the research of lattice dynamical systems which is more in line with nature originated from Bunimovich and Sinai [3] in 1988. After that, lattice dynamical models are widely used in biological issues, see for examples [7,10,16,27,29,30]. Generally speaking, it is more effective in case of the species live in patchy environments.

Obviously, the corresponding space-homogenous ordinary differential system of (1.1) is as follows,

$$\begin{cases} u'(t) = u(t)[r_1(t) - a_{11}(t)u(t) - a_{12}(t)v(t)], \\ v'(t) = v(t)[r_2(t) - b_{11}(t)v(t) - b_{12}(t)u(t) - b_{13}(t)w(t)], \\ w'(t) = w(t)[r_3(t) - c_{11}(t)w(t) - c_{12}(t)v(t)], t\in\mathbb{R}^+. \end{cases} \tag{1.4}$$

It is easy to see that system (1.4) at least has three nonnegative T-periodic solutions, which are the equilibrium points of (1.1). We denote them by $e_0 := (0,0,0), e_1 := (0, q(t), 0), e_2 := (p(t), 0, r(t))$ respectively, in which $p(t), q(t), r(t)$ can be expressed as

$$p(t) = \frac{p_0 e^{\int_0^t r_1(s)ds}}{p_0\int_0^t a_{11}(s)e^{\int_0^s r_1(\theta)d\theta}ds + 1}, p_0 = \frac{e^{\int_0^T r_1(s)ds} - 1}{\int_0^T a_{11}(s)e^{\int_0^s r_1(\theta)d\theta}ds},$$

$$q(t) = \frac{q_0 e^{\int_0^t r_2(s)ds}}{q_0\int_0^t b_{11}(s)e^{\int_0^s r_2(\theta)d\theta}ds + 1}, \ q_0 = \frac{e^{\int_0^T r_2(s)ds} - 1}{\int_0^T b_{11}(s)e^{\int_0^s r_2(\theta)d\theta}ds},$$

$$r(t)=\frac{r_0e^{\int_0^t r_3(s)ds}}{r_0\int_0^t c_{11}(s)e^{\int_0^s r_3(\theta)d\theta}ds+1}, r_0=\frac{e^{\int_0^T r_3(s)ds}-1}{\int_0^T c_{11}(s)e^{\int_0^s r_3(\theta)d\theta}ds}.$$

It is direct to check that $p(t), q(t)$ and $r(t)$ are T-periodic functions and satisfy $p(t+T)=p(t), q(t+T)=q(t)$ and $r(t+T)=r(t)$ for all $t\in\mathbb{R}^+$.

Since our main focus is on bistable waves of (1.1), we have to make the following assumption throughout this paper:

(A) $\int_0^T r_1(t)dt<\int_0^T a_{12}(t)q(t)dt$, $\int_0^T r_2(t)dt<\int_0^T b_{12}(t)p(t)+b_{13}(t)r(t)dt$ and $\int_0^T r_3(t)dt<\int_0^T c_{12}(t)q(t)dt$,

so that, e_1 and e_2 are linearly stable equilibrium points.

As mentioned above, we are concerned with periodic traveling wave of system (1.1), which bears the form of

$$\begin{pmatrix} u_j(t)\\ v_j(t)\\ w_j(t)\end{pmatrix}=\begin{pmatrix} U(t,j+ct)\\ V(t,j+ct)\\ W(t,j+ct)\end{pmatrix}=:\begin{pmatrix} U(t,z)\\ V(t,z)\\ W(t,z)\end{pmatrix},\ z=j+ct, \tag{1.5}$$

satisfying

$$\begin{pmatrix} U(t+T,z)\\ V(t+T,z)\\ W(t+T,z)\end{pmatrix}=\begin{pmatrix} U(t,z)\\ V(t,z)\\ W(t,z)\end{pmatrix},$$

and subjects to the boundary conditions

$$(U,V,W)(t,-\infty)=(0,0,0),(U,V,W)(t,+\infty)=(1,1,1), \tag{1.6}$$

where c is called as the wave speed, (U,V,W) are called as the wave profile. The limits in (1.6) holds uniformly in $t\in\mathbb{R}^+$.

After a substitution of (1.5), (1.1) can be rewritten as a wave profile system

$$\begin{cases} U_t+cU_z=d_1(t)\mathcal{D}_2[U](t,z)+U(r_1(t)-a_{11}(t)U-a_{12}(t)V),\\ V_t+cV_z=d_2(t)\mathcal{D}_2[V](t,z)+V(r_2(t)-b_{11}(t)V-b_{12}(t)U-b_{13}(t)W),\\ W_t+cW_z=d_3(t)\mathcal{D}_2[W](t,z)+W(r_3(t)-c_{11}(t)W-c_{12}(t)V),\end{cases} \tag{1.7}$$

where $\mathcal{D}_2[S](t,z)=S(t,z+1)+S(t,z-1)-2S(t,z)$ for $S=U,V,W$. Via the following changes

$$\Phi(t,z)=\frac{p(t)-U(t,z)}{p(t)},\Psi(t,z)=\frac{V(t,z)}{q(t)},\Theta(t,z)=\frac{r(t)-W(t,z)}{r(t)},$$

system (1.7) can be converted into a cooperative system

$$
\begin{cases}
d_1(t)\mathcal{D}_2[\Phi](t,z) - c\Phi_z + (1-\Phi)[a_{12}(t)q(t)\Psi - a_{11}(t)p(t)\Phi] = \Phi_t, \\
d_2(t)\mathcal{D}_2[\Psi](t,z) - c\Psi_z + \Psi[b_{11}(t)q(t)(1-\Psi) - b_{12}(t)p(t)(1-\Phi) - b_{13}(t)r(t)(1-\Theta)] = \Psi_t, \\
d_3(t)\mathcal{D}_2[\Theta](t,z) - c\Theta_z + (1-\Theta)[c_{12}(t)q(t)\Psi - c_{11}(t)r(t)\Theta] = \Theta_t,
\end{cases}
\tag{1.8}
$$

with periodic conditions and boundary conditions (1.6) become

$$
\begin{cases}
(\Phi,\Psi,\Theta)(t,z) = (\Phi,\Psi,\Theta)(t+T,z), \\
(\Phi,\Psi,\Theta)(t,-\infty) = (0,0,0), (\Phi,\Psi,\Theta)(t,+\infty) = (1,1,1).
\end{cases}
$$

For the sake of convenience, we shall call the first equation of (1.8) as Φ-equation, the second equation as Ψ-equation and the last one as Θ-equation throughout this paper. Note that the existence of bistable periodic traveling wave solution of (1.1) can be proved by following the ideas in [6, 15], or by the abstract theory established in [9].

The remainder of this paper is organized as follows. In Sect.2, we investigate the asymptotic behaviors of $\Phi(t,z), \Psi(t,z)$ and $\Theta(t,z)$ as the co-moving coordinate z tends to infinity, upon which the uniqueness of bistable wave speed is considered. In Sect.3, we derive two crucial theorems concerning the determination of the sign of the bistable wave speed by employing comparison principle. We will construct suitable upper/lower solutions to obtain explicit conditions in Sect.4 and the results of numerical simulation are shown in Sect.5.

2 Uniqueness of bistable wave-speed

To facilitate the forthcoming calculation and statement, we define some mathematical notations as follows:

$$
\begin{aligned}
&\overline{f(t)} := \frac{1}{T}\int_0^T f(t)dt, \\
&\Delta_1(t) := b_{11}(t)q(t) - b_{12}(t)p(t) - b_{13}(t)r(t), \\
&\Delta_2(t) := a_{11}(t)p(t) - a_{12}(t)q(t), \\
&\Delta_3(t) := c_{11}(t)r(t) - c_{12}(t)q(t), \\
&\Gamma_1(t,\mu) := d_1(t)(e^{\mu} + e^{-\mu} - 2) - c\mu - a_{11}(t)p(t), \\
&\Gamma_2(t,\mu) := d_3(t)(e^{\mu} + e^{-\mu} - 2) - c\mu - c_{11}(t)r(t), \\
&\Gamma_3(t,\mu) := d_2(t)(e^{\mu} + e^{-\mu} - 2) + c\mu - b_{11}(t)q(t).
\end{aligned}
$$

To investigate the asymptotic behavior of the bistable wave profile, we denote the unique positive solutions of the following equations

$$
\begin{aligned}
&\overline{d_2(t)}(e^{\mu} + e^{-\mu} - 2) - c\mu + \overline{\Delta_1(t)} = 0, \\
&\overline{d_1(t)}(e^{\mu} + e^{-\mu} - 2) - c\mu - \overline{a_{11}(t)p(t)} = 0, \\
&\overline{d_3(t)}(e^{\mu} + e^{-\mu} - 2) - c\mu - \overline{c_{11}(t)r(t)} = 0,
\end{aligned}
$$

by $\mu_1(c), \mu_2(c), \mu_3(c)$ respectively. Moreover, by a simple analysis it is not hard to find $\mu_1(c)$, $\mu_2(c)$, $\mu_3(c)$ are increasing functions in c. And $\mu_4(c)$, $\mu_5(c)$, $\mu_6(c)$ respectively express the unique positive roots of the following equations in turn:

$$\begin{aligned}
&\overline{d_1(t)}(e^{\mu}+e^{-\mu}-2)+c\mu+\overline{\Delta_2(t)}=0,\\
&\overline{d_3(t)}(e^{\mu}+e^{-\mu}-2)+c\mu+\overline{\Delta_3(t)}=0,\\
&\overline{d_2(t)}(e^{\mu}+e^{-\mu}-2)+c\mu-\overline{b_{11}(t)q(t)}=0.
\end{aligned}$$

Here, $\mu_4(c)$, $\mu_5(c)$, $\mu_6(c)$ are decreasing functions in c.

