A New Class of Curves of Rational B-Spline Type

Mohamed Allaoui [1];
Département de Sciences économiques, Université de Comores, Comores;
Jamal Adetola ;
Université Nationale des Sciences, Technologie, Ingénierie et Mathématiques (UNSTIM), Bénin;
Koffi Wilfrid Houédanou ;
Université d'Abomey-Calavi (UAC)-Département de Mathématiques, Bénin;
Aurélien Goudjo.
Université d'Abomey-Calavi (UAC)-Département de Mathématques, Bénin.

Abstract

A new class of rational parametrization has been developed and it was used to generate a new family of rational functions B-splines $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^{k}$ which depends on an index $\alpha \in]-\infty\,,\,0[\cup]1\,,\,+\infty[$. This family of functions verifies, among other things, the properties of positivity, of partition of the unit and, for a given degree k, constitutes a true basis approximation of continuous functions. We loose, however, the regularity classical optimal linked to the multiplicity of nodes, which we recover in the asymptotic case, when $\alpha \to \infty$. The associated B-splines curves verify the traditional properties particularly that of a convex hull and we see a certain "conjugated symmetry" related to α. The case of open knot vectors without an inner node leads to a new family of rational Bezier curves that will be separately, object of in-depth analysis.

Key words : Knot vector • Rational B-splines functions • Cox- de Boor recursion • de-Boor Algorithm • Computer Graphics.

1 Introduction

In this paper we will explore geometric objects very frequently used in the world of industrial design and graphic animation on computer. These are Bézier curves and B-spline curves. Their applications range from printing on paper and robotics to video games. In this introduction, we will present in turn a brief overview of the evolution of computer graphics, a bibliographic analysis and then our motivation which will situate the context of our work. The papers [1, 2] lay the groundwork for the approach to defining "normalized" B-spline functions commonly referred to as the Cox-de Boor recurrence relation although it was previously established by Lois Mansfield. Both papers show the numerical stability of this recurrence relation in spline approximation calculations as opposed to Schoenberg's initial approach which defined B-spline functions as divided differences of power functions truncated and which turns out to be very unstable. This numerical instability is very extensively illustrated in the article by Cox [2].

The Cox-de Boor recurrence relation will be used to formulate a new rational approach to B-spline functions from an algorithmic point of view. Although the founders of our approach to defining B-splines as basic functions of splines, these papers do not address the issue of curves generated by B-splines using control points.

David Rogers in [3], gives a very educational presentation of the different geometric objects ranging from Bezier curves to non-uniform rational B-spline curves. Surfaces were also well addressed. It gives us a synthetic view of the state of the art in the field of geometry applied to computer graphics, while indicating the contexts of its evolution as well as the actors of this evolution. The many examples which illustrate the various concepts here serve as a benchmark in our work. It should be noted that in this book, the emphasis has mainly been placed on the algorithmic aspects of the construction of curves and surfaces.

W. Tiller et al. [4] is the essential reference on the question of B-spline curves and surfaces. It offers in a single volume the essential proofs of the properties of these geometric objects which are the curves and surfaces of Bezier and B-splines and that the assisted design industry computer uses extensively today. It also contains some very interesting examples that we have borrowed to illustrate some properties in our work. Other works going in the direction of the use of polynomial B-spline functions and Nurbs are also approached in the references [5–19].

A standard B-spline curve G of degree $k \in \mathbb{N}^*$ in $\mathbb{R}^d$ with $d \in \mathbb{N}^*$, $1 \leq d \leq 3$ is defined by a polynomial basis $\left(\mathbf{G}_i^k\right)_{i=0}^{n}$ on a parametrization space $[a\,,\,b]$ subdivided by a vector of nodes $U = (t_i)_{i=0}^{m}$ with $m = n + k + 1$. The

[1] E-mail addresses :allamath9@gmail.com (M. Allaoui), adetolajamal58@yahoo.com (J. Adetola), khouedanou@yahoo.fr (K. W. Houédanou), aurelien.goudjo@uac.bj (A. Goudjo).

basis $\left(\mathbf{G}_i^k\right)_{i=0}^n$ is given by the recurrence relation of Cox/de Boor [3] as follows:

$$\begin{array}{rcl} \mathbf{G}_i^0(x) & = & \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1} \text{ for } i = 0, \ldots, m-1 \\ 0 & \text{otherwise} \end{cases} \\ \mathbf{G}_i^k(x) & = & w_i^k(x)\mathbf{G}_i^{k-1}(x) + \left(1 - w_{i+1}^k(x)\right)\mathbf{G}_{i+1}^{k-1}(x) \\ w_i^k(x) & = & \begin{cases} \dfrac{x - t_i}{t_{i+k} - t_i} & \text{if } t_i \leq x < t_{i+k} \text{ for } i = 0, \ldots, n \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{1.1}$$

If $(d_i)_{i=0}^n$ are the checkpoints of G, $d_i \in \mathbf{R}^d$ for all i then

$$G(x) = \sum_{i=0}^{n} d_i\, \mathbf{G}_i^k(x)\,,\ \forall x \in [a\,,\, b]$$

Likewise we have the rational B-spline basis $(R_i)_{i=0}^n$ of degree $k \in \mathbb{N}^*$ associated to the vector of nodes U and the weight vector $W = (\omega_i)_{i=0}^n$ which can be defined by

$$R_i(x) = \frac{\omega_i \mathbf{G}_i^n(x)}{\displaystyle\sum_{j=0}^{n} \omega_j\, \mathbf{G}_j^n(x)}$$

where $\omega_i > 0$, $\forall i = 0, \ldots, n$.

We can then define the rational B-spline curves replacing the polynomial basis by the rational basis [3, 18].

One has to notice that $w_i^k(x) = \varphi(x\,,\, t_i\,,\, t_{i+k})$ where φ is a real function defined on $\mathbf{R}^3$ satisfying the following properties:

1. $\varphi(x\,,\, a\,,\, b) \in [0\,,\, 1)$ for all $(x\,,\, a\,,\, b) \in \mathbf{R}^3$
2. For all $a, b \in \mathbf{R}$ such that $a < b$ the function $x \in \mathbf{R} \mapsto \varphi(x\,,\, a\,,\, b)$ is continuous, strictly increasing on $[a\,,\, b)$ and we have:
   - $\varphi(x\,,\, a\,,\, b) = 0$ for all $x \notin (a\,,\, b)$
   - $\lim\limits_{x \to b^-} \varphi(x\,,\, a\,,\, b) = 1$

The aim of this work is to maintain these properties while imposing that for all $a, b \in \mathbf{R}$ such that $a < b$, the function $x \in \mathbf{R} \mapsto \varphi(x\,,\, a\,,\, b)$ is homographic in order to build a natural B-spline basis composed of rational functions.

The outline of the paper is as follows. In Section 2, we study the new class of rational parametrization with their fundamental properties. The new class of rational B-spline basis has been developped in section 3, as well as the new properties obtained. The Section 4 studies the new class of B-spline curves. Some illustrations of properties of the new class of rational B-spline curve have been given in Section 5. We then offer our conclusion and the further works in Section 6.

2 A class of rational parametrization

2.1 Definition

The targeted class of parametrization is based on the following lemma which gives the foundation of a new class of curves of rational B-spline type.

Lemma 2.1. *Let a, $b \in \mathbf{R}$ verifying $a < b$. There exists a family $\mathcal{H}\left([a\,,\, b]\right)$ of homographic functions f strictly increasing on $[a\,,\, b]$ such that $f(a) = 0$ and $f(b) = 1$.*

More precisely, for all $f \in \mathcal{H}\left([a\,,\, b]\right)$ there exists a unique $\alpha \in (-\infty\,,\, 0) \cup (1\,,\, \infty)$ such that

$$f(x) = \frac{\alpha(x - a)}{x + (\alpha - 1)b - \alpha a}, \quad \forall x \in [a\,,\, b].$$

Proof. *(Existence)* Since f is homographic with $f(a) = 0$ there exists $\alpha \neq 0$ and $c \in \mathbf{R}\backslash\{-a, -b\}$ such that for all $x \in [a\,,\, b]$ we get: $f(x) = \dfrac{\alpha(x - a)}{x + c}$. As $f(b) = 1$ then $1 = \dfrac{\alpha(b - a)}{b + c}$. This leads to $c = (\alpha - 1)b - \alpha a$. Using the fact that $c \notin \{-a, -b\}$ then we have $\alpha \notin \{0, 1\}$. The strict increase of f yields $\alpha(\alpha - 1) > 0$, therefore $\alpha \in (-\infty\,,\, 0) \cup (1\,,\, \infty)$.

We then write

$$\mathcal{H}([a\,,\,b]) = \left\{ f_\alpha |\, f_\alpha(x) = \frac{\alpha(x-a)}{x+(\alpha-1)b-\alpha a},\, \alpha \in (-\infty\,,\,0)\cup(1\,,\,\infty),\, x\in[a\,,\,b] \right\}$$

(Uniqueness)
Let $\alpha,\,\beta \in (-\infty\,,\,0)\cup(1\,,\,\infty)$ and $f_\alpha,\, f_\beta \in \mathcal{H}([a\,,\,b])$ corresponding

$$f_\alpha = f_\beta \quad \text{implies} \quad \alpha = \beta$$

□

Remark 2.2.

1. *Let $x \in [a\,,\,b]$ and $\alpha \in (-\infty\,,\,0)\cup(1\,,\,\infty)$.*
   One has $D = x+(\alpha-1)b-\alpha a \neq 0$.
2. *Let $\alpha \in (-\infty\,,\,0)\cup(1\,,\,\infty)$ and $a<b$.*
   $f_\alpha \in \mathcal{H}([a\,,\,b])$ is continuous and strictly increasing on $[a\,,\,b]$ with $f_\alpha([a\,,\,b]) = [0\,,\,1]$.
   Moreover, the classical case as an asymptotic situation holds: $\lim\limits_{|\alpha|\to\infty} f_\alpha(x) = \lambda = \dfrac{x-a}{b-a}$.
   In addition, we have: $f_\alpha(a+b-x) = 1-f_{1-\alpha}(x)$ and $f_\alpha(x) = 1-f_{1-\alpha}(a+b-x)$.

Definition 2.3. *Let $\alpha \in (-\infty\,,\,0)\cup(1\,,\,\infty)$. A parametrization of index α is any real function φ_α defined for all $(x\,,\,a\,,\,b) \in \mathbf{R}^3$ by*

$$\varphi_\alpha(x\,,\,a\,,\,b) = \begin{cases} f_\alpha(x) & \text{if } a \le x < b \text{ with } f_\alpha \in \mathcal{H}([a\,,\,b]) \\ 0 & \text{otherwise} \end{cases}$$

2.2 Properties of the parametrization

Proposition 2.1. *Let $\alpha \in (-\infty\,,\,0)\cup(1\,,\,\infty)$ and φ_α the associated parametrization. Let T be an affine and bijective function of $\mathbf{R}$. The following properties hold: For all $(x\,,\,a\,,\,b) \in \mathbf{R}^3$*

1. $0 \le \varphi_\alpha(x\,,\,a\,,\,b) < 1$
2. *If T is strictly increasing then*
   $$\varphi_\alpha(T(x)\,,\,T(a)\,,\,T(b)) = \varphi_\alpha(x\,,\,a\,,\,b)$$
3. *If T is strictly decreasing then*
   $$\varphi_\alpha(T(x)\,,\,T(b)\,,\,T(a)) = 1-\varphi_{1-\alpha}(x\,,\,a\,,\,b)$$

Proof. Let T be an affine and bijective function of $\mathbf{R}$. There exists $(\lambda\,,\,\delta) \in \mathbf{R}^* \times \mathbf{R}$ such that, for all $x \in \mathbf{R}$, we have $T(x) = \lambda x + \delta$. By direct computation, the results follow.

□

Corollary 2.4. *Let $\alpha \in (-\infty\,,\,0)\cup(1\,,\,\infty)$ and φ_α be the associated parametrization. Let $a, b \in \mathbf{R}$ such that $a<b$. Let $a<t_1<t_2<b$. For all $x \in [a\,,\,b]$, we have*

$$\varphi_\alpha(a+b-x\,,\,t_1\,,\,t_2) = 1-\varphi_{1-\alpha}(x\,,\,a+b-t_2\,,\,a+b-t_1)$$

Proof. We apply Proposition 2.1 by taking $T(x) = a+b-x$ on $\mathbf{R}$. We observe that T is strictly decreasing and verifies $T\circ T(x) = x$ for all $x \in \mathbf{R}$. This gives the result.

□

Illustration 2.1. *The figures 1 and 2 illustrate $\varphi_\alpha(x\,,\,0\,,\,1)$ for $x \in (-1\,,\,2)$ with values of α conjugated respectively.*

We observe that on the subinterval $(0\,,\,1)$ which is the interior of its support, the function is convex for $\alpha<0$ and concave for $\alpha>1$.

The figure 3 which illustrates $\varphi_\alpha(x\,,\,1\,,\,3)$ for $x \in (0\,,\,6)$ confirms the previous observations and lets suspect the symmetrical role that the conjugated α are to play. It also shows that the effect of α is crucial in the neighborhood of 0 and of 1.