Based on the above notations, we are already to give the following lemma.

Lemma 2.1 *As $z \to -\infty$, the wave profile $(\Phi, \Psi, \Theta)(t, z)$ behave like*

$$\begin{pmatrix}\Phi(t,z)\\ \Psi(t,z)\\ \Theta(t,z)\end{pmatrix} \sim A_1\begin{pmatrix}\phi_{01}^*(t)\\ \psi_{01}(t)\\ \theta_{01}^*(t)\end{pmatrix}e^{\mu_1 z}+A_2\begin{pmatrix}\phi_{01}(t)\\ 0\\ 0\end{pmatrix}e^{\mu_2 z}+A_3\begin{pmatrix}0\\ 0\\ \theta_{01}(t)\end{pmatrix}e^{\mu_3 z}, \tag{2.1}$$

where $\mu_1 \neq \mu_2 \neq \mu_3$ and it holds uniformly in $t \in \mathbb{R}^+$. As $z \to \infty$, the wave profile $(\Phi, \Psi, \Theta)(t, z)$ behave like

$$\begin{pmatrix}\Phi(t,z)\\ \Psi(t,z)\\ \Theta(t,z)\end{pmatrix} \sim \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}-B_1\begin{pmatrix}\phi_{11}(t)\\ \psi_{11}^*(t)\\ 0\end{pmatrix}e^{-\mu_4 z}-B_2\begin{pmatrix}0\\ \psi_{11}^{**}(t)\\ \theta_{11}(t)\end{pmatrix}e^{-\mu_5 z}-B_3\begin{pmatrix}0\\ \psi_{11}(t)\\ 0\end{pmatrix}e^{-\mu_6 z}, \tag{2.2}$$

where $\mu_4 \neq \mu_5 \neq \mu_6$ and it holds uniformly in $t \in \mathbb{R}^+$. In the above formulas, $A_i, B_i, i = 1, 2, 3$ are nonnegative numbers. The functions $\psi_{01}(t), \phi_{01}(t), \theta_{01}(t), \phi_{01}^*(t)$, $\theta_{01}^*(t)$ are defined as (2.6),(2.10),(2.11),(2.13),(2.14) respectively; and $\phi_{11}(t), \theta_{11}(t), \psi_{11}(t), \psi_{11}^*(t)$, $\psi_{11}^{**}(t)$ are defined as (2.17),(2.18),(2.21),(2.22),(2.23) respectively.

Proof. First, we are concerned about the situation of $z \to -\infty$. It is clear that the linear system of (1.8) around the equilibrium $(0, 0, 0)$ can be represented by

$$\begin{cases}
d_1(t)\mathcal{D}_2[\widehat{\Phi}](t,z)-c\widehat{\Phi}_z+a_{12}(t)q(t)\widehat{\Psi}-a_{11}(t)p(t)\widehat{\Phi}-\widehat{\Phi}_t=0,\\
d_2(t)\mathcal{D}_2[\widehat{\Psi}](t,z)-c\widehat{\Psi}_z+[b_{11}(t)q(t)-b_{12}(t)p(t)-b_{13}(t)r(t)]\widehat{\Psi}-\widehat{\Psi}_t=0,\\
d_3(t)\mathcal{D}_2[\widehat{\Theta}](t,z)-c\widehat{\Theta}_z+c_{12}(t)q(t)\widehat{\Psi}-c_{11}(t)r(t)\widehat{\Theta}-\widehat{\Theta}_t=0.
\end{cases} \tag{2.3}$$

Substituting $\widehat{\Psi} = \psi_{01}(t)e^{\mu z}$ into the second equation of (2.3), we can obtain corresponding characteristic equation

$$d_2(t)(e^{\mu}+e^{-\mu}-2)-c\mu+\Delta_1(t)-\frac{\psi_{01}'(t)}{\psi_{01}(t)}=0, \tag{2.4}$$

where $\psi_{01}(t) > 0$ is a T-periodic function. Integrating both sides of equation (2.4) from 0 to T

gives

$$\overline{d_2(t)}(e^{\mu}+e^{-\mu}-2)-c\mu+\overline{\Delta_1(t)}=0. \tag{2.5}$$

Noticing $\int_0^T r_2(t)dt=\int_0^T b_{11}(t)q(t)dt$, and recalling the assumption **(A)**, it can be obtained that $\overline{\Delta_1(t)}<0$. Thereby equation (2.5) has a unique positive root defined as $\mu_1:=\mu_1(c)$. By putting $\mu=\mu_1$ into (2.4), $\psi_{01}(t)$ can be calculated as

$$\psi_{01}(t)=\psi_{01}\exp\left(\int_0^t\Big(d_2(s)(e^{\mu_1}+e^{-\mu_1}-2)-c\mu_1+\Delta_1(s)\Big)ds\right), \tag{2.6}$$

with $\psi_{01}(0)=\psi_{01}>0$. Thus, the asymptotic behavior of $\Psi(t,z)$ as $z\to-\infty$ can be expressed as

$$\Psi(t,z)\sim A_1\psi_{01}(t)e^{\mu_1 z}. \tag{2.7}$$

Using the same approach, ignoring $a_{12}(t)q(t)\widehat{\Psi}$ and $c_{12}(t)q(t)\widehat{\Psi}$. It is clear that the linear equations for $\widehat{\Phi}$ and $\widehat{\Theta}$ of (2.3) respectively are as following:

$$\begin{cases} d_1(t)\mathcal{D}_2[\widehat{\Phi}](t,z)-c\widehat{\Phi}_z-a_{11}(t)p(t)\widehat{\Phi}-\widehat{\Phi}_t=0,\\ d_3(t)\mathcal{D}_2[\widehat{\Theta}](t,z)-c\widehat{\Theta}_z-c_{11}(t)r(t)\widehat{\Theta}-\widehat{\Theta}_t=0. \end{cases} \tag{2.8}$$

Setting $\widehat{\Phi}=\phi_{01}(t)e^{\mu z}$ and $\widehat{\Theta}=\theta_{01}(t)e^{\mu z}$, (2.8) can be simplified as

$$\begin{cases} d_1(t)(e^{\mu}+e^{-\mu}-2)-c\mu-a_{11}(t)p(t)-\dfrac{\phi_{01}'(t)}{\phi_{01}(t)}=0,\\ d_3(t)(e^{\mu}+e^{-\mu}-2)-c\mu-c_{11}(t)r(t)-\dfrac{\theta_{01}'(t)}{\theta_{01}(t)}=0. \end{cases} \tag{2.9}$$

Likewise, we can obtain

$$\phi_{01}(t)=\phi_{01}\exp\left(\int_0^t\Gamma_1(s,\mu_2)ds\right), \tag{2.10}$$

$$\theta_{01}(t)=\theta_{01}\exp\left(\int_0^t\Gamma_2(s,\mu_3)ds\right). \tag{2.11}$$

In the first and third equation of (2.3), if the terms containing $\widehat{\Psi}$ are not considered, the asymptotic behaviors of $\widehat{\Phi}$ and $\widehat{\Theta}$ when $z\to-\infty$ can be expressed as $A_2\phi_{01}(t)e^{\mu_2 z}$ and $A_3\theta_{01}(t)e^{\mu_3 z}$. Next, we consider (2.3). Replacing $\widehat{\Psi}$ with $A_1\psi_{01}(t)e^{\mu_1 z}$, we get

$$\begin{cases} d_1(t)\mathcal{D}_2[\widehat{\Phi}](t,z)-c\widehat{\Phi}_z-a_{11}(t)p(t)\widehat{\Phi}-\widehat{\Phi}_t=-A_1a_{12}(t)q(t)\psi_{01}(t)e^{\mu_1 z},\\ d_3(t)\mathcal{D}_2[\widehat{\Theta}](t,z)-c\widehat{\Theta}_z-c_{11}(t)r(t)\widehat{\Theta}-\widehat{\Theta}_t=-A_1c_{12}(t)r(t)\psi_{01}(t)e^{\mu_1 z}. \end{cases}$$

It is not hard to obtain

$$\begin{cases} \Phi(t,z)\sim A_1\phi_{01}^*(t)e^{\mu_1 z}+A_2\phi_{01}(t)e^{\mu_2 z},\\ \Theta(t,z)\sim A_1\theta_{01}^*(t)e^{\mu_1 z}+A_3\theta_{01}(t)e^{\mu_3 z}. \end{cases} \tag{2.12}$$

Here

$$\phi_{01}^*(t)=\exp\bigg(\int_0^t\Gamma_1(s,\mu_1)ds\bigg)\cdot\bigg[\int_0^t a_{12}(s)q(s)\psi_{01}(s)\exp\bigg(-\int_0^s\Gamma_1(\tau,\mu_1)d\tau\bigg)ds+\phi_{01}^*(0)\bigg],\tag{2.13}$$

$$\theta_{01}^*(t)=\exp\bigg(\int_0^t\Gamma_2(s,\mu_1)ds\bigg)\cdot\bigg[\int_0^t c_{12}(s)r(s)\psi_{01}(s)\exp\bigg(-\int_0^s\Gamma_2(\tau,\mu_1)d\tau\bigg)ds+\theta_{01}^*(0)\bigg],\tag{2.14}$$

with

$$\phi_{01}^*(0)=\frac{\int_0^T a_{12}(s)q(s)\psi_{01}(s)\exp\bigg(-\int_0^s\Gamma_1(\tau,\mu_1)d\tau\bigg)ds}{\exp\bigg(-\int_0^T\Gamma_1(s,\mu_1)ds\bigg)-1},$$

$$\theta_{01}^*(0)=\frac{\int_0^T c_{12}(s)r(s)\psi_{01}(s)\exp\bigg(-\int_0^s\Gamma_2(\tau,\mu_1)d\tau\bigg)ds}{\exp\bigg(-\int_0^T\Gamma_2(s,\mu_1)ds\bigg)-1}.$$

By making use of the method of successive approximation (see, e.g. [20]), we conclude that (2.7) and (2.12) lead to (2.1).

Next, we intend to consider the asymptotic behavior of $(\Phi,\Psi,\Theta)(t,z)$ as $z\to\infty$. The linear system of (1.8) around the equilibrium $(1,1,1)$ can be expressed as following:

$$\begin{cases} d_1(t)\mathcal{D}_2[\widehat{\Phi}](t,z)-c\widehat{\Phi}_z+[a_{11}(t)p(t)-a_{12}(t)q(t)]\widehat{\Phi}-\widehat{\Phi}_t=0,\\ d_2(t)\mathcal{D}_2[\widehat{\Psi}](t,z)-c\widehat{\Psi}_z-b_{11}(t)q(t)\widehat{\Psi}+b_{12}(t)p(t)\widehat{\Phi}+b_{13}(t)r(t)\widehat{\Theta}-\widehat{\Psi}_t=0,\\ d_3(t)\mathcal{D}_2[\widehat{\Theta}](t,z)-c\widehat{\Theta}_z+[c_{11}(t)r(t)-c_{12}(t)q(t)]\widehat{\Theta}-\widehat{\Theta}_t=0.\end{cases}\tag{2.15}$$

In the same way, the characteristic equations of the first and last of (2.15) are given by

$$\begin{cases} d_1(t)(e^{-\mu}+e^{\mu}-2)+c\mu+\Delta_2(t)-\dfrac{\phi_{11}'(t)}{\phi_{11}(t)}=0,\\ d_3(t)(e^{-\mu}+e^{\mu}-2)+c\mu+\Delta_3(t)-\dfrac{\theta_{11}'(t)}{\theta_{11}(t)}=0,\end{cases}\tag{2.16}$$

where $\phi_{11}(t)>0,\theta_{11}(t)>0$ are T-periodic functions. From (2.16), we can solve that

$$\phi_{11}(t)=\phi_{11}\exp\bigg(\int_0^t\Big(d_1(s)(e^{\mu_4}-e^{-\mu_4}-2)+c\mu_4+\Delta_2(s)\Big)ds\bigg),\tag{2.17}$$

$$\theta_{11}(t)=\theta_{11}\exp\bigg(\int_0^t\Big(d_3(s)(e^{\mu_5}-e^{-\mu_5}-2)+c\mu_5+\Delta_3(s)\Big)ds\bigg),\tag{2.18}$$

with $\phi_{11}:=\phi_{11}(0)>0,\theta_{11}:=\theta_{11}(0)>0$. The asymptotic behaviors of $\Phi(t,z)$ and $\Theta(t,z)$ as

$z \to \infty$ are given by

$$\begin{cases} \Phi(t,z) \sim 1 - B_1\phi_{11}(t)e^{-\mu_4 z}, \\ \Theta(t,z) \sim 1 - B_2\theta_{11}(t)e^{-\mu_5 z}. \end{cases} \tag{2.19}$$

Following a similar argument for (2.12), we can get

$$\Psi(t,z) \sim 1 - B_1\psi_{11}^*(t)e^{-\mu_4 z} - B_2\psi_{11}^{**}(t)e^{-\mu_5 z} - B_3\psi_{11}(t)e^{-\mu_6 z}, \text{ as } z \to \infty. \tag{2.20}$$

Here

$$\psi_{11}(t) = \psi_{11}(0)\exp\left(\int_0^t \Gamma_3(s,\mu_6)ds\right), \tag{2.21}$$

$$\psi_{11}^*(t) = \exp\left(\int_0^t \Gamma_3(s,\mu_4)ds\right)\cdot\left[\int_0^t b_{12}(s)p(s)\phi_{11}(s)\exp\left(-\int_0^s \Gamma_3(\tau,\mu_4)d\tau\right)ds + \psi_{11}^*(0)\right], \tag{2.22}$$

$$\psi_{11}^{**}(t) = \exp\left(\int_0^t \Gamma_3(s,\mu_5)ds\right)\cdot\left[\int_0^t b_{13}(s)r(s)\theta_{11}(s)\exp\left(-\int_0^s \Gamma_3(\tau,\mu_5)d\tau\right)ds + \psi_{11}^{**}(0)\right], \tag{2.23}$$

with

$$\psi_{11}^*(0) = \frac{\int_0^T b_{12}(s)p(s)\phi_{11}(s)\exp\left(-\int_0^s \Gamma_3(\tau,\mu_4)d\tau\right)ds}{\exp\left(-\int_0^T \Gamma_3(s,\mu_4)ds\right) - 1},$$

$$\psi_{11}^{**}(0) = \frac{\int_0^T b_{13}(s)r(s)\theta_{11}(s)\exp\left(-\int_0^s \Gamma_3(\tau,\mu_5)d\tau\right)ds}{\exp\left(-\int_0^T \Gamma_3(s,\mu_5)ds\right) - 1}.$$

Again, by the method of successive approximation, we can get (2.2) from (2.19) and (2.20). The proof is thus complete. □

Remark 2.2 *We make some explanations for the symbol "$\sim$" appeared in (2.1) and (2.2). Take the first element, namely $\Phi(t,z)$, in (2.1) for an example. In the case of $\mu_2 < \mu_1 < \mu_3$ or $\mu_2 < \mu_3 < \mu_1$, we mean $\Phi(t,z) = A_2\phi_{01}(t)e^{\mu_2 z} + o(e^{\mu_2 z})$ uniformly in $t \in \mathbb{R}^+$ where the symbol o comes from the classic asymptotic definition.*

The uniqueness of the wave speed of the bistable wave solutions of (1.8) is presented in the following theorem. Instead of using the global stability of traveling wave front to prove the uniqueness, we employ the idea from [21].

Theorem 2.3 *Suppose that (1.8) has two bistable traveling wave solutions $(c_1, \Phi_1(t,z), \Psi_1(t,z), \Theta_1(t,z))$ with $z = x + c_1 t$ and $(c_2, \Phi_2(t,z), \Psi_2(t,z), \Theta_2(t.z))$ with $z = x + c_2 t$, then $c_1 = c_2$.*

Proof. To prove the theorem, we use a contradiction argument. Suppose that $c_2 > c_1$. Combining the monotonicity of $\mu_i(c), i = 1,2,3,4,5,6$ and asymptotic behavior established in Lemma

2.1, we know there exists a suitable positive constants z_0 (might be sufficiently large) such that

$$(\Phi_2, \Psi_2, \Theta_2)(t, z - z_0) < (\Phi_1, \Psi_1, \Theta_1)(t, z), \ (t, z) \in \mathbb{R}^+ \times \mathbb{R}.$$

Specially, when $t = 0$, the initial data satisfies

$$(\Phi_2, \Psi_2, \Theta_2)(0, j - z_0) < (\Phi_1, \Psi_1, \Theta_1)(0, j), \ j \in \mathbb{Z}.$$

By comparison principle, we have

$$(\Phi_2, \Psi_2, \Theta_2)(t, j + c_2 t - z_0) \le (\Phi_1, \Psi_1, \Theta_1)(t, j + c_1 t).$$

In particular, there holds

$$\Psi_2(t, j + c_2 t - z_0) \le \Psi_1(t, j + c_1 t).$$

Setting $\bar{z} = j + c_1 t$ so that $\Psi_1(t, \bar{z}) = \frac{1}{3}$, we get

$$\frac{1}{3} = \Psi_1(t, \bar{z}) \ge \Psi_2(t, \bar{z} + (c_2 - c_1)t - z_0) \to 1, \text{as } t \to \infty,$$

and a contradiction then follows, thus $c_2 \le c_1$. By a similar manner, it yields $c_2 \ge c_1$. In summary, $c_1 = c_2$. The proof is complete. $\square$

3 The determination of the sign of bistable wave speed

In this section, we aim at establishing two results so that the sign of bistable wave speed can be determined by comparison. To this end, we first make the following change

$$\widetilde{u}_j(t) = 1 - \frac{u_j(t)}{p(t)}, \widetilde{v}_j(t) = \frac{v_j(t)}{q(t)}, \widetilde{w}_j(t) = 1 - \frac{w_j(t)}{r(t)}, t \in \mathbb{R}^+, j \in \mathbb{Z},$$

such that system (1.1) can be rewritten as

$$\begin{cases} \widetilde{u}_j'(t) = d_1(t)\mathcal{D}_2[\widetilde{u}_j](t) + f(\widetilde{u}_j(t), \widetilde{v}_j(t), \widetilde{w}_j(t)), \\ \widetilde{v}_j'(t) = d_2(t)\mathcal{D}_2[\widetilde{v}_j](t) + g(\widetilde{u}_j(t), \widetilde{v}_j(t), \widetilde{w}_j(t)), \\ \widetilde{w}_j'(t) = d_3(t)\mathcal{D}_2[\widetilde{w}_j](t) + h(\widetilde{u}_j(t), \widetilde{v}_j(t), \widetilde{w}_j(t)), t \in \mathbb{R}^+, j \in \mathbb{Z}, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} f(\widetilde{u}_j(t), \widetilde{v}_j(t), \widetilde{w}_j(t)) :&= (1 - \widetilde{u}_j(t))[a_{12}(t)q(t)\widetilde{v}_j(t) - a_{11}(t)p(t)\widetilde{u}_j(t)], \\ g(\widetilde{u}_j(t), \widetilde{v}_j(t), \widetilde{w}_j(t)) :&= \widetilde{v}_j(t)[b_{11}(t)q(t)(1 - \widetilde{v}_j(t)) - b_{12}(t)p(t)(1 - \widetilde{u}_j(t)) \\ &\qquad - b_{13}(t)r(t)(1 - \widetilde{w}_j(t))], \\ h(\widetilde{u}_j(t), \widetilde{v}_j(t), \widetilde{w}_j(t)) :&= (1 - \widetilde{w}_j(t))[c_{12}(t)q(t)\widetilde{v}_j(t) - c_{11}(t)r(t)\widetilde{w}_j(t)]. \end{aligned}$$