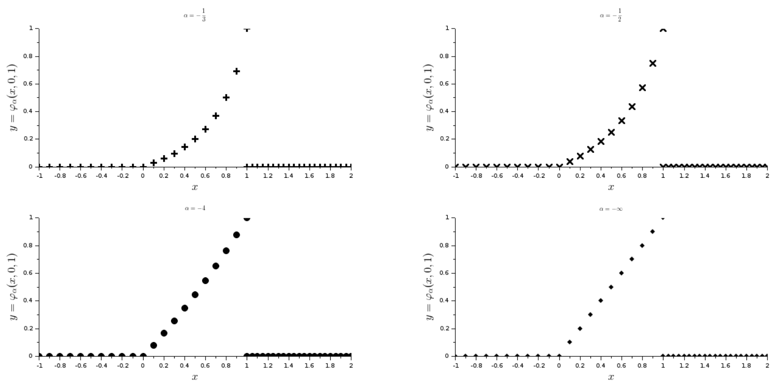

Figure 1: The curves of φ_α for $\alpha \in \{-\frac{1}{3}, -\frac{1}{2}, -4, \infty\}$

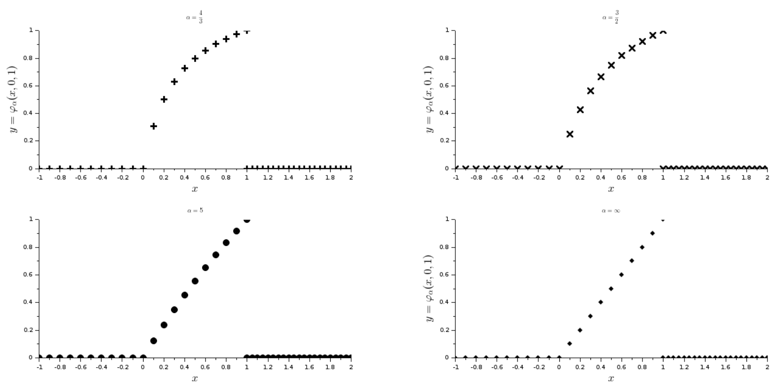

Figure 2: The curves φ_α for $\alpha \in \{\frac{4}{3}, \frac{3}{2}, 5, \infty\}$

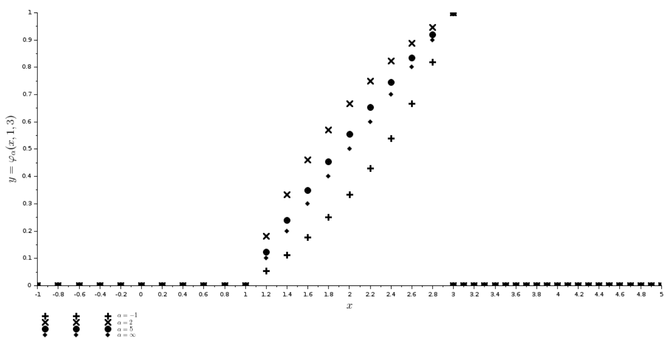

Figure 3: Comparaison of φ_α for conjugated α and large α

3 New class of rational B-splines basis

The B-splines curves are part of the family of curves obtained by concatenation of several generated pieces of curves using a family of basic functions of parametrization space $[a\ ,\ b]$ subdivised by a vector of nodes U and a set of points $(d_i)_{i=0}^n$ of $\mathbf{R}^d$ called control polygon.

The nature of chosen vector of nodes may strongly influence the properties of B-spline basis generated as well as the resulting curve. We must very quickly specify this object.

We follow the definitions of the book of D. F. Rogers entitled "An Introduction to NURBS with historical perspective" [3].

Definition 3.1. *Let $a, b \in \mathbf{R}$ such that $a < b$. A node vector or vector of nodes in $[a, b]$ is any increasing sequence $U = (t_i)_{i=0}^m$ in $[a\ ,\ b]$.*

The node vectors fall into two categories: the open node vectors and periodic node vectors. Each category is divided in two variants: uniform and non-uniform.

Definition 3.2. *Let $a, b \in \mathbf{R}$ such that $a < b$ and $m, k \in \mathbf{N}^*$ such that $m > 2k$. We consider the node vector $U = (t_i)_{i=0}^m$ such that $t_k = a$ and $t_{m-k} = b$.*

1. End nodes:

   The nodes $t_0, t_1, \ldots, t_k$ and the nodes $t_{m-k}, t_{m-k+1}, \ldots, t_m$ are called end nodes.

   The nodes $t_{k+1}, t_{k+2}, \ldots, t_{m-k-1}$ are called interior nodes.

2. Open node vector:

   The vector of nodes is said to be open if its end nodes coincide; we then have $t_0 = t_1 = \ldots = t_k = a$ and $t_{m-k} = t_{m-k+1} = \ldots = t_m = b$.

   Otherwise U is said to be periodic.

3. Uniform node vector:

   U is uniform if its interior nodes are equidistant; that is, there exists $h > 0$ such that $t_{i+1} - t_i = h$ for all $k \leq i \leq m - k - 1$.

   Otherwise U is non-uniform.

4. Multiple node (multiplicity of a node) :

   Let $p \in \mathbf{N}^*$ and t_i be a node of U. We say that t_i is a node of multiplicity p if there exists a unique $j \in [0, \ldots, m-1] \cap \mathbf{N}$ such that the subsequence $U_i = (t_{j+l})_{l=0}^{p-1}$ with $j \leq i \leq j + p - 1$ is constant.

   If $p > 1$, we say that t_i is multiple node.

5. Stop nodes:

   The set $(u_i)_{i=0}^r$ of distinct nodes of $U = (t_i)_{i=0}^m$ constitutes the stop nodes. We have $u_0 = t_0 < u_1 < \ldots < u_r = t_m$ and there exists a unique sequence of nonnegative integers $p = (p_i)_{i=0}^r$ such that for all $i = 0, \ldots, r$, u_i is of multiplicity p_i.

   We shall remark that $\displaystyle\sum_{i=0}^{r} p_i = m + 1$. On the other hand, these nodes define the different segments of studied curves and the interior stop nodes define the transition between its segments.

6. Symmetrical node vector:

   $U = (t_i)_{i=0}^m$ is a symmetrical node vector if for all $i = 0, \ldots, m$, $t_{m-i} = t_0 + t_m - t_i$.

Definition 3.3. *Let $a, b \in \mathbf{R}$ such that $a < b$ and $m, n, k \in \mathbf{N}^*$ such that $n \geq k$ and $m = n + k + 1$. Let $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$ and φ_α the parametrization of index α. Let $U = (t_i)_{i=0}^m$ be a node vector of the interval $[a\,,\,b]$.*

A B-spline basis of index α and of degree k on the node vector U is the real functions $\left({}^\alpha\mathbf{G}_i^k\right)_{i=0}^n$ defined by the recurrence relation:

$$
\begin{array}{rcl}
{}^\alpha\mathbf{G}_i^0(x) & = & \left\{ \begin{array}{ll} 1 & \textit{if } t_i \leq x < t_{i+1} \textit{ for } i = 0, \ldots, m-1 \\ 0 & \textit{otherwise} \end{array} \right. \\
{}^\alpha\mathbf{G}_i^k(x) & = & w_i^k(x)\,{}^\alpha\mathbf{G}_i^{k-1}(x) + \left(1 - w_{i+1}^k(x)\right){}^\alpha\mathbf{G}_{i+1}^{k-1}(x) \\
w_i^k(x) & = & \varphi_\alpha(x\,,\,t_i\,,\,t_{i+k})
\end{array}
\tag{3.1}
$$

This relation is said to be of Cox/de Boor.

Definition 3.4. *Let $a, b \in \mathbf{R}$ such that $a < b$. Let $m, n, k \in \mathbf{N}^*$ such that $n > k$ and $m = n + k + 1$. Let $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$. Let $U = (t_i)_{i=0}^m$ be a node vector of interval $[a\,,\,b]$. Let $d \in \mathbf{N}^*$ such that $d \leq 3$, and $\Pi = (d_i)_{i=0}^n \subset \mathbf{R}^d$.*

Let $\left({}^\alpha\mathbf{G}_i^k\right)_{i=0}^n$ be the B-spline basis of index α, of degree k and of node vector U.

A B-spline curve of index α, with node vector U and control points $(d_i)_{i=0}^n$ is the $\mathbf{R}^d$ valued function G_α defined by:

$$
x \in [t_0\,,\,t_m] \mapsto G_\alpha(x) = \sum_{i=0}^{n} d_i\,{}^\alpha\mathbf{G}_i^k(x)
$$

Π is called control polygon of the curve G_α.

3.1 Fundamental properties of the new class of basis

Theorem 3.5. *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n + k + 1$. Let $U = (t_i)_{i=0}^{m}$ be a vector of nodes and $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

The rational B-spline basis of index α with vector of nodes U and of degree k, $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^{n}$, verifies the following properties:

1. Local support property:
   For all $x \notin (t_i\,,\,t_{i+k+1})$, ${}^{\alpha}\mathbf{G}_i^k(x) = 0$
2. Positivity property:
   For all $i = 0, \ldots, n$ and $x \in (t_i\,,\,t_{i+k+1})$, ${}^{\alpha}\mathbf{G}_i^k(x) > 0$
3. Unit partition property:
   For all j such that $t_j < t_{j+1}$, for all $x \in [t_j\,,\,t_{j+1})$, we have
   $$\sum_{i=0}^{n} {}^{\alpha}\mathbf{G}_i^k(x) = \sum_{i=j-k}^{j} {}^{\alpha}\mathbf{G}_i^k(x) = 1$$
4. Symmetry property:
   If U is a symmetrical node vector then for all $x \in [t_0\,,\,t_m]$ and $i = 0, \ldots, n$ we have
   $${}^{\alpha}\mathbf{G}_i^k(t_0 + t_m - x) = {}^{1-\alpha}\mathbf{G}_{n-i}^k(x)$$

Proof. Let $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$ and φ_α be the parametrization of index α.

We will proceed by recurrence on k.

1. (*Local support and Positivity:*)
   - For $k = 0$, we have by definition: for all $i = 0, \ldots, m-1$
     $${}^{\alpha}\mathbf{G}_i^0(x) = \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1} \text{ for } i = 0, \ldots, m-1 \\ 0 & \text{otherwise} \end{cases}$$
     hence we have
     $$\begin{array}{ll} {}^{\alpha}\mathbf{G}_i^k(x) = 0 & \text{if } x \notin (t_i\,,\,t_{i+k+1}) \\ {}^{\alpha}\mathbf{G}_i^k(x) > 0 & \text{if } x \in (t_i\,,\,t_{i+k+1}) \neq \emptyset \end{array}$$
   - Let $k > 0$ and assume that for all $0 \leq j < k$ we have
     $$\begin{array}{ll} {}^{\alpha}\mathbf{G}_i^j(x) = 0 & \text{if } x \notin (t_i\,,\,t_{i+j+1}) \\ {}^{\alpha}\mathbf{G}_i^j(x) > 0 & \text{if } x \in (t_i\,,\,t_{i+j+1}) \neq \emptyset \end{array}$$
     By definition we have
     $${}^{\alpha}\mathbf{G}_i^k(x) = w_i^k(x)\,{}^{\alpha}\mathbf{G}_i^{k-1}(x) + \left(1 - w_{i+1}^k(x)\right){}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x)$$
     with
     $$\begin{cases} {}^{\alpha}\mathbf{G}_i^{k-1}(x) = 0 & \text{if } x \notin (t_i\,,\,t_{i+k}) \\ {}^{\alpha}\mathbf{G}_i^{k-1}(x) > 0 & \text{if } x \in (t_i\,,\,t_{i+k}) \neq \emptyset \end{cases}$$
     and
     $$\begin{cases} {}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) = 0 & \text{if } x \notin (t_{i+1}\,,\,t_{i+k+1}) \\ {}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) > 0 & \text{if } x \in (t_{i+1}\,,\,t_{i+k+1}) \neq \emptyset \end{cases}$$
     - Let $x \notin (t_i\,,\,t_{i+k+1}) = (t_i\,,\,t_{i+k}) \cup (t_{i+1}\,,\,t_{i+k+1})$. Then we have $x \notin (t_i\,,\,t_{i+k})$ and $x \notin (t_{i+1}\,,\,t_{i+k+1})$ which gives ${}^{\alpha}\mathbf{G}_i^{k-1}(x) = 0$, ${}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) = 0$ and ${}^{\alpha}\mathbf{G}_i^k(x) = 0$
     - Let $x \in (t_i\,,\,t_{i+k+1}) = (t_i\,,\,t_{i+k}) \cup (t_{i+1}\,,\,t_{i+k+1}) \neq \emptyset$. Then we have $x \in (t_i\,,\,t_{i+k}) \neq \emptyset$; or $x \in (t_{i+1}\,,\,t_{i+k+1}) \neq \emptyset$.
       If $x \in (t_i\,,\,t_{i+k}) \neq \emptyset$ then one has ${}^{\alpha}\mathbf{G}_i^{k-1}(x) > 0$ and ${}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) \geq 0$. But from proposition 2.1 we have
       $$\begin{cases} w_i^k(x) = \varphi_\alpha(x\,,\,t_i\,,\,t_{i+k}) \in (0\,,\,1) \\ w_{i+1}^k(x) = \varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+k+1}) \geq 0 \end{cases}$$
       We conclude that
       $${}^{\alpha}\mathbf{G}_i^k(x) \geq w_i^k(x)\,{}^{\alpha}\mathbf{G}_i^{k-1}(x) > 0$$

Similarly if $x \in (t_{i+1}\,,\, t_{i+k+1}) \neq \emptyset$ then ${}^{\alpha}\mathbf{G}_i^{k-1}(x) \geq 0$ and ${}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) > 0$. By using once more proposition 2.1 we have

$$\begin{cases} w_i^k(x) = \varphi_\alpha(x\,,\, t_i\,,\, t_{i+k}) \geq 0 \\ w_{i+1}^k(x) = \varphi_\alpha(x\,,\, t_{i+1}\,,\, t_{i+k+1}) \in (0\,,\,1) \end{cases}$$

We then conclude that

$${}^{\alpha}\mathbf{G}_i^k(x) \geq \left(1 - w_{i+1}^k(x)\right){}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) > 0$$

Hence ${}^{\alpha}\mathbf{G}_i^k(x) > 0$ if $x \in (t_i\,,\, t_{i+k+1})$

2. (*Unit partition*)

   Let $m, k, n \in \mathbb{N}^*$ such that $n > k$ and $m = n + k + 1$.

   - Let j such that $t_j < t_{j+1}$. Let $i = 0, \ldots, n$.

   $$[t_i\,,\, t_{i+k+1}) \cap [t_j\,,\, t_{j+1}) \neq \emptyset \;\Leftrightarrow\; j - k \leq i \leq j$$

   - Let $x \in [t_j\,,\, t_{j+1})$ and $i = 0, \ldots, n$.

   $${}^{\alpha}\mathbf{G}_i^k(x) \neq 0 \;\Leftrightarrow\; j - k \leq i \leq j$$

   Thus we have $\displaystyle\sum_{i=0}^{n} {}^{\alpha}\mathbf{G}_i^k(x) = \sum_{i=j-k}^{j} {}^{\alpha}\mathbf{G}_i^k(x)$.

   As ${}^{\alpha}\mathbf{G}_i^k(x) = w_i^k(x){}^{\alpha}\mathbf{G}_i^{k-1}(x) + \left[1 - w_{i+1}^k(x)\right]{}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x)$ then

   $$\begin{aligned}
   \sum_{i=j-k}^{j} {}^{\alpha}\mathbf{G}_i^k(x) &= \sum_{i=j-k}^{j} w_i^k(x){}^{\alpha}\mathbf{G}_i^{k-1}(x) + \sum_{i=j-k}^{j} \left[1 - w_{i+1}^k(x)\right]{}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) \\
   &= \sum_{i=j-k}^{j} w_i^k(x){}^{\alpha}\mathbf{G}_i^{k-1}(x) + \sum_{i=j-k+1}^{j+1} \left[1 - w_i^k(x)\right]{}^{\alpha}\mathbf{G}_i^{k-1}(x) \\
   &= w_{j-k}^k(x){}^{\alpha}\mathbf{G}_{j-k}^{k-1}(x) + \sum_{i=j-k+1}^{j} {}^{\alpha}\mathbf{G}_i^{k-1}(x) \\
   &+ \left[1 - w_{j+1}^k(x)\right]{}^{\alpha}\mathbf{G}_{j+1}^{k-1}(x) \\
   &= \sum_{i=j-k+1}^{j} {}^{\alpha}\mathbf{G}_i^{k-1}(x)
   \end{aligned}$$

   because

   $$\begin{cases} \mathbf{supp}\,{}^{\alpha}\mathbf{G}_{j-k}^{k-1} \cap [t_j\,,\, t_{j+1}) &= [t_{j-k}\,,\, t_j) \cap [t_j\,,\, t_{j+1}) = \emptyset \\ \mathbf{supp}\,{}^{\alpha}\mathbf{G}_{j+1}^{k-1} \cap [t_j\,,\, t_{j+1}) &= [t_{j+1}\,,\, t_{j+k+1}) \cap [t_j\,,\, t_{j+1}) = \emptyset \end{cases}$$

   - Let us show that for all $0 \leq r \leq k - 1$ we have

   $$\sum_{i=j-k+r}^{j} {}^{\alpha}\mathbf{G}_i^{k-r}(x) = \sum_{i=j-k+r+1}^{j} {}^{\alpha}\mathbf{G}_i^{k-r-1}(x)$$

     - For $r = 0$, it is verified.
     - Let $0 < r \leq k - 1$. Suppose that the property is satisfied for all $0 \leq s < r$, i.e.

     $$\sum_{i=j-k+s}^{j} {}^{\alpha}\mathbf{G}_i^{k-s}(x) = \sum_{i=j-k+s+1}^{j} {}^{\alpha}\mathbf{G}_i^{k-s-1}(x)$$

     Then, since

     $${}^{\alpha}\mathbf{G}_i^{k-r}(x) = w_i^{k-r}(x){}^{\alpha}\mathbf{G}_i^{k-r-1}(x) + \left[1 - w_{i+1}^{k-r}(x)\right]{}^{\alpha}\mathbf{G}_{i+1}^{k-r-1}(x)$$

     we have

$$\begin{aligned}
\sum_{i=j-k+r}^{j} {}^{\alpha}\mathbf{G}_i^{k-r}(x) &= \sum_{i=j-k+r}^{j} w_i^{k-r}(x)^{\alpha}\mathbf{G}_i^{k-r-1}(x) \\
&+ \sum_{i=j-k+r}^{j} \left[1 - w_{i+1}^{k-r}(x)\right]^{\alpha}\mathbf{G}_{i+1}^{k-r-1}(x) \\
&= w_{j-k+r}^{k-r}(x)^{\alpha}\mathbf{G}_{j-k+r}^{k-r-1}(x) + \sum_{i=j-k+r+1}^{j} {}^{\alpha}\mathbf{G}_i^{k-r-1}(x) \\
&+ \left[1 - w_{j+1}^{k-r}(x)\right]^{\alpha}\mathbf{G}_{j+1}^{k-r-1}(x) \\
&= \sum_{i=j-k+r+1}^{j} {}^{\alpha}\mathbf{G}_i^{k-r-1}(x)
\end{aligned}$$

because

$$\begin{cases} \mathbf{supp}\, {}^{\alpha}\mathbf{G}_{j-k+r}^{k-r-1} \cap [t_j\,,\, t_{j+1}) & = [t_{j-k+r}\,,\, t_j) \cap [t_j\,,\, t_{j+1}) = \emptyset \\ \mathbf{supp}\, {}^{\alpha}\mathbf{G}_{j+1}^{k-r-1} \cap [t_j\,,\, t_{j+1}) & = [t_{j+1}\,,\, t_{j+k-r+1}) \cap [t_j\,,\, t_{j+1}) = \emptyset \end{cases}$$

Therefore the result follows.

- By setting $r = k - 1$ we obtain

$$\sum_{i=j-k}^{j} {}^{\alpha}\mathbf{G}_i^{k}(x) = \sum_{i=j}^{j} {}^{\alpha}\mathbf{G}_i^{0}(x) = {}^{\alpha}\mathbf{G}_j^{0}(x) = 1$$

3. (*Symmetry*)
Consider the symmetrical vector of nodes $U = (t_i)_{i=0}^{m}$, let $x \in [t_0\,,\, t_m]$, let us show that for all $k \geq 0$ and all $i \leq m - k - 1$, we have

$${}^{\alpha}\mathbf{G}_i^{k}(t_0 + t_m - x) = {}^{\alpha}\mathbf{G}_{m-k-1-i}^{k}(x)$$

Let T be the affine function on $\mathbb{R}$ defined by $T(x) = t_0 + t_m - x$. T is strictly decreasing.

- For all $j_1 < j_2$ such that $t_{j_1} < t_{j_2}$

$$\begin{aligned}
x \in (t_{j_1}\,,\, t_{j_2}) &\Leftrightarrow T(x) \in (T(t_{j_2})\,,\, T(t_{j_1})) \\
&\Leftrightarrow T(x) \in (t_{m-j_2}\,,\, t_{m-j_1}) \text{ because } U \text{ is symmetric}
\end{aligned}$$

- We begin by checking for $k = 0$, i.e.

$${}^{\alpha}\mathbf{G}_i^{0}(T(x)) = {}^{1-\alpha}\mathbf{G}_{m-1-i}^{0}(x)$$

$$\begin{aligned}
{}^{\alpha}\mathbf{G}_i^{0}(T(x)) \neq 0 &\Rightarrow t_i < T(x) < t_{i+1} \\
&\Leftrightarrow t_{m-i-1} = T(t_{i+1}) < x < T(t_i) = t_{m-i} \\
&\Rightarrow {}^{1-\alpha}\mathbf{G}_{m-1-i}^{0}(x) \neq 0
\end{aligned}$$

and conversely. The result follows as a consequence of the definition.

- Let $k \in \mathbb{N}^*$. We suppose that for all $j < k$ one has

$${}^{\alpha}\mathbf{G}_i^{j}(T(x)) = {}^{1-\alpha}\mathbf{G}_{m-j-1-i}^{j}(x)$$

We first observe that

$$T(x) \in (t_i\,,\, t_{i+k+1}) \Leftrightarrow x \in (T(t_{i+k+1})\,,\, T(t_i)) = (t_{m-i-k-1}\,,\, t_{m-i})$$

By definition:

$$\begin{aligned}
{}^{\alpha}\mathbf{G}_i^{k}(T(x)) &= \varphi_\alpha(T(x)\,,\, t_i\,,\, t_{i+k})^{\alpha}\mathbf{G}_i^{k-1}(T(x)) \\
&+ \left[1 - \varphi_\alpha(T(x)\,,\, t_{i+1}\,,\, t_{i+k+1})\right]{}^{\alpha}\mathbf{G}_{i+1}^{k-1}(T(x))
\end{aligned}$$

By using corollary 2.4

$$\begin{array}{rcl} {}^{\alpha}\mathbf{G}_i^k(T(x)) & = & \varphi_\alpha(T(x)\,,\,t_i\,,\,t_{i+k})^{\alpha}\mathbf{G}_i^{k-1}(T(x)) \\ & + & [1-\varphi_\alpha(T(x)\,,\,t_{i+1}\,,\,t_{i+k+1})]\,{}^{\alpha}\mathbf{G}_{i+1}^{k-1}(T(x)) \\ & = & [1-\varphi_{1-\alpha}(x\,,\,T(t_{i+k})\,,\,T(t_i))]\,{}^{\alpha}\mathbf{G}_i^{k-1}(T(x)) \\ & + & \varphi_{1-\alpha}(x\,,\,T(t_{i+k+1})\,,\,T(t_{i+1}))^{\alpha}\mathbf{G}_{i+1}^{k-1}(T(x)) \\ & = & \varphi_{1-\alpha}(x\,,\,t_{m-i-k-1}\,,\,t_{m-i-1})^{\alpha}\mathbf{G}_{i+1}^{k-1}(T(x)) \\ & + & [1-\varphi_{1-\alpha}(x\,,\,t_{m-i-k}\,,\,t_{m-i})]\,{}^{\alpha}\mathbf{G}_i^{k-1}(T(x)) \end{array}$$

By using the recurrence hypothesis for $j = k-1$ we obtain:

$$\begin{array}{rcl} {}^{\alpha}\mathbf{G}_i^k(T(x)) & = & \varphi_{1-\alpha}(x\,,\,t_{m-i-k-1}\,,\,t_{m-i-1})^{\alpha}\mathbf{G}_{i+1}^{k-1}(T(x)) \\ & + & [1-\varphi_{1-\alpha}(x\,,\,t_{m-i-k}\,,\,t_{m-i})]\,{}^{\alpha}\mathbf{G}_i^{k-1}(T(x)) \\ & = & \varphi_{1-\alpha}(x\,,\,t_{m-i-k-1})\,,\,t_{m-i-1})^{1-\alpha}\mathbf{G}_{m-k-i-1}^{k-1}(x) \\ & + & [1-\varphi_{1-\alpha}(x\,,\,t_{m-i-k}\,,\,t_{m-i})]\,{}^{1-\alpha}\mathbf{G}_{m-k-i}^{k-1}(x) \\ & = & {}^{1-\alpha}\mathbf{G}_{m-k-i-1}^{k}(x) \text{ by definition} \end{array}$$

This completes the proof of the property.