To proceed, we investigate two eigen-problems of the ODE system of (3.1) around $(0, 0, 0)$

and $(1,1,1)$. Denote λ_0, λ_1 by the eigenvalues of the following systems respectively

$$\begin{cases} \dfrac{d\phi}{dt} - a_{12}(t)q(t)\psi(t) + a_{11}(t)p(t)\phi(t) = \lambda\phi(t), \\ \dfrac{d\psi}{dt} - [b_{11}(t)q(t) - b_{12}(t)p(t) - b_{13}(t)r(t)]\psi(t) = \lambda\psi(t), \\ \dfrac{d\theta}{dt} - c_{12}(t)q(t)\psi(t) + c_{11}(t)r(t)\theta(t) = \lambda\theta(t), \\ \phi(t+T) = \phi(t),\ \psi(t+T) = \psi(t),\ \theta(t+T) = \theta(t), \end{cases}$$

and

$$\begin{cases} \dfrac{d\phi}{dt} - [a_{11}(t)p(t) - a_{12}(t)q(t)]\phi(t) = \lambda\phi(t), \\ \dfrac{d\psi}{dt} + b_{11}(t)q(t)\psi(t) - b_{12}(t)p(t)\phi(t) - b_{13}(t)r(t)\theta(t) = \lambda\psi(t), \\ \dfrac{d\theta}{dt} - [c_{11}(t)r(t) - c_{12}(t)q(t)]\theta(t) = \lambda\theta(t), \\ \phi(t+T) = \phi(t),\ \psi(t+T) = \psi(t),\ \theta(t+T) = \theta(t). \end{cases}$$

Let $(\phi_0(t), \psi_0(t), \theta_0(t))$ and $(\phi_1(t), \psi_1(t), \theta_1(t))$ be the eigenfunctions corresponding to λ_0 and λ_1, respectively. It is easy to calculate that

$$\begin{cases} \phi_0(t) = (a_0(t) + \phi_0(0)) \exp\left(\lambda_0 t - \displaystyle\int_0^t a_{11}(s)p(s)ds\right), \\ \psi_0(t) = \exp\left(\displaystyle\int_0^t (b_{11}(s)q(s) - b_{12}(s)p(s) - b_{13}(s)r(s))ds + \lambda_0 t\right), \\ \theta_0(t) = (b_0(t) + \theta_0(0)) \exp\left(\lambda_0 t - \displaystyle\int_0^t c_{11}(s)r(s)ds\right), \end{cases}$$

where

$$\begin{aligned} &\lambda_0 = -\overline{\Delta_1(t)}, \quad \psi_0(0) = 1, \\ &\phi_0(0) = \frac{\int_0^T a_{12}(t)q(t)\psi_0(t)\exp(\int_0^t a_{11}(\tau)p(\tau)d\tau) - \lambda_0 t)dt}{\exp\left(\int_0^T a_{11}(t)q(t)dt - \lambda_0 T\right) - 1}, \\ &\theta_0(0) = \frac{\int_0^T c_{12}(t)q(t)\psi_0(t)\exp(\int_0^t c_{11}(\tau)r(\tau)d\tau) - \lambda_0 t)dt}{\exp\left(\int_0^T c_{11}(t)r(t)dt - \lambda_0 T\right) - 1}, \\ &a_0(t) = \int_0^t a_{12}(s)q(s)\psi_0(s)\exp\left(\int_0^s a_{11}(\tau)p(\tau)d\tau - \lambda_0 s\right)ds, \\ &b_0(t) = \int_0^t c_{12}(s)q(s)\psi_0(s)\exp\left(\int_0^s c_{11}(\tau)r(\tau)d\tau - \lambda_0 s\right)ds, \end{aligned}$$

and

$$
\begin{cases}
\phi_1(t) = \exp\left(\displaystyle\int_0^t (a_{11}(s)p(s) - a_{12}(s)q(s))ds + \lambda_1 t\right),\\
\psi_1(t) = (c_1(t) + \psi_1(0))\exp\left(\lambda_1 t - \displaystyle\int_0^t b_{11}(s)q(s)ds\right),\\
\theta_1(t) = \exp\left(\displaystyle\int_0^t (c_{11}(s)r(s) - c_{12}(s)q(s))ds + \lambda_1 t\right),
\end{cases}
$$

where

$$
\begin{aligned}
&\lambda_1 = -\overline{\Delta_2(t)} = -\overline{\Delta_3(t)}, \quad \phi_0(0) = \theta_0(0) = 1,\\
&\psi_1(0) = \frac{\int_0^T (b_{12}(t)p(t)\phi_1(t) + b_{13}(t)r(t)\theta_1(t))\exp(\int_0^t b_{11}(s)q(s)ds - \lambda_1 t)dt}{\exp\left(\int_0^T b_{11}(t)q(t)dt - \lambda_1 T\right) - 1},\\
&c_1(t) = \int_0^t (b_{12}(s)p(s)\phi_1(s) + b_{13}(s)r(s)\theta_1(s))\exp\left(\int_0^s b_{11}(\tau)q(\tau)d\tau - \lambda_1 s\right)ds.
\end{aligned}
$$

Next, to construct a pair of crucial upper and lower solutions, we define the transition functions as follows

$$
\begin{aligned}
p_1(t,x) &= \zeta(x)\phi_1(t) + (1-\zeta(x))\phi_0(t),\\
p_2(t,x) &= \zeta(x)\psi_1(t) + (1-\zeta(x))\psi_0(t),\\
p_3(t,x) &= \zeta(x)\theta_1(t) + (1-\zeta(x))\theta_0(t),
\end{aligned}
$$

where $\zeta(x)$ is a smooth function with $\zeta(x) = 0$ for $x \le -2$ and $\zeta(x) = 1$ for $x \ge 2$.

In order to discuss the sign of bistable wave speed, we give the following two lemmas.

Lemma 3.1 *For any $\xi^{\pm} \in \mathbb{R}$, there exist positive numbers β, σ, δ such that $(u_j^+, v_j^+, w_j^+)(t)$ and $(u_j^-, v_j^-, w_j^-)(t)$ defined as*

$$
\begin{cases}
u_j^{\pm}(t) = \Phi(t, j+ct+\xi^{\pm} \pm \sigma\delta(1-e^{-\beta t})) \pm \delta p_1(t, j+ct+\xi^{\pm} \pm \sigma\delta(1-e^{-\beta t}))e^{-\beta t},\\
v_j^{\pm}(t) = \Psi(t, j+ct+\xi^{\pm} \pm \sigma\delta(1-e^{-\beta t})) \pm \delta p_2(t, j+ct+\xi^{\pm} \pm \sigma\delta(1-e^{-\beta t}))e^{-\beta t},\\
w_j^{\pm}(t) = \Theta(t, j+ct+\xi^{\pm} \pm \sigma\delta(1-e^{-\beta t})) \pm \delta p_3(t, j+ct+\xi^{\pm} \pm \sigma\delta(1-e^{-\beta t}))e^{-\beta t},
\end{cases}
\tag{3.2}
$$

form a generalized upper/lower solution of the system (3.1).

Proof. The proof is similar to the ideas in Lemma 3.1 in article [2], Thus we omit it for simplicity here. □

Lemma 3.2 *Suppose that the initial data $(\widetilde{u}_j(0), \widetilde{v}_j(0), \widetilde{w}_j(0))$ satisfies*

$$
0 < \widetilde{u}_j(0) < 1, 0 < \widetilde{v}_j(0) < 1, 0 < \widetilde{w}_j(0) < 1,
$$

and

$$
u_j^-(0) \le \widetilde{u}_j(0) \le u_j^+(0), v_j^-(0) \le \widetilde{v}_j(0) \le v_j^+(0), w_j^-(0) \le \widetilde{w}_j(0) \le w_j^+(0),
$$

then the solution $(\widetilde{u}_j(t), \widetilde{v}_j(t), \widetilde{w}_j(t))$ *of (3.1) fulfills*

$$u_j^-(t) \le \widetilde{u}_j(t) \le u_j^+(t), v_j^-(t) \le \widetilde{v}_j(t) \le v_j^+(t), w_j^-(t) \le \widetilde{w}_j(t) \le w_j^+(t)$$

for all $t \in \mathbb{R}^+, j \in \mathbb{Z}$.

Next, we use the comparison principle based on the above two lemmas to establish the two crucial theorems.