□

Lemma 3.6. *Let $m,k,n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n+k+1$. Let $U = (t_i)_{i=0}^m$ be an open node vector and $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

Consider the rational B-spline basis of index α with node vector U and of degree k, $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$. For all $0 \leq r \leq k-1$ we have:

$$\begin{array}{c} {}^{\alpha}\mathbf{G}_r^{k-r}(t_0) = {}^{\alpha}\mathbf{G}_{r+1}^{k-r-1}(t_0) \\ {}^{\alpha}\mathbf{G}_{r+1}^{k-r}(t_0) = {}^{\alpha}\mathbf{G}_{r+2}^{k-r-1}(t_0) \end{array} \tag{3.2}$$

Proof.
- For $r = 0$, we have

$$\begin{array}{rcl} {}^{\alpha}\mathbf{G}_r^{k-r}(t_0) & = & {}^{\alpha}\mathbf{G}_0^k(t_0) \\ & = & \varphi_\alpha(t_0\,,\,t_0\,,\,t_k)^{\alpha}\mathbf{G}_0^{k-1}(t_0) \\ & + & [1-\varphi_\alpha(t_0\,,\,t_1\,,\,t_{k+1})]^{\alpha}\mathbf{G}_1^{k-1}(t_0) \\ & = & {}^{\alpha}\mathbf{G}_1^{k-1}(t_0) = {}^{\alpha}\mathbf{G}_{r+1}^{k-r-1}(t_0) \end{array}$$

Besides

$$\begin{array}{rcl} {}^{\alpha}\mathbf{G}_{r+1}^{k-r}(t_0) & = & {}^{\alpha}\mathbf{G}_1^k(t_0) \\ & = & \varphi_\alpha(t_0\,,\,t_1\,,\,t_{k+1})^{\alpha}\mathbf{G}_1^{k-1}(t_0) \\ & + & [1-\varphi_\alpha(t_0\,,\,t_2\,,\,t_{k+2})]^{\alpha}\mathbf{G}_2^{k-1}(t_0) \\ & = & {}^{\alpha}\mathbf{G}_2^{k-1}(t_0) = {}^{\alpha}\mathbf{G}_{r+2}^{k-r-1}(t_0) \end{array}$$

because

$$\begin{array}{c} \varphi_\alpha(t_0\,,\,t_1\,,\,t_{k+1}) = \varphi_\alpha(t_0\,,\,t_0\,,\,t_{k+1}) = 0 \\ \varphi_\alpha(t_0\,,\,t_2\,,\,t_{k+1}) = \varphi_\alpha(t_k\,,\,t_k\,,\,t_{k+1}) = 0 \end{array}$$

since U is open.

- Let $0 < r < k$.

We assume that for all $0 \leq j < r$ we have

$$\begin{array}{c} {}^{\alpha}\mathbf{G}_j^{k-j}(t_0) = {}^{\alpha}\mathbf{G}_{j+1}^{k-j-1}(t_0) \\ {}^{\alpha}\mathbf{G}_{j+1}^{k-j}(t_0) = {}^{\alpha}\mathbf{G}_{j+2}^{k-j-1}(t_0) \end{array}$$

Then

$$\begin{array}{rcl} {}^{\alpha}\mathbf{G}_r^{k-r}(t_0) & = & \varphi_\alpha(t_0\,,\,t_r\,,\,t_k)^{\alpha}\mathbf{G}_r^{k-r-1}(t_0) \\ & + & [1-\varphi_\alpha(t_0\,,\,t_{r+1}\,,\,t_{k+1})]^{\alpha}\mathbf{G}_{r+1}^{k-r-1}(t_0) \\ & = & {}^{\alpha}\mathbf{G}_{r+1}^{k-r-1}(t_0) \end{array}$$

and

$$\begin{array}{rcl} {}^{\alpha}\mathbf{G}_{r+1}^{k-r}(t_0) & = & \varphi_\alpha(t_0\,,\,t_{r+1}\,,\,t_{k+1})^{\alpha}\mathbf{G}_{r+1}^{k-r-1}(t_0) \\ & + & [1-\varphi_\alpha(t_0\,,\,t_{r+2}\,,\,t_{k+2})]^{\alpha}\mathbf{G}_{r+2}^{k-r-1}(t_0) \\ & = & {}^{\alpha}\mathbf{G}_{r+2}^{k-r-1}(t_0) \end{array}$$

because

$$\begin{array}{c} \varphi_\alpha(t_0\,,\,t_{r+1}\,,\,t_{k+1}) = \varphi_\alpha(t_0\,,\,t_0\,,\,t_{k+1}) = 0 \\ \varphi_\alpha(t_0\,,\,t_{r+2}\,,\,t_{k+2}) = \varphi_\alpha(t_k\,,\,t_{k+1}\,,\,t_{k+2}) = 0 \end{array}$$

since U is open.

The result follows.

□

Lemma 3.7. *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n + k + 1$. Let $U = (t_i)_{i=0}^{m}$ be an open node vector and $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

Consider the rational B-spline basis of index α with node vector U and of degree k, $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^{n}$. For all $0 \leq r \leq k-1$ we have

$$\begin{aligned} \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-r}(x) &= \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-r-1}(x) \\ \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-r}(x) &= \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-r-1}(x) \text{ for } k \geq 2 \end{aligned} \tag{3.3}$$

Proof. • For $r = 0$, we have

$$\begin{aligned} \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-r}(x) &= \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k}(x) \\ &= \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_n\,,\,t_{n+k}) \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-1}(x) \\ &+ \lim_{x \to t_m^-} [1 - \varphi_\alpha(x\,,\,t_{n+1}\,,\,t_m)] \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n+1}^{k-1}(x) \\ &= \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_n\,,\,t_m) \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-1}(x) \\ &= \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-1}(x) = \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-r-1}(x) \end{aligned}$$

since $\mathbf{supp}\, {}^{\alpha}\mathbf{G}_{n+1}^{k-1} = [t_{n+1}\,,\,t_m) = \emptyset$

and

$$\begin{aligned} \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-r}(x) &= \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k}(x) \\ &= \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_{n-1}\,,\,t_{n+k-1}) \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-1}(x) \\ &+ \lim_{x \to t_m^-} [1 - \varphi_\alpha(x\,,\,t_n\,,\,t_{n+k})] \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-1}(x) \\ &= \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-1}(x) = \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-r-1}(x) \end{aligned}$$

since for $k \geq 2$ one has

$$\begin{aligned} \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_{n-1}\,,\,t_{n+k-1}) &= \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_{n-1}\,,\,t_m) = 1 \\ \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_n\,,\,t_{n+k}) &= \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_n\,,\,t_m) = 1 \end{aligned}$$

.

• Let $0 < r < k$.

We suppose that for all $0 \leq j \leq r$ we have $\lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{j-r}(x) = \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{j-r-1}(x)$. Then

$$\begin{aligned} \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-r}(x) &= \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_n\,,\,t_{n+k-r}) \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-r-1}(x) \\ &+ \lim_{x \to t_m^-} [1 - \varphi_\alpha(x\,,\,t_{n+1}\,,\,t_{m-r})] \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n+1}^{k-r-1}(x) \\ &= \lim_{x \to t_m^-} \varphi_\alpha(x\,,\,t_n\,,\,t_m) \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-r-1}(x) \\ &= \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^{k-r-1}(x) \end{aligned}$$

because $\mathbf{supp}\, {}^{\alpha}\mathbf{G}_{n+1}^{k-1} = [t_{n+1}\,,\,t_{m-r}) = [t_{n+1}\,,\,t_m) = \emptyset$

The result then follows.

On the other hand we assume that for all $0 \leq j \leq r$ with $k \geq 2$, one has

$$\lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{j-r}(x) = \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{j-r-1}(x)$$

Then we get

$$
\begin{aligned}
\lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-r}(x) &= \lim_{x \to t_m^-} \varphi_\alpha(x\,,\, t_{n-1}\,,\, t_{n+k-r-1}) \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-r-1}(x) \\
&+ \lim_{x \to t_m^-} [1 - \varphi_\alpha(x\,,\, t_n\,,\, t_{n+k-r})] \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n}^{k-r-1}(x) \\
&= \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_{n-1}^{k-r-1}(x)
\end{aligned}
$$

because for $k \geq 2$ we have

$$
\begin{aligned}
\lim_{x \to t_m^-} \varphi_\alpha(x\,,\, t_{n-1}\,,\, t_{n+k-r-1}) = \lim_{x \to t_m^-} \varphi_\alpha(x\,,\, t_{n-1}\,,\, t_m) = 1 \\
\lim_{x \to t_m^-} \varphi_\alpha(x\,,\, t_n\,,\, t_{n+k-r}) = \lim_{x \to t_m^-} \varphi_\alpha(x\,,\, t_n\,,\, t_m) = 1
\end{aligned}
$$

□

Proposition 3.1 (Continuity property). *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n + k + 1$. Let $U = (t_i)_{i=0}^m$ be a vector of nodes, let $\alpha \in (-\infty\,,\, 0) \cup (1\,,\, \infty)$.*

Consider the rational B-spline basis of index α, with node vector U and of degree k, $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$. The following properties hold:

1. *For all $i = 0, \ldots, n$, ${}^{\alpha}\mathbf{G}_i^k$ is a piecewise rational function.*
2. *For all $i = 0, \ldots, n$, ${}^{\alpha}\mathbf{G}_i^k$ is of class $\mathcal{C}^0$ if the nodes vector U does not have any interior nodes with multiplicity strictly greater than k.*
3. *If the node vector U is open we have*

$$
\begin{aligned}
{}^{\alpha}\mathbf{G}_0^k(t_0) &= 1 \\
{}^{\alpha}\mathbf{G}_i^k(t_0) &= 0 \text{ for all } 0 < i \leq n \\
{}^{\alpha}\mathbf{G}_i^k(t_m) &\equiv \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_i^k(x) = 0 \text{ for all } 0 \leq i < n \\
{}^{\alpha}\mathbf{G}_n^k(t_m) &\equiv \lim_{x \to t_m^-} {}^{\alpha}\mathbf{G}_n^k(x) = 1
\end{aligned}
$$

Proof. Let $n, k \in \mathbb{N}^*$ such that $n \geq k$, let $m = n + k + 1$ and $U = (t_i)_{i=0}^m$ be a node vector. Let t_i be an interior node with multiplicity m_i. Assume that $1 \leq m_i \leq k$

1. We shall show simultaneously the two properties by recurrence on the degree k
2. We make use of the recurrence for $k \geq 1$.
   - For $k = 1$, we suppose a multiplicity $m_i = 1$ for all interior node t_i.

$$
\begin{aligned}
{}^{\alpha}\mathbf{G}_i^1(x) &= \varphi_\alpha(x\,,\, t_i\,,\, t_{i+1}) {}^{\alpha}\mathbf{G}_i^0(x) + [1 - \varphi_\alpha(x\,,\, t_{i+1}\,,\, t_{i+2})] {}^{\alpha}\mathbf{G}_{i+1}^0(x) \\
&= \begin{cases} \varphi_\alpha(x\,,\, t_i\,,\, t_{i+1}) & \text{if } x \in [t_i\,,\, t_{i+1}) \neq \emptyset \\ 1 - \varphi_\alpha(x\,,\, t_{i+1}\,,\, t_{i+2}) & \text{if } x \in [t_i\,,\, t_{i+1}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Since $x \in [t_i\,,\, t_{i+1}) \mapsto \varphi_\alpha(x\,,\, t_i\,,\, t_{i+1})$ is homographic on $[t_i\,,\, t_{i+1}) \neq \emptyset$ then ${}^{\alpha}\mathbf{G}_i^1$ is rational on $[t_i\,,\, t_{i+1}) \neq \emptyset$ and $[t_{i+1}\,,\, t_{i+2}) \neq \emptyset$ as well. We then deduce that ${}^{\alpha}\mathbf{G}_i^1$ is $\mathcal{C}^\infty$ on $[t_i\,,\, t_{i+1}) \neq \emptyset$ and also on $[t_{i+1}\,,\, t_{i+2}) \neq \emptyset$. Let show that ${}^{\alpha}\mathbf{G}_i^1$ is continuous at the nodes t_i, t_{i+1} et t_{i+2}

$$
\begin{aligned}
\lim_{x \to t_i^-} {}^{\alpha}\mathbf{G}_i^1(x) &= 0 \text{ because } x \notin (t_i\,,\, t_{i+2}) \\
\lim_{x \to t_i^+} {}^{\alpha}\mathbf{G}_i^1(x) &= \lim_{x \to t_i^+} \varphi_\alpha(x\,,\, t_i\,,\, t_{i+1}) = 0 \text{ if } [t_i\,,\, t_{i+1}) \neq \emptyset \\
&= {}^{\alpha}\mathbf{G}_i^1(t_i) \\
\lim_{x \to t_{i+1}^-} {}^{\alpha}\mathbf{G}_i^1(x) &= \lim_{x \to t_{i+1}^-} \varphi_\alpha(x\,,\, t_i\,,\, t_{i+1}) = 1 \text{ if } [t_i\,,\, t_{i+1}) \neq \emptyset \\
\lim_{x \to t_{i+1}^+} {}^{\alpha}\mathbf{G}_i^1(x) &= \lim_{x \to t_{i+1}^+} [1 - \varphi_\alpha(x\,,\, t_{i+1}\,,\, t_{i+2})] = 1 \\
&\quad \text{if } [t_{i+1}\,,\, t_{i+2}) \neq \emptyset \\
&= {}^{\alpha}\mathbf{G}_i^1(t_{i+1}) \\
\lim_{x \to t_{i+2}^-} {}^{\alpha}\mathbf{G}_i^1(x) &= \lim_{x \to t_{i+2}^-} [1 - \varphi_\alpha(x\,,\, t_{i+1}\,,\, t_{i+2})] = 0 \\
&\quad \text{if } [t_{i+1}\,,\, t_{i+2}) \neq \emptyset \\
\lim_{x \to t_{i+2}^+} {}^{\alpha}\mathbf{G}_i^1(x) &= {}^{\alpha}\mathbf{G}_i^1(t_{i+2}) = 0 \text{ because } x \notin (t_i\,,\, t_{i+2}) \neq \emptyset
\end{aligned}
$$

We conclude that ${}^\alpha\mathbf{G}_i^1$ is piecewise rational and of class $\mathcal{C}^0$.