Theorem 3.3 *Assume that (1.8) has a nonnegative non-decreasing upper solution* $(\overline{\Phi}(t,z), \overline{\Psi}(t,z), \overline{\Theta}(t,z))$ *with speed* $\bar{c} < 0$ *and* $\overline{\Phi}(t,z), \overline{\Psi}(t,z)$ *and* $\overline{\Theta}(t,z)$ *are* T*-period functions relative to* t*, satisfying*

$$(\overline{\Phi}, \overline{\Psi}, \overline{\Theta})(t, -\infty) < (1,1,1), \ (\overline{\Phi}, \overline{\Psi}, \overline{\Theta})(t, \infty) \ge (1,1,1), \tag{3.3}$$

then

$$c \le \bar{c} < 0.$$

Proof. For contradiction, we assume that $c > \bar{c}$ on the contrary and choose the initial datum $(\widetilde{u}_j(0), \widetilde{v}_j(0), \widetilde{w}_j(0))$ of (3.1) which is continuous, nondecreasing and satisfies

$$\widetilde{u}_j(0) = \widetilde{v}_j(0) = \widetilde{w}_j(0) = 0, \quad \text{for } j \le -J,$$

and

$$\widetilde{u}_j(0) = \widetilde{v}_j(0) = \widetilde{w}_j(0) = 1 - \eta, \quad \text{for } j \ge J,$$

for a sufficiently large positive integer J and a small enough number $\eta > 0$. This together with (3.3) enables us to further suppose that

$$\widetilde{u}_j(0) \le \overline{\Phi}(0,j), \widetilde{v}_j(0) \le \overline{\Psi}(0,j), \widetilde{w}_j(0) \le \overline{\Theta}(0,j), \quad \text{for } j \in \mathbb{Z}.$$

Then, by the comparison principle, we have

$$\widetilde{u}_j(t) \le \overline{\Phi}(t,z) = \overline{\Phi}(t, j + \bar{c}t), \widetilde{v}_j(t) \le \overline{\Psi}(t,z) = \overline{\Psi}(t, j + \bar{c}t), \widetilde{w}_j(t) \le \overline{\Theta}(t,z) = \overline{\Theta}(t, j + \bar{c}t) \tag{3.4}$$

for all $(t,j) \in \mathbb{R}^+ \times \mathbb{Z}$. On the other hand, by Lemma 3.2, we have

$$\begin{aligned}
\widetilde{u}_j(t) &\ge \Phi(t, j + ct + \xi^- - \sigma\delta(1 - e^{-\beta t})) - \delta p_1(t, j + ct + \xi^- - \sigma\delta(1 - e^{-\beta t}))e^{-\beta t}, \\
\widetilde{v}_j(t) &\ge \Psi(t, j + ct + \xi^- - \sigma\delta(1 - e^{-\beta t})) - \delta p_2(t, j + ct + \xi^- - \sigma\delta(1 - e^{-\beta t}))e^{-\beta t}, \\
\widetilde{w}_j(t) &\ge \Theta(t, j + ct + \xi^- - \sigma\delta(1 - e^{-\beta t})) - \delta p_3(t, j + ct + \xi^- - \sigma\delta(1 - e^{-\beta t}))e^{-\beta t}.
\end{aligned} \tag{3.5}$$

Again, in view of (3.3), we know that there exists a number $z_0 = j + \bar{c}t$ such that $\overline{\Phi}(t, z_0) < 1$. Combining (3.4) and (3.5), we can derive

$$1 > \overline{\Phi}(t, z_0) \ge \Phi(t, z_0 + (c - \bar{c})t + \xi^- - \sigma\delta(1 - e^{-\beta t})) - \delta p_1(t, j + ct + \xi^- - \sigma\delta(1 - e^{-\beta t}))e^{-\beta t} \to 1,$$

as $t \to \infty$, which gives a contradiction. Hence, $c \le \bar{c} < 0$. The proof is complete. $\square$

Theorem 3.4 *Suppose that (1.8) has a nonnegative non-decreasing lower solution* ($\underline{\Phi}(t,z)$, $\underline{\Psi}(t,z), \underline{\Theta}(t,z)$) *with speed* $\underline{c} > 0$ *and* $\underline{\Phi}(t,z), \underline{\Psi}(t,z)$ *and* $\underline{\Theta}(t,z)$ *are* T*-period functions relative to* t*, satisfying*

$$(\underline{\Phi}, \underline{\Psi}, \underline{\Theta})(t, -\infty) = (0,0,0) < (\underline{\Phi}, \underline{\Psi}, \underline{\Theta})(t, \infty) \le (1,1,1), \tag{3.6}$$

then

$$c \ge \underline{c} > 0.$$

Proof. The proof is similar to that of Theorem 3.3. By choosing proper initial data (depending on (3.6)) and assume $c < \underline{c}$ for contradiction, we can obtain

$$\underline{\Phi}(t, j + \underline{c}t) \le \Phi(t, j + ct + \xi^+ + \sigma\delta(1 - e^{-\beta t})) + \delta p_1(t, j + ct + \xi^+ + \sigma\delta(1 - e^{-\beta t}))e^{-\beta t}.$$

On the plane $z = z_1 := j + \underline{c}t$, we set $\underline{\Phi}(t, z_1) = \frac{1}{3}$. Hence

$$\frac{1}{3} = \underline{\Phi}(t, z_1) \le \Phi(t, z_1 + (c - \underline{c})t + \xi^+ + \sigma\delta(1 - e^{-\beta t})) + \delta p_1(t, j + ct + \xi^+ + \sigma\delta(1 - e^{-\beta t}))e^{-\beta t} \to 0,$$

as $t \to \infty$. Thus, we reach a contradiction. In short, $c \ge \underline{c} > 0$. The proof is complete. $\square$

4 Sign of bistable wave speed with specific conditions

Although Theorems 3.3 and 3.4 provide two criteria about how to predict the sign of bistable wave speed, explicit condition expressed by the model-parameter does not be presented. This part aims to gain some of such conditions via constructing explicit upper and lower solutions which seems to be nontrivial in contrast with the classic constructions, namely, the joint of a constant function and an exponential function.

Theorem 4.1 *The speed* c *of the bistable traveling wave solution of (1.8) is negative, if there exist constants* k_1, k_2 *such that*

$$-2d_2(t)\tau_{10} + d_2(t)\tau_{10}^2\chi_{10} + b_{12}(t)q(t)k_1 + b_{13}(t)r(t)k_2 \le 0, \tag{4.1}$$

and

$$1 < \frac{a_{12}(t)q(t)}{a_{11}(t)p(t) + \Delta_1(t) + [d_2(t) - d_1(t)]\tau_{10}} < k_1 < \min_{t\in[0,T]} \left\{ \frac{d_1(t)\tau_{10}(2 - \tau_{10}\chi_{10})}{[d_1(t) - d_2(t)]\tau_{10} - \Delta_1(t)} \right\}, \tag{4.2}$$

$$1 < \frac{c_{12}(t)q(t)}{c_{11}(t)r(t) + \Delta_1(t) + [d_2(t) - d_3(t)]\tau_{10}} < k_2 < \min_{t\in[0,T]} \left\{ \frac{d_3(t)\tau_{10}(2 - \tau_{10}\chi_{10})}{[d_3(t) - d_2(t)]\tau_{10} - \Delta_1(t)} \right\}, \tag{4.3}$$

where

$$\tau_{10} = e^{\mu_1(0)} + e^{-\mu_1(0)} - 2, \chi_{10} = \frac{1}{\tau_{10} + 4 + 2\sqrt{\tau_{10} + 4}}.$$

Proof. To make the sign of the bistable wave speed to be negative, by Theorem 3.3, we only need to construct an upper solution to (1.8). Let

$$\overline{\Psi}(t,z) = \frac{\psi_{01}(t)}{\psi_{01}(t) + e^{-\mu_1(-\epsilon)z}},$$

and redefine $\overline{\Phi}(t,z), \overline{\Theta}(t,z)$, which are continuous functions, as follows

$$\overline{\Phi}(t,z) = \min\{1, k_1\overline{\Psi}(t,z)\} = \begin{cases} k_1\overline{\Psi}(t,z), & z \le z_1(t), \\ 1, & z > z_1(t), \end{cases} \tag{4.4}$$

$$\overline{\Theta}(t,z) = \min\{1, k_2\overline{\Psi}(t,z)\} = \begin{cases} k_2\overline{\Psi}(t,z), & z \le z_2(t), \\ 1, & z > z_2(t). \end{cases}$$

Here, $0 < \epsilon \ll 1$. For any fixed $t \in \mathbb{R}^+$, $z_1(t)$ and $z_2(t)$ are uniquely determined by $k_1\overline{\Psi}(t, z_1(t)) = 1$ and $k_2\overline{\Psi}(t, z_2(t)) = 1$ respectively. Without loss of generality, we may assume that $k_1 > k_2$ which implies $z_1(t) < z_2(t), t \in \mathbb{R}^+$, according to the monotonicity of $\overline{\Psi}(t,z)$ in z.

To proceed, we note that $\mathcal{D}_2[\overline{\Psi}]$ can be reduced to

$$\mathcal{D}_2[\overline{\Psi}] = \tau_1\overline{\Psi}(1-\overline{\Psi})(1-2\overline{\Psi}) + \tau_1^2\overline{\Psi}^2(1-\overline{\Psi})H_1(t,z), \tag{4.5}$$

where

$$\tau_1 = e^{\mu_1(-\epsilon)} + e^{-\mu_1(-\epsilon)} - 2,\ H_1(t,z) = \frac{e^{-\mu_1(-\epsilon)z}/\psi_{01}(t)(1 - e^{-\mu_1(-\epsilon)z}/\psi_{01}(t))}{(1+e^{-\mu_1(-\epsilon)(z+1)}/\psi_{01}(t))(1+e^{-\mu_1(-\epsilon)(z-1)}/\psi_{01}(t))}.$$

It is easy to check that $H_1(t,z) \le \chi_1$ with

$$\chi_1 = \frac{1}{\tau_1 + 4 + 2\sqrt{\tau_1+4}}.$$

We first concentrate on the Ψ-equation. Substituting

$$\overline{\Psi}_z = \mu_1\overline{\Psi}(1-\overline{\Psi}), \quad \overline{\Psi}_t = \frac{\psi_{01}'(t)}{\psi_{01}(t)}\overline{\Psi}(1-\overline{\Psi})$$

and (4.5) into $\overline{\Psi}$-equation, we have

$$\begin{aligned}
&d_2(t)\mathcal{D}_2[\overline{\Psi}](t,z) + \epsilon\overline{\Psi}_z + \overline{\Psi}[b_{11}(t)q(t)(1-\overline{\Psi}) - b_{12}(t)p(t)(1-\overline{\Phi}) - b_{13}(t)r(t)(1-\overline{\Theta})] - \overline{\Psi}_t \\
&\le \overline{\Psi}(1-\overline{\Psi})\bigg\{d_2(t)\tau_1 + \epsilon\mu_1 + \Delta_1(t) - \frac{\psi_{01}'(t)}{\psi_{01}(t)} + \overline{\Psi}\Big(-2d_2(t)\tau_1 + d_2(t)\tau_1^2\chi_1 + Y(t,z)\Big)\bigg\} \\
&\le \overline{\Psi}^2(1-\overline{\Psi})\bigg\{-2d_2(t)\tau_1 + d_2(t)\tau_1^2\chi_1 + Y(t,z)\bigg\},
\end{aligned}$$

where

$$Y(t,z) = \frac{b_{12}(t)p(t)(\overline{\Phi} - \overline{\Psi}) + b_{13}(t)r(t)(\overline{\Theta} - \overline{\Psi})}{\overline{\Psi}(1-\overline{\Psi})}.$$

Next, we have to discuss the maximum of $Y(t,z)$ in the following cases.
(1) When $z > z_2(t)$, it is easy to realize that $\overline{\Phi}(t,z) = 1, \overline{\Theta}(t,z) = 1, \frac{1}{k_2} \le \overline{\Psi}(t,z) \le 1$. Then

$$Y(t,z) = \frac{b_{12}(t)p(t) + b_{13}(t)r(t)}{\overline{\Psi}} \le k_2\Big(b_{12}(t)p(t) + b_{13}(t)r(t)\Big). \tag{4.6}$$

(2) When $z \le z_1(t)$, it follows that $\overline{\Phi}(t,z) = k_1\overline{\Psi}(t,z)$ and $\overline{\Theta}(t,z) = k_2\overline{\Psi}(t,z)$. From (4.4), we can infer that $\overline{\Psi} \le \frac{1}{k_1}$. Therefore, $Y(t,z)$ can be rewritten as

$$Y(t,z) = \frac{b_{12}(t)p(t)(k_1-1) + b_{13}(t)r(t)(k_2-1)}{1-\overline{\Psi}} \le \frac{b_{12}(t)p(t)(k_1-1) + b_{13}(t)r(t)(k_2-1)}{1-\frac{1}{k_1}}. \tag{4.7}$$

(3) When $z_1(t) < z \le z_2(t)$, we have $\overline{\Phi}(t,z) = 1$ and $\overline{\Theta}(t,z) = k_2\overline{\Psi}(t,z)$. Then

$$Y(t,z) = \frac{b_{12}(t)p(t)}{\overline{\Psi}} + \frac{b_{13}(t)r(t)(k_2-1)}{1-\overline{\Psi}}.$$

It is easy to check that $\frac{1}{k_1} \le \overline{\Psi} \le \frac{1}{k_2}$, which results in

$$Y(t,z) \le b_{12}(t)q(t)k_1 + b_{13}(t)r(t)k_2. \tag{4.8}$$

By comparing (4.6),(4.7), and (4.8), we find the maximum among them is $b_{12}(t)q(t)k_1+b_{13}(t)r(t)k_2$. Thus, by assumption (4.1), we have

$$-2d_2(t)\tau_1 + d_2(t)\tau_1^2\chi_1 + Y(t,z) \le -2d_2(t)\tau_1 + d_2(t)\tau_1^2\chi_1 + b_{12}(t)q(t)k_1 + b_{13}(t)r(t)k_2 \le 0. \tag{4.9}$$

Next, we consider the Φ-equation. There are four subcases needed to be discussed.
(i) When $z \ge z_1(t)+1$, we get $\overline{\Phi}(t,z) = 1$ and hence

$$d_1(t)\mathcal{D}_2[\overline{\Phi}](t,z) + \epsilon\overline{\Phi}_z + (1-\overline{\Phi})[a_{12}(t)q(t)\overline{\Psi} - a_{11}(t)p(t)\overline{\Phi}] - \overline{\Phi}_t = 0.$$

(ii) When $z_1(t) < z < z_1(t)+1$, we notice that $\overline{\Phi}(t,z-1) = k_1\overline{\Psi}(t,z-1), \overline{\Phi}(t,z+1) = \overline{\Phi}(t,z) = 1$. Therefore, the Φ-equation can be evaluated by

$$d_1(t)\mathcal{D}_2[\overline{\Phi}](t,z) + \epsilon\overline{\Phi}_z + (1-\overline{\Phi})[a_{12}(t)q(t)\overline{\Psi} - a_{11}(t)p(t)\overline{\Phi}] - \overline{\Phi}_t = d_1(t)[k_1\overline{\Psi}(t,z-1) - 1] \le 0,$$

using $k_1\overline{\Psi}(t,z-1) \le 1$.
(iii) The case $z_1(t)-1 < z \le z_1(t)$ can be discussed together with the last case.

(iv) When $z \le z_1(t) - 1$, it follows from (4.4) that $\overline{\Phi}(t,z) = k_1\overline{\Psi}(t,z)$. Thus,

$$\begin{aligned}
&d_1(t)\mathcal{D}_2[\overline{\Phi}](t,z) + \epsilon\overline{\Phi}_z + (1-\overline{\Phi})[a_{12}(t)q(t)\overline{\Psi} - a_{11}(t)p(t)\overline{\Phi}] - \overline{\Phi}_t \\
&\le k_1\overline{\Psi}\Bigg\{(1-\overline{\Psi})\Bigg[\tau_1(1-2\overline{\Psi})d_1(t) + \tau_1^2\chi_1\overline{\Psi}d_1(t) + \epsilon\mu_1 - \frac{\psi_{01}'(t)}{\psi_{01}(t)}\Bigg] \\
&\qquad\qquad + (1-k_1\overline{\Psi})\Bigg[\frac{a_{12}(t)q(t)}{k_1} - a_{11}(t)p(t)\Bigg]\Bigg\} \\
&\le k_1\overline{\Psi}F_1(\overline{\Psi}),
\end{aligned}$$

where

$$\begin{aligned}
F_1(\overline{\Psi}) := (1-\overline{\Psi})&\left[\tau_1(1-2\overline{\Psi})d_1(t) + \tau_1^2\chi_1\overline{\Psi}d_1(t) + \epsilon\mu_1 - \frac{\psi_{01}'(t)}{\psi_{01}(t)}\right] \\
&+ (1-k_1\overline{\Psi})\left[\frac{a_{12}(t)q(t)}{k_1} - a_{11}(t)p(t)\right].
\end{aligned}$$

It is obvious that $F_1''(\overline{\Psi}) = 2d_1(t)\tau_1(2-\tau_1\chi_1) \ge 0$ (using $\tau_1\chi_1 < 1$), where the derivative is respect to the variable $\overline{\Psi}$. Therefore, $F_1(\overline{\Psi})$ is concave for $\overline{\Psi} \in [0, \frac{1}{k_1}]$. In can be easily calculated that

$$\begin{aligned}
F_1(0) &= d_1(t)\tau_1 + \epsilon\mu_1 - \frac{\psi_{01}'(t)}{\psi_{01}(t)} + \frac{a_{12}(t)q(t)}{k_1} - a_{11}(t)p(t) \\
&= [d_1(t) - d_2(t)]\tau_1 - \Delta_1(t) + \frac{a_{12}(t)q(t)}{k_1} - a_{11}(t)p(t),
\end{aligned} \tag{4.10}$$

$$F_1(\frac{1}{k_1}) = (1-\frac{1}{k_1})\Bigg[d_1(t)\tau_1 + \frac{1}{k_1}(\tau_1^2\chi_1 - 2\tau_1)d_1(t) + \epsilon\mu_1 - \frac{\psi_{01}'(t)}{\psi_{01}(t)}\Bigg].$$

For the purpose of proving $F_1(\overline{\Psi}) < 0$ for $\overline{\Psi} \in [0, \frac{1}{k_1}]$, we only need to check that $F_1(0) < 0$ and $F_1(\frac{1}{k_1}) < 0$ which are ensured by (4.2) as $\epsilon \to 0^+$. To sum up the cases (i)-(iv), we have

$$d_1(t)\mathcal{D}_2[\overline{\Phi}](t,z) + \epsilon\overline{\Phi}_z + (1-\overline{\Phi})[a_{12}(t)q(t)\overline{\Psi} - a_{11}(t)p(t)\overline{\Phi}] - \overline{\Phi}_t \le 0.$$

By a similar manner, we can infer from (4.3) that

$$d_3(t)\mathcal{D}_2[\overline{\Theta}](t,z) + \epsilon\overline{\Theta}_z + (1-\overline{\Theta})[c_{12}(t)q(t)\overline{\Psi} - c_{11}(t)r(t)\overline{\Theta}] - \overline{\Theta}_t \le 0.$$

As such, it is proved that $(\overline{\Phi}, \overline{\Psi}, \overline{\Theta})(t,z)$ is an upper solution of (1.8). By Theorem 3.3, the proof is complete. □