- For $k>1$ we suppose a multiplicity $1 \le m_i \le k$ for all interior node t_i.
  Suppose that for all $1 \le j < k$ ${}^\alpha\mathbf{G}_i^j$ is piecewise rational and of class $\mathcal{C}^0$. Let us show that ${}^\alpha\mathbf{G}_i^k$ is piecewise rational and of class $\mathcal{C}^0$ on $[t_0\,,\,t_m]$.
  By definition we know that

$$\begin{aligned} {}^\alpha\mathbf{G}_i^k &= \varphi_\alpha(x\,,\,t_i\,,\,t_{i+k})^\alpha\mathbf{G}_i^{k-1}(x) \\ &+ [1-\varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+k+1})]^\alpha\mathbf{G}_{i+1}^{k-1}(x) \end{aligned}$$

  Thus ${}^\alpha\mathbf{G}_i^k$ is piecewise rational as product and sum of piecewise rational functions. As the ${}^\alpha\mathbf{G}_i^{k-1}$ are $\mathcal{C}^0$ on $[t_0\,,\,t_m)$ and if the multiplicity of interior nodes is at most k,

$$\begin{aligned} &x \mapsto \varphi_\alpha(x\,,\,t_i\,,\,t_{i+k}) \text{ is continuous on } [t_0\,,\,t_{k+i}) \cup (t_{k+i}\,,\,t_m) \\ &x \mapsto \varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+k+1}) \text{ is continuous on } [t_0\,,\,t_{k+i+1}) \cup (t_{k+i+1}\,,\,t_m) \end{aligned}$$

  with

$$\begin{aligned} &\lim_{x\to t_{i+k}^-} \varphi_\alpha(x\,,\,t_i\,,\,t_{i+k}) = 1 \\ &\lim_{x\to t_{i+k}^+} \varphi_\alpha(x\,,\,t_i\,,\,t_{i+k}) = 0 \\ &\lim_{x\to t_{i+k+1}^-} \varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+k+1}) = 1 \\ &\lim_{x\to t_{i+k+1}^+} \varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+k+1}) = 0 \end{aligned}$$

  then ${}^\alpha\mathbf{G}_i^k$ is continuous on $[t_0\,,\,t_{k+i}) \cup (t_{k+i}\,,\,t_m)$ since

$$\begin{aligned} &\mathbf{supp}\,{}^\alpha\mathbf{G}_i^{k-1} \cap (t_{k+i+1}\,,\,t_m) = \emptyset \\ &\mathbf{supp}\,{}^\alpha\mathbf{G}_{i+1}^{k-1} \cap (t_{k+i+1}\,,\,t_m) = \emptyset \end{aligned}$$

  It is left with checking the continuity at t_{k+i}, which is obvious.
  We can conclude that ${}^\alpha\mathbf{G}_i^k$ is of class $\mathcal{C}^0$ on $[t_0\,,\,t_m)$

3. For the endpoints values of the node vector U, we have

$$\begin{aligned} &{}^\alpha\mathbf{G}_k^0(t_0) = {}^\alpha\mathbf{G}_k^0(t_k) = 1 \\ &\lim_{x\to t_m^-} {}^\alpha\mathbf{G}_n^0(x) = \lim_{x\to t_{n+1}^-} {}^\alpha\mathbf{G}_n^0(x) = 1 \end{aligned}$$

By using successively, for $r=0$ and $r=k-1$, the recurrence 3.2 of lemma 3.6 and the recurrence 3.3 of lemma 3.7, one can deduce that:

$$\begin{aligned} &{}^\alpha\mathbf{G}_0^k(t_0) = {}^\alpha\mathbf{G}_k^0(t_0) = {}^\alpha\mathbf{G}_k^0(t_k) = 1 \\ &\lim_{x\to t_m^-} {}^\alpha\mathbf{G}_n^k(x) = \lim_{x\to t_m^-} {}^\alpha\mathbf{G}_n^0(x) = \lim_{x\to t_{n+1}^-} {}^\alpha\mathbf{G}_n^0(x) = 1 \end{aligned}$$

From the property of unit partition, we have

$$\sum_{i=0}^{n} {}^\alpha\mathbf{G}_i^k(x) = 1 \quad \forall x \in [t_0\,,\,t_m) = [t_k\,,\,t_{n+1})$$

Thus

$$\begin{aligned} &\sum_{i=1}^{n} {}^\alpha\mathbf{G}_i^k(t_0) = 0 \\ &\sum_{i=0}^{n-1} \left(\lim_{x\to t_m^-} {}^\alpha\mathbf{G}_i^k(x) \right) = \lim_{x\to t_m^-} \sum_{i=0}^{n-1} {}^\alpha\mathbf{G}_i^k(x) = 0 \end{aligned}$$

From the fact that the ${}^\alpha\mathbf{G}_i^k$ are positive, we obtain

$$\begin{aligned} &{}^\alpha\mathbf{G}_i^k(t_0) = 0 && \text{for all } i = 1, \ldots, n \\ &\lim_{x\to t_m^-} {}^\alpha\mathbf{G}_i^k(x) = 0 && \text{for all } i = 0, \ldots, n-1 \end{aligned}$$

Each ${}^{\alpha}\mathbf{G}_i^k$ admits a continuous extension at t_m

□

Using Lemmas 3.6, 3.7 and the Proposition 3.1, we obtain the following lemma:

Lemma 3.8. *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n + k + 1$. Let $U = (t_i)_{i=0}^m$ be an open node vector and $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

Consider the rational B-spline basis $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ of index α with node vector U and of degree k. For all $0 \leq r \leq k-1$ and all $i \geq 2$ we have:

$$\begin{aligned}\lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_r^{k-r}(x) &= \lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{r+1}^{k-r-1}(x) - \lim_{x\to t_0^+}\frac{d}{dx}w_{r+1}^{k-r}(x)\\ \lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{r+1}^{k-r}(x) &= \lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{r+2}^{k-r-1}(x) + \lim_{x\to t_0^+}\frac{d}{dx}w_{r+1}^{k-r}(x)\\ \lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{i+r}^{k-r}(x) &= \lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{i+r+1}^{k-r-1}(x)\end{aligned} \tag{3.4}$$

with $w_i^j(x) = \varphi_\alpha(x\,,\,t_i\,,\,t_{i+j})$

By the Lemma 3.8, we easily proof the regularity result given by the following lemmas:

Lemma 3.9. *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n + k + 1$. Let $U = (t_i)_{i=0}^m$ be an open node vector and $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

Consider the rational B-spline basis $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ of index α with node vector U and of degree k. For all $0 \leq r \leq k-1$ we have:

$$\begin{aligned}\lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_n^{k-r}(x) &= \lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_n^{k-r-1}(x) + \lim_{x\to t_m^-}\frac{d}{dx}w_n^{k-r}(x)\\ \lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{n-1}^{k-r}(x) &= \lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{n-1}^{k-r-1}(x) - \lim_{x\to t_m^-}\frac{d}{dx}w_n^{k-r}(x)\end{aligned} \tag{3.5}$$

with $w_i^j(x) = \varphi_\alpha(x\,,\,t_i\,,\,t_{i+j})$

Lemma 3.10. *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n + k + 1$. Let $U = (t_i)_{i=0}^m$ be an open node vector, let $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

Consider the rational B-spline basis $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ of index α, U as a vector of nodes and of degree k. For all $i \leq n-2$, $k \geq 2$ we have:

$$\lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x) = 0 \tag{3.6}$$

with $w_i^j(x) = \varphi_\alpha(x\,,\,t_i\,,\,t_{i+j})$

Theorem 3.11 (Regularity property)**.** *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n + k + 1$. Let $U = (t_i)_{i=0}^m$ be a vector of nodes, let $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

Consider the rational B-spline $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ of index α, U as node vector and of degree k. We have the following properties:

1. *For all $i = 0, \ldots, n$, ${}^{\alpha}\mathbf{G}_i^k$ is of class $\mathcal{C}^\infty$ on all $(t_j\,,\,t_{j+1})$ if $t_j < t_{j+1}$.*
2. *For all $i = 0, \ldots, n$, ${}^{\alpha}\mathbf{G}_i^k$ is left and right differentiable at all t_j for all j.*
3. *If U is an open node vector then we have*

   (a)

$$\begin{aligned}\lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_0^k(x) &= -\lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_1^k(x)\\ &= -\frac{\alpha k}{(\alpha-1)(t_{k+1}-t_0)}\\ \lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x) &= 0 \text{ for all } 2 \leq i \leq n\end{aligned}$$

   (b)

$$\begin{aligned}\lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_n^k(x) &= -\lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{n-1}^k(x)\\ &= \frac{(\alpha-1)k}{\alpha(t_m-t_n)}\\ \lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x) &= 0 \text{ for all } 0 \leq i \leq n-2\end{aligned}$$

By definition, for all $0 \le i \le n$,

$$\begin{aligned} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(t_0) &= \lim_{x \to t_0^+} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x) \\ \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(t_m) &= \lim_{x \to t_m^-} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x) \end{aligned}$$

.

Proof. 1. $\mathcal{C}^\infty$ regularity except on the nodes is a consequence of the fact that ${}^{\alpha}\mathbf{G}_i^k$ is picewise rational function, as stated in proposition 3.1 on continuity property.

2. The basis functions ${}^{\alpha}\mathbf{G}_i^k$ are of $\mathcal{C}^0$ on $[t_0\,,\,t_m]$ and $\mathcal{C}^1$ on $\bigcup_{i=0}^{m-1}(t_i\,,\,t_{i+1})$. It is sufficient to prove that for all $i = 0,\ldots,n$ and all $j = 0,\ldots,m-1$ such that $t_j < t_{j+1}$, we have $\lim_{x\to t_j^+} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x) \in \mathbf{R}$ and $\lim_{x\to t_{j+1}^-} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x) \in \mathbf{R}$.

We will proceed by recurrence on k.

- Let $k = 1$. Assume a multiplicity$m_i = 1$ for all interior node t_i. Thus

$$ {}^{\alpha}\mathbf{G}_i^1(x) = \begin{cases} \varphi_\alpha(x\,,\,t_i\,,\,t_{i+1}) & \text{if } x \in [t_i\,,\,t_{i+1}) \neq \emptyset \\ 1 - \varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+2}) & \text{if } x \in [t_{i+1}\,,\,t_{i+2}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} $$

  One deduces that

$$ \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^1(x) = \begin{cases} \frac{d}{dx}\varphi_\alpha(x\,,\,t_i\,,\,t_{i+1}) & \text{if } x \in (t_i\,,\,t_{i+1}) \neq \emptyset \\ -\frac{d}{dx}\varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+2}) & \text{if } x \in (t_{i+1}\,,\,t_{i+2}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} $$

  From this we obtain:

$$\begin{aligned} \lim_{x\to t_i^-} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^1(x) &= 0 \\ \lim_{x\to t_i^+} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^1(x) &= \lim_{x\to t_i^+} \frac{d}{dx}\varphi_\alpha(x\,,\,t_i\,,\,t_{i+1}) \\ &= \frac{\alpha}{(\alpha-1)\,(t_{i+1}-t_i)} \in \mathbf{R} \\ \lim_{x\to t_{i+1}^-} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^1(x) &= \lim_{x\to t_{i+1}^-} \frac{d}{dx}\varphi_\alpha(x\,,\,t_i\,,\,t_{i+1}) \\ &= \frac{\alpha-1}{\alpha\,(t_{i+1}-t_i)} \in \mathbf{R} \\ \lim_{x\to t_{i+1}^+} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^1(x) &= -\lim_{x\to t_{i+1}^+} \frac{d}{dx}\varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+2}) \\ &= -\frac{\alpha}{(\alpha-1)\,(t_{i+2}-t_{i+1})} \in \mathbf{R} \\ \lim_{x\to t_{i+2}^-} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^1(x) &= -\lim_{x\to t_{i+2}^-} \frac{d}{dx}\varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+2}) \\ &= -\frac{\alpha-1}{\alpha\,(t_{i+2}-t_{i+1})} \in \mathbf{R} \\ \lim_{x\to t_{i+2}^+} \frac{d}{dx}{}^{\alpha}\mathbf{G}_i^1(x) &= 0 \end{aligned}$$

  We can conclude that ${}^{\alpha}\mathbf{G}_i^1$ is left and right differentiable at any point if U only admits interior points of multiplicity 1.

- Let $k > 1$ and suppose that for all $1 \le s \le k-1$ and all $i = 0,\ldots,m-s-1$ ${}^{\alpha}\mathbf{G}_i^s$ is left and right differentiable at all node of multiplicity at most s.
  As for all $x \in \mathbf{R}$

$$\begin{aligned} {}^{\alpha}\mathbf{G}_i^k(x) &= \varphi_\alpha(x\,,\,t_i\,,\,t_{i+k})^{\alpha}\mathbf{G}_i^{k-1}(x) \\ &+ (1-\varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{i+k+1}))\,{}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) \end{aligned}$$

  then if for all i ${}^{\alpha}\mathbf{G}_i^{k-1}$ is left and right differentiable at a certain node t_j, ${}^{\alpha}\mathbf{G}_i^k$ is also left differentiable at t_j as product and sum of left differentiable functions at t_j because from remark **??**, all $\varphi_\alpha(.\,,\,t_i\,,\,t_{i+k})$ is left and right differentiable at any point of $\mathbf{R}$

It is also the case for the right differentiability.