Theorem 4.2 *The speed c of the bistable traveling wave solution of (1.8) satisfies $c \ge \epsilon > 0$ provided that*

$$\max\{\Pi_1(t), \Pi_2(t)\} < \min_{t\in[0,T]}\left\{1 - \frac{d_2(t)(2\tau_{20} + \tau_{20}^2)}{b_{11}(t)q(t)}\right\}. \tag{4.11}$$

where

$$\Pi_1(t) := \frac{a_{11}(t)p(t) + [d_1(t) + d_1(t)\tau_{20} + d_2(t)]\tau_{20} + \Delta_1(t)}{a_{12}(t)q(t)},$$

$$\Pi_2(t) := \frac{c_{11}(t)r(t) + [d_3(t) + d_3(t)\tau_{20} + d_2(t)]\tau_{20} + \Delta_1(t)}{c_{12}(t)q(t)},$$

and

$$\tau_{20} = e^{\mu_1(0)} + e^{-\mu_1(0)} - 2.$$

Proof. We intend to construct a lower solution to show that the wave speed c is positive. Define

$$\underline{\Psi}(t,z) = \frac{\underline{k}\psi_{01}(t)}{\psi_{01}(t) + e^{-\mu_1(\epsilon)z}},\ \underline{\Phi}(t,z) = \underline{\Theta}(t,z) = \frac{\underline{\Psi}(t,z)}{\underline{k}}$$

with $0 < \epsilon \ll 1$ and $\underline{k}$ satisfing

$$\max\{\Pi_1(t), \Pi_2(t)\} < \underline{k} < \min_{t\in[0,T]} \left\{1 - \frac{d_2(t)(2\tau_2 + \tau_2^2)}{b_{11}(t)q(t)}\right\}. \tag{4.12}$$

By a similar computation with (4.5), we obtain

$$\mathcal{D}_2[\underline{\Psi}] = \tau_2\underline{\Psi}(1 - \frac{\underline{\Psi}}{\underline{k}})(1 - \frac{2\underline{\Psi}}{\underline{k}}) + \tau_2^2\frac{\underline{\Psi}^2}{\underline{k}}(1 - \frac{\underline{\Psi}}{\underline{k}})H_2(t,z)$$

with

$$\tau_2 = e^{\mu_1(\epsilon)} + e^{-\mu_1(\epsilon)} - 2,\ H_2(t,z) = \frac{e^{-\mu_1(\epsilon)z}/\psi_{01}(t)(1 - e^{-\mu_1(\epsilon)z}/\psi_{01}(t))}{(1 + e^{-\mu_1(\epsilon)(z+1)}/\psi_{01}(t))(1 + e^{-\mu_1(\epsilon)(z-1)}/\psi_{01}(t))}.$$

On account of the lower bound of $H_2(t,z)$ is -1, we have

$$\begin{aligned}
&d_2(t)\mathcal{D}_2[\underline{\Psi}](t,z) - \epsilon\underline{\Psi}_z + \underline{\Psi}[b_{11}(t)q(t)(1 - \underline{\Psi}) - b_{12}(t)p(t)(1 - \underline{\Phi}) - b_{13}(t)r(t)(1 - \underline{\Theta})] - \underline{\Psi}_t \\
&\geq \frac{\underline{\Psi}^2}{\underline{k}}(1 - \frac{\underline{\Psi}}{\underline{k}})\left\{-2d_2(t)\tau_2 - d_2(t)\tau_2^2 + b_{11}(t)q(t)(1 - \underline{k})\right\}.
\end{aligned}$$

Thanks to (4.12), we get

$$d_2(t)\mathcal{D}_2[\underline{\Psi}](t,z) - \epsilon\underline{\Psi}_z + \underline{\Psi}[b_{11}(t)q(t)(1 - \underline{\Psi}) - b_{12}(t)p(t)(1 - \underline{\Phi}) - b_{13}(t)r(t)(1 - \underline{\Theta})] - \underline{\Psi}_t \geq 0.$$

As for the Φ-equation and Θ-equation, we have the following estimation:

$$\begin{aligned}
&d_1(t)\mathcal{D}_2[\underline{\Phi}](t,z) - \epsilon\underline{\Phi}_z + (1 - \underline{\Phi})[a_{12}(t)q(t)\underline{\Psi} - a_{11}(t)p(t)\underline{\Phi}] - \underline{\Phi}_t \\
&\qquad \geq \underline{\Phi}(1 - \underline{\Phi})\left\{-d_1(t)\tau_2 - d_1(t)\tau_2^2 - d_2(t)\tau_2 - \Delta_1(t) + a_{12}(t)q(t)\underline{k} - a_{11}(t)p(t)\right\},
\end{aligned}$$

and

$$\begin{aligned}
&d_3(t)\mathcal{D}_2[\underline{\Theta}](t,z) - \epsilon\underline{\Theta}_z + (1 - \underline{\Theta})[c_{12}(t)q(t)\underline{\Psi} - c_{11}(t)r(t)\underline{\Theta}] - \underline{\Theta}_t \\
&\qquad \geq \underline{\Theta}(1 - \underline{\Theta})\left\{-d_3(t)\tau_2 - d_3(t)\tau_2^2 - d_2(t)\tau_2 - \Delta_1(t) + c_{12}(t)q(t)\underline{k} - c_{11}(t)r(t)\right\},
\end{aligned}$$

in which the assumption (4.12) is used. Let $\epsilon \to 0^+$, we can derive that

$$d_1(t)\mathcal{D}_2[\underline{\Phi}](t,z) - \epsilon\underline{\Phi}_z + (1-\underline{\Phi})[a_{12}(t)q(t)\underline{\Psi} - a_{11}(t)p(t)\underline{\Phi}] - \underline{\Phi}_t \geq 0,$$

and

$$d_3(t)\mathcal{D}_2[\underline{\Theta}](t,z) - \epsilon\underline{\Theta}_z + (1-\underline{\Theta})[c_{12}(t)q(t)\underline{\Psi} - c_{11}(t)r(t)\underline{\Theta}] - \underline{\Theta}_t \geq 0.$$

Thus, we proved that $(\underline{\Phi}, \underline{\Psi}, \underline{\Theta})(t,z)$ is a lower solution of (1.8). By Theorem 3.4, the proof is complete. □

As applications of Theorems 4.1 and 4.2, we want to provide partially the answer to the open problem proposed in [13] associating to the following constant-coefficient system of (1.1):

$$\begin{cases} u_j'(t) = d_1\mathcal{D}_2[u_j](t) + u_j(t)(r_1 - a_{11}u_j(t) - a_{12}v_j(t)), \\ v_j'(t) = d_2\mathcal{D}_2[v_j](t) + v_j(t)(r_2 - b_{11}v_j(t) - b_{12}u_j(t) - b_{13}w_j(t)), \\ w_j'(t) = d_3\mathcal{D}_2[w_j](t) + w_j(t)(r_3 - c_{11}w_j(t) - c_{12}v_j(t)),\ j \in \mathbb{Z}, t > 0. \end{cases} \tag{4.13}$$

For system (4.13), the equilibrium points and bistable condition $(\mathbf{A})$ become respectively

$$e_0 := (0,0,0), e_1 := (0, \frac{r_2}{b_{11}}, 0), e_2 := (\frac{r_1}{a_{11}}, 0, \frac{r_3}{c_{11}}),$$

and

$$b_{11}r_1 < a_{12}r_2, a_{11}c_{11}r_2 < b_{12}c_{11}r_1 + a_{11}b_{13}r_3, b_{11}r_3 < c_{12}r_2. \tag{4.14}$$

Applying Theorems 4.1 and 4.2 to (4.13), we have the following two corollaries:

Corollary 4.3 *The speed c of the bistable traveling wave solution of (4.13) is negative, if there exist positive constants k_1, k_2 such that*

$$-2d_2\tau_{10} + d_2\tau_{10}^2\chi_{10} + b_{12}\frac{r_2}{b_{11}}k_1 + b_{13}\frac{r_3}{c_{11}}k_2 \leq 0, \tag{4.15}$$

and

$$1 < \frac{a_{12}\frac{r_2}{b_{11}}}{r_1 + r_2 - \frac{b_{12}r_1}{a_{11}} - \frac{b_{13}r_3}{c_{11}} + (d_2 - d_1)\tau_{10}} < \frac{d_1\tau_{10}(2 - \tau_{10}\chi_{10})}{(d_1 - d_2)\tau_{10} - r_2 + \frac{b_{12}r_1}{a_{11}} + \frac{b_{13}r_3}{c_{11}}}, \tag{4.16}$$

$$1 < \frac{c_{12}\frac{r_2}{b_{11}}}{r_3 + r_2 - \frac{b_{12}r_1}{a_{11}} - \frac{b_{13}r_3}{c_{11}} + (d_2 - d_3)\tau_{10}} < \frac{d_3\tau_{10}(2 - \tau_{10}\chi_{10})}{(d_3 - d_2)\tau_{10} - r_2 + \frac{b_{12}r_1}{a_{11}} + \frac{b_{13}r_3}{c_{11}}}. \tag{4.17}$$

Corollary 4.4 *The speed c of the bistable traveling wave solution of (4.13) is positive provided* *that*

$$\max\Big\{\frac{r_1+[d_1+d_1\tau_{20}+d_2]\tau_{20}+r_2-\frac{b_{12}r_1}{a_{11}}-\frac{b_{13}r_3}{c_{11}}}{a_{12}\frac{r_2}{b_{11}}},\ \frac{r_3+[d_3+d_3\tau_{20}+d_2]\tau_{20}+r_2-\frac{b_{12}r_1}{a_{11}}-\frac{b_{13}r_3}{c_{11}}}{c_{12}\frac{r_2}{b_{11}}}\Big\} < \min_{t\in[0,T]}\Big\{1-\frac{d_2(2\tau_{20}+\tau_{20}^2)}{r_2}\Big\}. \tag{4.18}$$

We can learn from Corollaries 4.3 and 4.4 that almost all of the parameters appeared in (4.13) should be taken into account in the determination of bistable wave speed sign. Hence, one can analyze the effect of different coefficients on this determination. For instance, if one of the diffusivity coefficients $d_i, i = 1, 2, 3$ are sufficiently small, then one of the conditions (4.15), (4.16) and (4.17) would not be valid any more. While we fixed d_1 and d_3 and let d_3 be sufficiently large, the condition (4.18) is not true. We emphasize that one of our main contributions is that we proposed a method and obtained some conditions for the determination of bistable wave speed sign. One can get more criteria by constructing different upper-lower solutions.