3. (a) Using lemma 3.8 one can prove that:
   - on one hand,

$$\begin{aligned}\lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_0^k(x) &= \lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_k^0(x)-\sum_{i=0}^{k-1}\lim_{x\to t_0^+}\frac{d}{dx}w_{i+1}^{k-i}(x)\\ &= -\sum_{i=0}^{k-1}\lim_{x\to t_0^+}\frac{d}{dx}\varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{k+1})\\ &= -\sum_{i=0}^{k-1}\lim_{x\to t_0^+}\frac{d}{dx}\varphi_\alpha(x\,,\,t_0\,,\,t_{k+1})\\ &= -k\frac{\alpha}{(\alpha-1)\,(t_{k+1}-t_0)}\end{aligned}$$

   - on other hand

$$\begin{aligned}\lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_1^k(x) &= \lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{k+1}^0(x)+\sum_{i=0}^{k-1}\lim_{x\to t_0^+}\frac{d}{dx}w_{i+1}^{k-i}(x)\\ &= \sum_{i=0}^{k-1}\lim_{x\to t_0^+}\frac{d}{dx}\varphi_\alpha(x\,,\,t_{i+1}\,,\,t_{k+1})\\ &= \sum_{i=0}^{k-1}\lim_{x\to t_0^+}\frac{d}{dx}\varphi_\alpha(x\,,\,t_0\,,\,t_{k+1})\end{aligned}$$

   - and finally for $i\geq 2$ we obtain

$$\lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x)=\lim_{x\to t_0^+}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{i+k}^0(x)=0$$

     because $\mathbf{supp}\,{}^{\alpha}\mathbf{G}_{i+k}^0\cap[t_0\,,\,t_{k+1})=\emptyset$

   (b) Similarly by using lemma 3.9 one shows that:
   - from one hand,

$$\begin{aligned}\lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_n^k(x) &= \lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_n^0(x)+\sum_{i=0}^{k-1}\lim_{x\to t_m^-}\frac{d}{dx}w_n^{k-i}(x)\\ &= \sum_{i=0}^{k-1}\lim_{x\to t_m^-}\frac{d}{dx}\varphi_\alpha(x\,,\,t_n\,,\,t_{n+k-i})\\ &= \sum_{i=0}^{k-1}\lim_{x\to t_m^-}\frac{d}{dx}\varphi_\alpha(x\,,\,t_n\,,\,t_m)\\ &= k\frac{\alpha-1}{\alpha\,(t_m-t_n)}\end{aligned}$$

   - On another hand, we have

$$\begin{aligned}\lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{n-1}^k(x) &= \lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_{n-1}^0(x)-\sum_{i=0}^{k-1}\lim_{x\to t_m^-}\frac{d}{dx}w_n^{k-i}(x)\\ &= -\sum_{i=0}^{k-1}\lim_{x\to t_m^-}\frac{d}{dx}\varphi_\alpha(x\,,\,t_n\,,\,t_{n+k-i})\\ &= -\sum_{i=1}^{k}\lim_{x\to t_m^-}\frac{d}{dx}\varphi_\alpha(x\,,\,t_n\,,\,t_m)\\ &= -k\frac{\alpha-1}{\alpha\,(t_m-t_n)}\end{aligned}$$

   - Finally for $i\leq n-2$ by directly applying lemma 3.10 we have:

$$\lim_{x\to t_m^-}\frac{d}{dx}{}^{\alpha}\mathbf{G}_i^k(x)=0$$

□

Remarque 3.1. *As shown by the illustrations of appendix, for $k \geq 1$ the functions $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ are not of class $\mathcal{C}^1$, even when the nodes are of multiplicity 1, this perfectly contradicts the classical results [4] page 57.*

Conjecture 3.1 (Existence property and unicity of a maximum). *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n+k+1$. Let $U = (t_i)_{i=0}^m$ be a node vectors, let $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

Any element of the rational B-spline $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ of index α with node vector U and of degree k admits one and only one maximum.

Remarque 3.2. *We admit for any useful purpose this conjecture which is widely illustrated by numerical experience and cited in classical review [3] to the page 58 and [4] to the page 45.*

Proposition 3.2 (Linear independence property). *Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n+k+1$. Let $U = (t_i)_{i=0}^m$ be an open node vector with interior nodes of multiplicity at most k, let $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.*

The rational B-spline basis $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ of index α with node vector U and of degree k is a free system in the vector space $\mathcal{C}^0([t_0\,,\,t_m])$ of continuous functions on $[t_0\,,\,t_m]$.

Proof. To show that the B-spline basis $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ is linear independent, we will proceed by recurrence on the degree k.

- Let $k = 1$ we search $(\lambda_i)_{i=0}^{m-k-1} \subset \mathbf{R}$ such that $\displaystyle\sum_{i=0}^{m-k-1} \lambda_i {}^{\alpha}\mathbf{G}_i^k = 0$

  Let $x \in [t_0\,,\,t_m]$ by setting $w_i^r(x) = \varphi_\alpha(x\,,\,t_i\,,\,t_{i+r})$

$$\begin{aligned}
0 &= \sum_{i=0}^{m-k-1} \lambda_i {}^{\alpha}\mathbf{G}_i^k(x) = \sum_{i=0}^{m-2} \lambda_i {}^{\alpha}\mathbf{G}_i^1(x) \\
&= \sum_{i=0}^{m-2} \lambda_i w_i^1(x) {}^{\alpha}\mathbf{G}_i^0(x) \\
&+ \sum_{i=0}^{m-2} \lambda_i \big(1 - w_{i+1}^1(x)\big) {}^{\alpha}\mathbf{G}_{i+1}^0(x) \\
&= \lambda_0 w_0^1(x) {}^{\alpha}\mathbf{G}_0^0(x) + \lambda_{m-2}\big(1 - w_{m-1}^1(x)\big) {}^{\alpha}\mathbf{G}_{m-1}^0(x) \\
&+ \sum_{i=1}^{m-2} \left[\lambda_i w_i^1(x) + \lambda_{i-1}\big(1 - w_i^1(x)\big)\right] {}^{\alpha}\mathbf{G}_i^0(x) \\
&= \sum_{i=1}^{m-2} \left[\lambda_i w_i^1(x) + \lambda_{i-1}\big(1 - w_i^1(x)\big)\right] {}^{\alpha}\mathbf{G}_i^0(x)
\end{aligned}$$

  since U is open and

$$\begin{aligned}
&\mathbf{supp}\, w_0^1 = [t_0\,,\,t_1) = \emptyset \\
&\mathbf{supp}\, w_{m-1}^1 = [t_{m-1}\,,\,t_m) = \emptyset
\end{aligned}$$

  As the interior nodes of U are of multiplicity at most $k = 1$ then for all $1 \leq j \leq m-2$ $[t_j\,,\,t_{j+1}) \neq \emptyset$.

  Thus for all $1 \leq j \leq m-2$ and all $x \in [t_j\,,\,t_{j+1})$ we have

$$\begin{aligned}
0 &= \sum_{i=1}^{m-2} \left[\lambda_i w_i^1(x) + \lambda_{i-1}\big(1 - w_i^1(x)\big)\right] {}^{\alpha}\mathbf{G}_i^0(x) \\
&= \lambda_j w_j^1(x) + \lambda_{j-1}\big(1 - w_j^1(x)\big)
\end{aligned}$$

  Moreover we have $0 = \displaystyle\sum_{i=0}^{m-2} \lambda_i {}^{\alpha}\mathbf{G}_i^1(t_0) = \lambda_0$

  All in all we get this linear system:

$$\left\{\begin{aligned}
\lambda_0 &= 0 \\
\lambda_{j-1}\big(1 - w_j^1(x_j)\big) + \lambda_j w_j^1(x_j) &= 0 \text{ for } j = 1, \ldots, m-2 \\
&\quad\ \text{and } x_j \in \,]t_j\,,\,t_{j+1}[
\end{aligned}\right.$$

  where $w_j^1(x_j > 0$ and $1 - w_j^1(x_j > 0$ for all $1 \leq j \leq m-2$. Since the system is lower-triangular with null diagonal terms and homogeneous then we have $\lambda_j = 0$ for all $j = 0, \ldots, m-2$. We conclude that $\left({}^{\alpha}\mathbf{G}_i^1\right)_{i=0}^{m-2}$ is a free system.

- let $k > 1$ and suppose that for all $1 \leq p \leq k-1$ $\left({}^{\alpha}\mathbf{G}_i^p\right)_{i=0}^{m-p-1}$ is a free system. Let show that $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^{m-k-1}$ is a free system.

$$\begin{aligned}
0 &= \sum_{i=0}^{m-k-1} \lambda_i {}^{\alpha}\mathbf{G}_i^k(x) \\
&= \sum_{i=0}^{m-k-1} \lambda_i w_i^k(x) {}^{\alpha}\mathbf{G}_i^{k-1}(x) \\
&+ \sum_{i=0}^{m-k-1} \lambda_i \big(1 - w_{i+1}^k(x)\big) {}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) \\
&= \lambda_0 w_0^k(x) {}^{\alpha}\mathbf{G}_0^{k-1}(x) + \lambda_{m-k-1}\big(1 - w_{m-k}^k(x)\big) {}^{\alpha}\mathbf{G}_{m-k}^{k-1}(x) \\
&+ \sum_{i=1}^{m-k-1} \left[\lambda_i w_i^k(x) + \lambda_{i-1}\big(1 - w_i^k(x)\big)\right] {}^{\alpha}\mathbf{G}_i^{k-1}(x) \\
&= \sum_{i=1}^{m-k-1} \left[\lambda_i w_i^k(x) + \lambda_{i-1}\big(1 - w_i^k(x)\big)\right] {}^{\alpha}\mathbf{G}_i^{k-1}(x)
\end{aligned}$$

since U is open and

$$\begin{aligned}
\mathbf{supp}\, w_0^k &= [t_0\,,\, t_k) = \emptyset \\
\mathbf{supp}\, w_{m-k}^k &= [t_{m-k}\,,\, t_m) = \emptyset
\end{aligned}$$

As by hypothesis $\left({}^{\alpha}\mathbf{G}_i^{k-1}\right)_{i=0}^{m-k}$ is a free system and the multiplicity of a node of U is at most k, then for all $1 \leq j \leq m-k-1$ and all $x_j \in (t_j\,,\, t_{j+k}) \neq \emptyset$ we have $\lambda_j w_j^k(x_j) + \lambda_{j-1}\big(1 - w_j^k(x_j)\big) = 0$ with $w_j^k(x_j) > 0$ and $1 - w_j^k(x_j) > 0$.

Moreover we have $0 = \displaystyle\sum_{i=0}^{m-k-1} \lambda_i {}^{\alpha}\mathbf{G}_i^k(t_0) = \lambda_0$

We then obtain the following linear system:

$$\left\{\begin{array}{rcl}
\lambda_0 &=& 0 \\
\lambda_{j-1}\big(1 - w_j^k(x_j)\big) + \lambda_j w_j^k(x_j) &=& 0 \text{ for } j = 1, \ldots, m-k-1 \\
&& \text{ and } x_j \in \,]t_j\,,\, t_{j+1}[
\end{array}\right.$$

This lower-triangular system with positive diagonal terms admits a unique solution $\lambda_j = 0$ for all $0 \leq j \leq m-k-1$. Hence $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^{m-k-1}$ is free.

□

3.2 Case of an open node vector with no interior node

Proposition 3.3. *Let $a, b \in \mathbf{R}$ such that $a < b$. Let $m, k, n \in \mathbf{N}^*$ such that $n = k$ and $m = 2k+1$. Let $U_k = \left(t_i^k\right)_{i=0}^{2k+1}$ be the open node vector such that $t_k^k = a$ and $t_{k+1}^k = b$ let $\alpha \in (-\infty\,,\, 0) \cup (1\,,\, \infty)$.*

Let $\left({}^{\alpha}\mathbf{B}_i^k\right)_{i=0}^{k}$ be the rational B-spline basis of index α with node vectors U_k and of degree k, let $\left({}^{\alpha}\mathbf{B}_i^{k-1}\right)_{i=0}^{k-1}$ be the rational B-spline basis of index α with node vectors U_{k-1} and of degree $k-1$.

For all $x \in [a\,,\, b]$ and by setting $w(x) = \varphi_\alpha(x\,,\, a\,,\, b)$ we have the following:

1. Recurrence relation
$$ {}^{\alpha}\mathbf{B}_i^k(x) = w(x) {}^{\alpha}\mathbf{B}_{i-1}^{k-1}(x) + (1 - w(x)) {}^{\alpha}\mathbf{B}_i^{k-1}(x) \tag{3.7}$$

2. Explicit formula
$$ {}^{\alpha}\mathbf{B}_i^k(x) = C_k^i (w(x))^i (1 - w(x))^{k-i}$$

By definition $\left({}^{\alpha}\mathbf{B}_i^k\right)_{i=0}^{k}$ will be called Bernstein basis of index α and of degree k on the parametrization space $[a\,,\, b]$.