5 Numerical Simulation

We can derive that the bistable wave speed is negative in Theorem 4.1, which implies the bistable wave speed propagates to the right and u and w will win the competition. On the contrary, Theorem 4.2 ensures that the bistable wave speed is positive, which means the bistable wave speed propagates to the left and v will win the competition.

In order to illustrate our theoretical results Corollaries 4.3 and 4.4, we choose the initial data in the form of

$$u_j(0)=\begin{cases}0, & 1\le j\le N_j,\\ 1, & N_j+1\le j\le N_L,\end{cases}$$

$$v_j(0)=\begin{cases}1, & 1\le j\le N_j,\\ 0, & N_j+1\le j\le N_L,\end{cases}$$

$$w_j(0)=\begin{cases}0, & 1\le j\le N_j,\\ 1, & N_j+1\le j\le N_L,\end{cases}$$

with the boundary conditions

$$\begin{cases}u_1(t)-u_2(t)=u_{N_L}(t)-u_{N_{L-1}}(t)=0,\\ v_1(t)-v_2(t)=v_{N_L}(t)-v_{N_{L-1}}(t)=0,\\ w_1(t)-w_2(t)=w_{N_L}(t)-w_{N_{L-1}}(t)=0,\end{cases}$$

where N_j and N_L are two integers.

In (4.13), we choose

$$
\begin{aligned}
&a_{11} = b_{11} = c_{11} = 1,\ a_{12} = 1.2,\ b_{12} = 0.8,\ b_{13} = 0.7, \\
&c_{12} = 1.2,\ d_1 = 1,\ d_2 = 2,\ d_3 = 1.3,\ r_1 = r_2 = r_3 = 1.
\end{aligned}
\tag{5.1}
$$

From which, we can compute $\tau_{10} = 0.250, \chi_{10} = 0.119$. It is easy to see that the set of such chosen parameters make (4.14)-(4.17) valid. As a result, one may except the bistable wave speed to be negative. This fact is exactly verified by the numerical results, see Fig. 5.1.

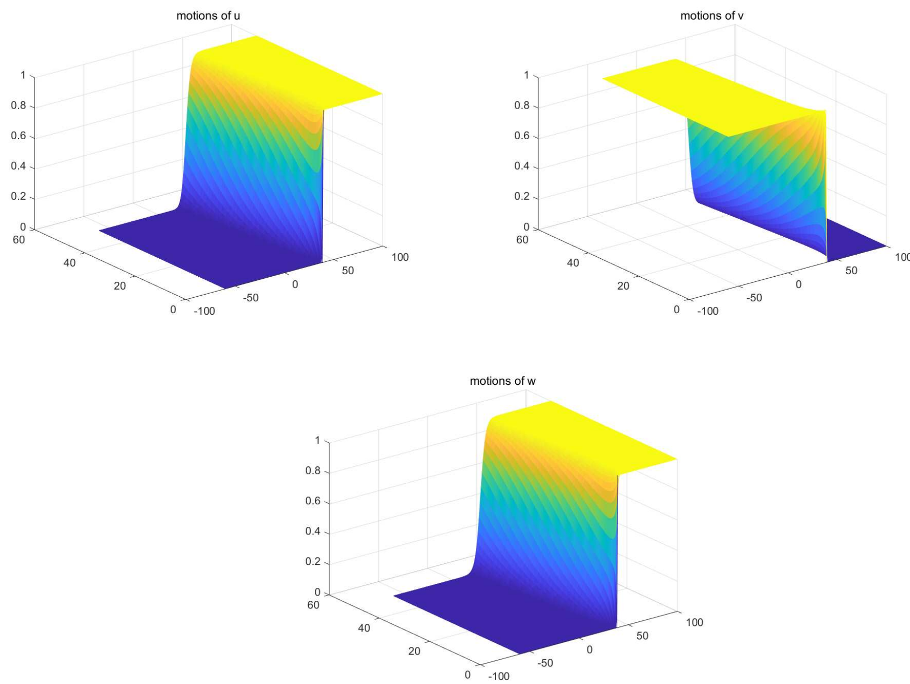

Figure 5.1: The simulation of (4.13) for the setting of (5.1).

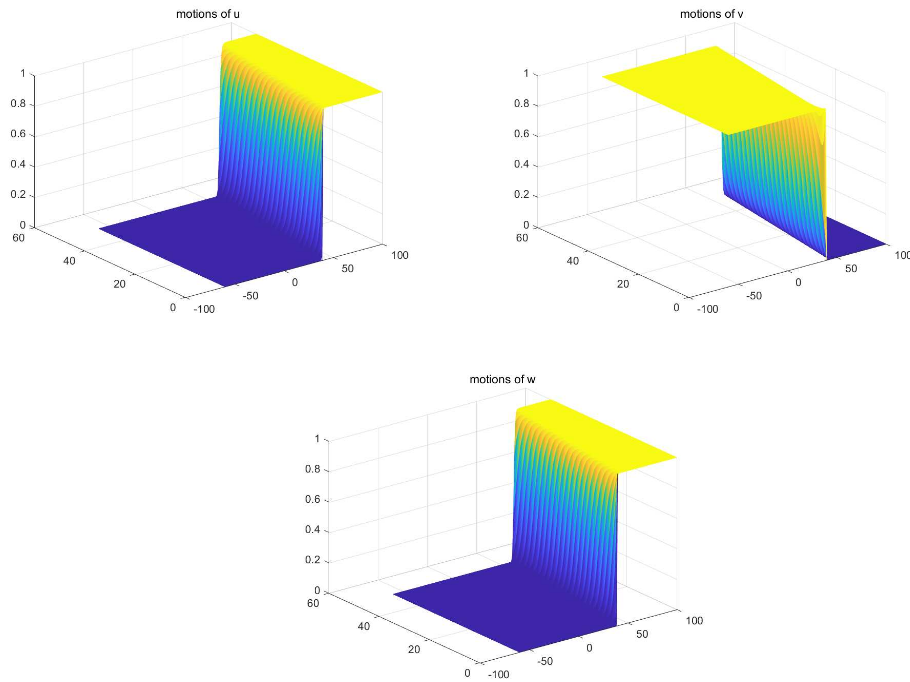

Figure 5.2: The simulation of (4.13) for the setting of (5.2)

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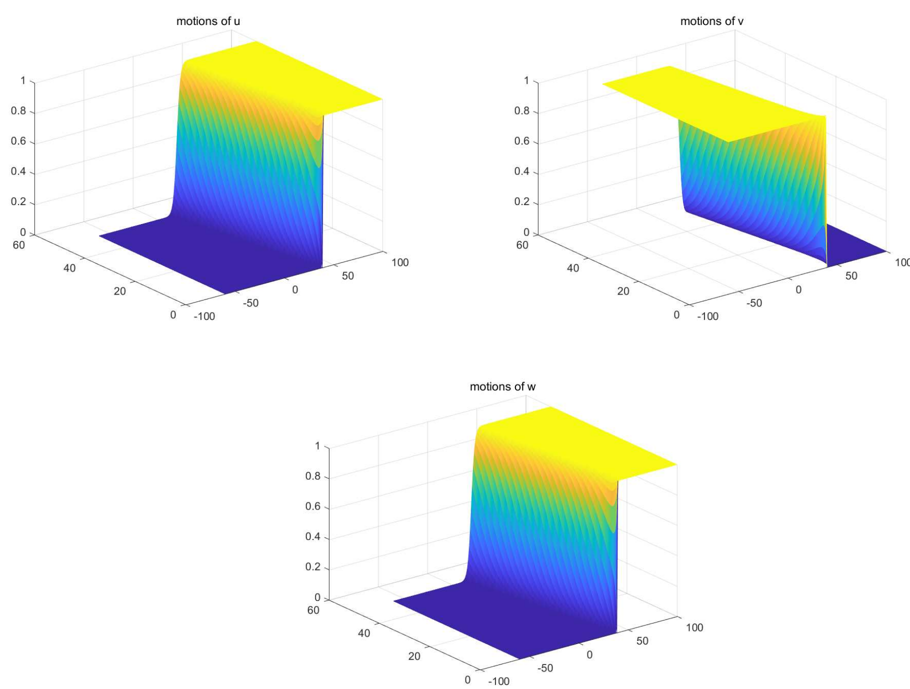

Figure 5.3: The simulation of (4.13) for the setting of (5.3).

In (4.13), we choose

$$\begin{aligned} &a_{11}=b_{11}=c_{11}=1,\ a_{12}=10,\ b_{12}=1.2,\ b_{13}=1.2,\\ &c_{12}=8,\ d_1=1,\ d_2=0.5,\ d_3=1.2,\ r_1=r_2=r_3=1. \end{aligned} \tag{5.2}$$

For the above set of parameters, one can derive that $\tau_{20}=2.800$. Meanwhile, they fulfill (4.14) and (4.18), so the bistable wave speed would be positive according to Corollary 4.4. This is demonstrated in Fig 5.2.

We all know that the competitive ability of a strong species will be greater than that of a weak species indicating that the strong species can wipe out the weak one. However, when more than two species are involved, the outcome may be not that simple. Indeed, the Theorem 3.4 in Guo [13] proves that it is possible for two weak species to outcompete a strong species in model (1.2) under certain conditions. Naturally, we want to wonder whether the same phenomenon can be observed in model (4.13). To this end, we choose

$$\begin{aligned} &a_{11}=b_{11}=c_{11}=1,\ a_{12}=c_{12}=1.1,\ b_{12}=b_{13}=0.9,\\ &r_1=r_2=r_3=1,\ d_1=d_2=d_3=1. \end{aligned} \tag{5.3}$$

Fig 5.3 tells us that such a phenomenon still exists.

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