Proof. 1. *Recurrence relation*

Consider the open node vectors:
$U_k = \left(t_i^k\right)_{i=0}^{2k+1}$ and $U_{k-1} = \left(t_i^{k-1}\right)_{i=0}^{2k-1}$ satisfy

$$\begin{array}{lcl} t_k^k = a & \text{and} & t_{k+1}^k = b \\ t_{k-1}^{k-1} = a & \text{and} & t_k^{k-1} = b \end{array}$$

Let $g_k : i \in \mathbf{Z} \mapsto g_k(i) = i - 1 \in \mathbf{Z}$. Based on this bijection, we have

$$t_i^k = t_{g_k(i)}^{k-1} \quad \forall i = 0, \ldots, 2k+1$$

by imposing $t_0^k = t_{-1}^{k-1} = t_0^{k-1}$ and $t_{2k+1}^k = t_{2k}^{k-1}$.
Thus U_k is seen as a natural extension of U_{k-1}.

Consider the family $\left({}^{\alpha}\mathbf{G}_i^j\right)_{i=0}^{2k-j}$ of B-spline basis of index α with node vector U_k and of degree j with $0 \leq j \leq k$.

Let $\left({}^{\alpha}\mathbf{B}_i^k\right)_{i=0}^{k}$ be the B-spline basis of index α with node vector U_k and of degree k.

Let $\left({}^{\alpha}\mathbf{B}_i^{k-1}\right)_{i=0}^{k-1}$ be the B-spline basis of index α with node vector U_{k-1} and of degree $k-1$.
From the definition, for all $i = 0, \ldots, k$ and all $x \in [a\,,\, b]$ we have

$$\begin{array}{rcl} {}^{\alpha}\mathbf{B}_i^k(x) & = & {}^{\alpha}\mathbf{G}_i^k(x) \\ & = & w_i^k(x)\,{}^{\alpha}\mathbf{G}_i^{k-1}(x) + \left(1 - w_{i+1}^k(x)\right){}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x) \end{array}$$

${}^{\alpha}\mathbf{G}_i^{k-1}$ is of degree $k-1$ respect to the node vector U_k which is an extension of the node vector U_{k-1}.
Relative to the node vector U_{k-1} by imposing

$${}^{\alpha}\mathbf{B}_{-1}^{k-1} = {}^{\alpha}\mathbf{B}_k^{k-1} \equiv 0$$

we have for all $i = 0, \ldots, k+1$

$${}^{\alpha}\mathbf{G}_i^{k-1} = {}^{\alpha}\mathbf{B}_{g_k(i)}^{k-1} = {}^{\alpha}\mathbf{B}_{i-1}^{k-1}$$

Thus we have

$${}^{\alpha}\mathbf{B}_i^k(x) = w_i^k(x)\,{}^{\alpha}\mathbf{B}_{i-1}^{k-1}(x) + \left(1 - w_{i+1}^k(x)\right){}^{\alpha}\mathbf{B}_i^{k-1}(x)$$

As

$$\begin{array}{rcl} w_i^k(x) & = & \varphi_{\alpha}(x\,,\, t_i\,,\, t_{i+k}) \\ & = & \left\{\begin{array}{rl} \varphi_{\alpha}(x\,,\, a\,,\, b) & \text{if } 1 \leq i \leq k \\ 0 & \text{otherwise} \end{array}\right. \end{array}$$

we can set $w(x) = \varphi_{\alpha}(x\,,\, a\,,\, b)$ and obtain for all $k \in \mathbb{N}^*$ and all $0 \leq i \leq k$, the recurrence relation

$${}^{\alpha}\mathbf{B}_i^k(x) = w(x)\,{}^{\alpha}\mathbf{B}_{i-1}^{k-1}(x) + (1 - w(x))\,{}^{\alpha}\mathbf{B}_i^{k-1}(x)$$

2. *Explicit formula*

We will now show that the recurrence relation 3.7 leads to

$$\left\{\begin{array}{rcl} {}^{\alpha}\mathbf{B}_0^k(x) & = & (1 - w(x))^k \\ {}^{\alpha}\mathbf{B}_k^k(x) & = & (w(x))^k \\ {}^{\alpha}\mathbf{B}_i^k(x) & = & C_k^i\,(w(x))^i\,(1 - w(x))^{k-i} \text{ for } 1 \leq i \leq k-1 \end{array}\right.$$

- For all $k \in \mathbb{N}^*$, if $i = 0$ then the equation 3.7 becomes

$${}^{\alpha}\mathbf{B}_0^k(x) = (1 - w(x))\,{}^{\alpha}\mathbf{B}_0^{k-1}(x)$$

The sequence $\left({}^{\alpha}\mathbf{B}_0^k(x)\right)_{k\geq 0}$ is geometric with common ratio $1 - w(x)$. We deduce that

$${}^{\alpha}\mathbf{B}_0^k(x) = (1 - w(x))^k\,{}^{\alpha}\mathbf{B}_0^0(x) = (1 - w(x))^k$$

since ${}^{\alpha}\mathbf{B}_0^0(x) = {}^{\alpha}\mathbf{G}_0^0(x) = 1$ for all $x \in [a\,,\, b)$.

We remark that for all $x \in (a\,,\,b)$ ${}^{\alpha}\mathbf{B}_0^k(x) = C_k^0\,(w(x))^0\,(1-w(x))^k$ since $C_k^0 = 1$, $w(x) > 0$ and $1 - w(x) > 0$

- For all $k \in \mathbb{N}^*$, if $i = k$ then the equation 3.7 gives

$$ {}^{\alpha}\mathbf{B}_k^k(x) = (w(x))\,{}^{\alpha}\mathbf{B}_{k-1}^{k-1}(x) $$

The sequence $\left({}^{\alpha}\mathbf{B}_k^k(x)\right)_{k\geq 0}$ is geometric with common ratio $w(x)$. We deduce that

$$ {}^{\alpha}\mathbf{B}_k^k(x) = (w(x))^k\,{}^{\alpha}\mathbf{B}_0^0(x) = (w(x))^k $$

As previously we observe that for $x \in (a\,,\,b)$ ${}^{\alpha}\mathbf{B}_k^k(x) = C_k^k\,(w(x))^k\,(1-w(x))^0$ because $C_k^k = 1$

- For all $k \in \mathbb{N}^*$, if $1 \leq i < k$ then the equation 3.7 gives

$$ {}^{\alpha}\mathbf{B}_i^k(x) = (w(x))\,{}^{\alpha}\mathbf{B}_{i-1}^{k-1}(x) + (1-w(x))\,{}^{\alpha}\mathbf{B}_i^{k-1}(x) $$

Let us prove by recurrence on k that ${}^{\alpha}\mathbf{B}_i^k(x) = C_k^i\,(w(x))^i\,(1-w(x))^{k-i}$

  - The relation is true for $k = 1$.
  - Let $k > 1$. Suppose that for all $1 \leq j < k$, one has for all $0 \leq i \leq j$ ${}^{\alpha}\mathbf{B}_i^j(x) = C_j^i\,(w(x))^i\,(1-w(x))^{j-i}$. For all $1 \leq i \leq k-1$, we have

$$ \begin{array}{rcl} {}^{\alpha}\mathbf{B}_i^k(x) & = & (w(x))\,{}^{\alpha}\mathbf{B}_{i-1}^{k-1}(x) + (1-w(x))\,{}^{\alpha}\mathbf{B}_i^{k-1}(x) \\ & = & (w(x))C_{k-1}^{i-1}\,(w(x))^{i-1}\,(1-w(x))^{k-i} \\ & + & (1-w(x))C_{k-1}^{i}\,(w(x))^{i}\,(1-w(x))^{k-1-i} \\ & = & C_{k-1}^{i-1}\,(w(x))^i\,(1-w(x))^{k-i} \\ & + & C_{k-1}^{i}\,(w(x))^i\,(1-w(x))^{k-i} \\ & = & \left[C_{k-1}^{i-1} + C_{k-1}^{i}\right](w(x))^i\,(1-w(x))^{k-i} \\ & = & C_k^i\,(w(x))^i\,(1-w(x))^{k-i} \end{array} $$

because $C_k^i = C_{k-1}^{i-1} + C_{k-1}^i$.

□

4 New class of B-spline curves

Let $m, k, n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n+k+1$. Let $U = (t_i)_{i=0}^m$ be an open node vector, let $\alpha \in (-\infty\,,\,0) \cup (1\,,\,\infty)$.

Consider the rational B-spline basis $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^n$ of index α with node vector U and of degree k,

Consider the B-spline curve G_α of index α,of node vector U, of control points $(d_i)_{i=0}^n \subset \mathbb{R}^d$ and defined for all $x \in [t_0\,,\,t_m]$ by

$$ G_\alpha(x) = \sum_{i=0}^{n} d_i\,{}^{\alpha}\mathbf{G}_i^k(x) $$

4.1 Geometric properties

The curves of this new class verify the classical properties of B-spline curve. They also show some exotic properties namely related to the symmetry. These properties are given in the following propositions.

Proposition 4.1. *We have the following properties:*

1. Local control property:
   Let $j \in \mathbb{N}$ such that $0 \leq j \leq n$. Any variation of the control point d_j does influence $G_\alpha(x)$ only for $x \in [t_j\,,\,t_{j+k+1})$
2. Second local control property:
   Let $j \in \mathbb{N}$ such that $k \leq j \leq n$ and $t_j < t_{j+1}$. For all $x \in [t_j\,,\,t_{j+1})$, we have

$$ G_\alpha(x) = \sum_{i=j-k}^{j} d_i\,{}^{\alpha}\mathbf{G}_i^k(x) $$

   This computation uses only the $k+1$ control points $(d_i)_{i=j-k}^{j}$.

3. Convex hull property:
   G_α is in convex hull of its control points $(d_i)_{i=0}^{n}$.
   In other words, for all $x \in [a\,,\,b]$, there exists $(\lambda_i)_{i=0}^{n} \subset \mathbf{R}_+$ such that $G_\alpha(x) = \sum_{i=0}^{n} \lambda_i d_i$ with $\sum_{i=0}^{n} \lambda_i = 1$

4. Invariance by affine transformation property:
   For any affine transformation T in $\mathbf{R}^d$, we have

$$T\left(G_\alpha(x)\right) = \sum_{i=0}^{n} T(d_i)^\alpha \mathbf{G}_i^k(x)$$

Proof. 1. *Local control property:*

Consider the control polygons $\Pi = (d_i)_{i=0}^{n} \subset \mathbf{R}^d$ and $\hat{\Pi} = \left(\hat{d}_i\right)_{i=0}^{n} \subset \mathbf{R}^d$. Suppose that for a fixed $0 \le j \le n$ we have

$$\begin{cases} \hat{d}_i &= d_i \text{ if } i \neq j \\ \hat{d}_j &\neq d_j \end{cases}$$

Let G_α and $\hat{G}_\alpha$ be the B-spline curves of index α of degree k and of control polygons Π and $\hat{\Pi}$ respectively.
For $x \in [t_0\,,\,t_m]$ we have

$$\begin{cases} G_\alpha(x) &= \sum_{i=0}^{n} d_i{}^\alpha \mathbf{G}_i^k(x) \\ \hat{G}_\alpha(x) &= \sum_{i=0}^{n} \hat{d}_i{}^\alpha \mathbf{G}_i^k(x) \end{cases}$$

The variation $\Delta d_j = d_j - \hat{d}_j$ of the control point d_j induces a variation at x of the curve G_α denoted by $\Delta G_\alpha(x) = G_\alpha(x) - \hat{G}_\alpha(x)$.
One has

$$\Delta G_\alpha(x) = G_\alpha(x) - \hat{G}_\alpha(x) = \left(d_j - \hat{d}_j\right){}^\alpha \mathbf{G}_j^k(x) = \Delta d_j{}^\alpha \mathbf{G}_j^k(x)$$

Thus

$$\Delta G_\alpha(x) \neq 0 \;\Leftrightarrow\; {}^\alpha\mathbf{G}_j^k(x) \neq 0 \;\Leftrightarrow\; x \in (t_j\,,\,t_{j+k+1})$$

The effect of the variation Δd_j can then only be viewed on the computation of $G_\alpha(x)$ for $x \in (t_j\,,\,t_{j+k+1})$.

2. *Second local control property:*
   Let $j \in \mathbb{N}$. Since $U = (t_i)_{i=0}^{m}$ is open,

$$t_j < t_{j+1} \;\Rightarrow\; j \ge k \text{ and } j \le n = m-k-1 \;\Leftrightarrow\; k \le j \le n = m-k-1$$

Let then $k \le j \le n$ such that $t_j < t_{j+1}$ and $x \in [t_j\,,\,t_{j+1}]$.

A control point d_s influences the computation of $G_\alpha(x) = \sum_{i=0}^{n} d_i{}^\alpha \mathbf{G}_i^k(x)$ if and only if ${}^\alpha\mathbf{G}_s^k(x) \neq 0$

$$\begin{array}{rcl} {}^\alpha\mathbf{G}_s^k(x) \neq 0 & \Leftrightarrow & \mathbf{supp}\,{}^\alpha\mathbf{G}_s^k \cap [t_j\,,\,t_{j+1}) \neq \emptyset \\ & \Leftrightarrow & \emptyset \neq [t_j\,,\,t_{j+1}) \subset [t_s\,,\,t_{s+k+1}) \\ & \Leftrightarrow & t_s \le t_j < t_{j+1} \le t_{s+k+1} \\ & \Leftrightarrow & s \le j < j+1 \le s+k+1 \\ & \Leftrightarrow & j-k \le s \le j \end{array}$$

We deduce that

$$G_\alpha(x) = \sum_{i=0}^{n} d_i{}^\alpha \mathbf{G}_i^k(x) = \sum_{i=j-k}^{j} d_i{}^\alpha \mathbf{G}_i^k(x)$$

This computation does use only the $k+1$ control points $(d_i)_{i=j-k}^{j}$.
This result gives another point of view of local control.

3. *Convex hull property:*
   Let $x \in [t_0\,,\,t_m]$

$$\begin{aligned} G_\alpha(x) &= \sum_{i=0}^{n} d_i{}^\alpha\mathbf{G}_i^n(x) \\ &= \sum_{i=0}^{n} \lambda_i d_i \\ \text{where} & \\ \lambda_i &= {}^\alpha\mathbf{G}_i^n(x) \in \mathbf{R}_+ \; \forall i \end{aligned}$$

   But from unit partition property, one gets $\sum_{i=0}^{n} \lambda_i = \sum_{i=0}^{n} {}^\alpha\mathbf{G}_i^n(x) = 1$. $G_\alpha(x)$ is in the convex hull of control polygon $(d_i)_{i=0}^n$

4. *Invariance by affine transformation property:*
   Let T be an affine transformation in $\mathbf{R}^d$. There exists a square matrix M of order d and a point $C \in \mathbf{R}^d$ such that for all $X \in \mathbf{R}^d$, $T(X) = M\,X + C$. Let $x \in [t_0\,,\,t_m]$. Since $G_\alpha(x) \in \mathbf{R}^d$ then we have

$$\begin{aligned} T(G_\alpha(x)) &= T\left(\sum_{i=0}^{n} d_i{}^\alpha\mathbf{G}_i^n(x)\right) \\ &= M\left(\sum_{i=0}^{n} d_i{}^\alpha\mathbf{G}_i^n(x)\right) + C \\ &= \sum_{i=0}^{n} M(d_i{}^\alpha\mathbf{G}_i^n(x)) + \left(\sum_{i=0}^{n} {}^\alpha\mathbf{G}_i^n(x)\right) C \\ &= \sum_{i=0}^{n} (M d_i{}^\alpha\mathbf{G}_i^n(x)) + \sum_{i=0}^{n} (C^\alpha\mathbf{G}_i^n(x)) \\ &= \sum_{i=0}^{n} (M d_i + C)^\alpha\mathbf{G}_i^n(x) = \sum_{i=0}^{n} T(d_i)^\alpha\mathbf{G}_i^n(x) \end{aligned}$$

   what is expected.

□

Proposition 4.2. *The following properties hold:*

1. Interpolation property of extreme points:
   The curve G_α interpolates the extreme points of is control polygon, that is $G_\alpha(t_0) = d_0$ and $G_\alpha(t_m) = d_n$

2. Tangent property at extreme points:
   The curve G_α is tangent to its control polygon at extreme points. More precisely, we have

$$\begin{cases} \dfrac{dG_\alpha}{dx}(t_0) = \dfrac{k\alpha}{(\alpha-1)(t_{k+1}-t_0)}(d_1 - d_0) \\[2ex] \dfrac{dG_\alpha}{dx}(t_m) = \dfrac{k(\alpha-1)}{\alpha(t_m - t_n)}(d_n - d_{n-1}) \end{cases}$$

Proof. We draw attention on the fact that once the node vector $U = (t_i)_{i=0}^m$ has no interior node of multiplicity greater than k, the associated basis $\left({}^\alpha\mathbf{G}_i^k\right)_{i=0}^n$ is of class $\mathcal{C}^0$. We have a curve $G_\alpha = \sum_{i=0}^{n} d_i{}^\alpha\mathbf{G}_i^k$ which is $\mathcal{C}^0$ on $[t_0\,,\,t_m]$ for all control polygon $\Pi = (d_i)_{i=0}^n \subset \mathbf{R}^d$.

1. *Interpolation property of extreme points:*
   By using proposition 3.1 we have

$$G_\alpha(t_0) = \sum_{i=0}^{n} d_i{}^\alpha\mathbf{G}_i^k(t_0) = d_0{}^\alpha\mathbf{G}_0^k(t_0) = d_0$$

and

$$G_\alpha(t_m) = \sum_{i=0}^{n} d_i {}^\alpha\mathbf{G}_i^k(t_m) = d_n {}^\alpha\mathbf{G}_n^k(t_m) = d_n$$

2. *Tangent property at extreme points:*

   By making use of proposition 3.11 we obtain

$$\begin{aligned}
\frac{d}{dx}G_\alpha(t_0) &= \sum_{i=0}^{n} d_i \frac{d}{dx}{}^\alpha\mathbf{G}_i^k(t_0) \\
&= d_0 \frac{d}{dx}{}^\alpha\mathbf{G}_0^k(t_0) + d_1 \frac{d}{dx}{}^\alpha\mathbf{G}_1^k(t_0) \\
&= (d_1 - d_0)\frac{d}{dx}{}^\alpha\mathbf{G}_1^k(t_0) \\
&= (d_1 - d_0)\frac{k\alpha}{(\alpha-1)(t_{k+1}-t_0)}
\end{aligned}$$

   and

$$\begin{aligned}
\frac{d}{dx}G_\alpha(t_m) &= \sum_{i=0}^{n} d_i \frac{d}{dx}{}^\alpha\mathbf{G}_i^k(t_m) \\
&= d_{n-1} \frac{d}{dx}{}^\alpha\mathbf{G}_{n-1}^k(t_m) + d_n \frac{d}{dx}{}^\alpha\mathbf{G}_n^k(t_m) \\
&= (d_n - d_{n-1})\frac{d}{dx}{}^\alpha\mathbf{G}_n^k(t_m) \\
&= (d_n - d_{n-1})\frac{k(\alpha-1)}{\alpha(t_m-t_n)}
\end{aligned}$$

□

Proposition 4.3 (Symmetry property). *If the node vector $U = (t_i)_{i=0}^n$ is symmetric and the control polygon $\Pi = (d_i)_{i=0}^n$ is also symmetric with respect to the perpendicular bisector $\mathcal{D}$ of segment $(d_0\,,\,d_n)$ then the curves of degree k : G_α and $G_{1-\alpha}$ of the same node vector U and of the same control polygon Π are symmetric with respect to the line $\mathcal{D}$*

Proof. Let $U = (t_i)_{i=0}^m$ be symmetric.

We suppose that $\mathbf{R}^d$ is endowed with orthonormed coordinate system $\mathcal{R} = (O\,,\,\vec{e}_1\,,\,\ldots\,,\,\vec{e}_d)$.

Let $\Pi = (d_i)_{i=0}^n \subset \mathbf{R}^d$ be a symmetric control polygon with respect to the perpendicular bisector $\mathcal{D}$ of segment $(d_0\,,\,d_n)$.

Then for all $0 \le i \le n$, $\mathcal{D}$ is the perpendicular bisector of $(d_i\,,\,d_{n-i})$; there exists a unique $M_i \in \mathcal{D}$ such that $\overrightarrow{M_i d_i} = -\overrightarrow{M_i d_{n-i}}$ and $\mathcal{D}$ orthogonal to $(d_i\,,\,d_{n-i})$. Without loss of generality, suppose that $\{O\} = \mathcal{D} \cap (d_0\,,\,d_n)$, $\mathcal{D}$ is the line $(O\,,\,\vec{e}_d)$ and $\mathcal{R}$ the canonical coordinate system. Hence for all $0 \le i \le n$, there exists $\hat{d}_i \in \mathbf{R}^{d-1}$ and $z_i \in \mathbf{R}$ both unique such that

$$\begin{cases}
d_i &= \left(\hat{d}_i\,,\,z_i\right) \equiv \hat{d}_i + z_i\vec{e}_d \\
d_{n-i} &= \left(-\hat{d}_i\,,\,z_i\right) \equiv -\hat{d}_i + z_i\vec{e}_d
\end{cases}$$

Consider the B-spline curves G_α and $G_{1-\alpha}$ of degree k, of node vector U which is symmetric and of symmetric control polygon Π.

For all $x \in [t_0\,,\,t_m]$, we have

$$\begin{aligned}
G_\alpha(x) &= \sum_{i=0}^{n} d_i {}^\alpha\mathbf{G}_i^k(x) \\
&= \sum_{i=0}^{n} \left(\hat{d}_i + z_i\vec{e}_d\right) {}^\alpha\mathbf{G}_i^k(x) \\
&= \sum_{i=0}^{n} \hat{d}_i {}^\alpha\mathbf{G}_i^k(x) + \left(\sum_{i=0}^{n} z_i {}^\alpha\mathbf{G}_i^k(x)\right)\vec{e}_d
\end{aligned}$$

Also

$$\begin{aligned}
G_{1-\alpha}(t_0+t_m-x) &= \sum_{i=0}^{n} d_i{}^{1-\alpha}\mathbf{G}_i^k(t_0+t_m-x) \\
&= \sum_{i=0}^{n} d_i{}^{\alpha}\mathbf{G}_{n-i}^k(x) \\
&= \sum_{i=0}^{n} \left(\hat{d}_i + z_i\vec{e}_d\right)^{\alpha}\mathbf{G}_{n-i}^k(x) \\
&= \sum_{i=0}^{n} \hat{d}_i{}^{\alpha}\mathbf{G}_{n-i}^k(x) + \left(\sum_{i=0}^{n} z_i{}^{\alpha}\mathbf{G}_{n-i}^k(x)\right)\vec{e}_d \\
&= -\sum_{i=0}^{n} \hat{d}_{n-i}{}^{\alpha}\mathbf{G}_{n-i}^k(x) + \left(\sum_{i=0}^{n} z_{n-i}{}^{\alpha}\mathbf{G}_{n-i}^k(x)\right)\vec{e}_d \\
&= -\sum_{i=0}^{n} \hat{d}_i{}^{\alpha}\mathbf{G}_i^k(x) + \left(\sum_{i=0}^{n} z_i{}^{\alpha}\mathbf{G}_i^k(x)\right)\vec{e}_d
\end{aligned}$$

We deduce that

$$\begin{aligned}
\frac{1}{2}\left[G_\alpha(x) + G_{1-\alpha}(t_0+t_m-x)\right] &= \left(\sum_{i=0}^{n} z_i{}^{\alpha}\mathbf{G}_i^k(x)\right)\vec{e}_d \in \mathcal{D} \\
\frac{1}{2}\left[G_\alpha(x) - G_{1-\alpha}(t_0+t_m-x)\right].\vec{e}_d &= \sum_{i=0}^{n} \left(\hat{d}_i.\vec{e}_d\right)^{\alpha}\mathbf{G}_i^k(x) = 0
\end{aligned}$$

Thus $\mathcal{D}$ is the perpendicular bisector of segment $[G_\alpha(x)\,,\,G_{1-\alpha}(t_0+t_m-x)]$, we can then conclude that both G_α and $G_{1-\alpha}$ are symmetric with respect to $\mathcal{D}$. □

4.2 Algorithms of computation of B-spline curve

These algorithms show that it is possible to compute a point of B-spline curve or all of them without making use of the explicit construction of the associated B-spline basis. The fundamental algorithm is of deBoor and can be defined as follows:

Theorem 4.1 (de-Boor algorithm). *Let $m,k,n \in \mathbb{N}^*$ such that $n \geq k$ and $m = n+k+1$. Let $U = (t_i)_{i=0}^m$ be a node vector. Let $\Pi = (d_i)_{i=0}^n \subset \mathbb{R}^d$ be a control polygon.*

For all $j = k,\ldots,m-k-1$ such that $t_j < t_{j+1}$ and for all $x \in [t_j\,,\,t_{j+1})$

$$G_\alpha(x) = \sum_{i=j-k+r}^{j} d_i^r(x)^{\alpha}\mathbf{G}_i^{k-r}(x)$$

with

$$\begin{cases}
d_i^0(x) = d_i & \forall i = 0,\ldots,n \\
d_i^{r+1}(x) = w_i^{k-r}(x)d_{i+1}^r(x) + \left(1-w_i^{k-r}(x)\right)d_i^r(x) & \forall r = 0,\ldots,k-1 \\
& \forall i = j-k+r,\ldots,j
\end{cases}$$

where $w_i^{k-r}(x) = \varphi_\alpha(x\,,\,t_i\,,\,t_{i+k-r})$

Moreover we have $G_\alpha(x) = d_j^k(x)$

Proof. Let $j = k,\ldots,m-k-1$ such that $t_j < t_{j+1}$ and $x \in [t_j\,,\,t_{j+1})$. Since for all i

$$^{\alpha}\mathbf{G}_i^k(x) = w_i^k(x)^{\alpha}\mathbf{G}_i^{k-1}(x) + \left(1-w_{i+1}^k(x)\right){}^{\alpha}\mathbf{G}_{i+1}^{k-1}(x)$$

then

$$
\begin{aligned}
G_\alpha(x) &= \sum_{i=j-k}^{j} d_i{}^\alpha\mathbf{G}_i^k(x) \\
&= \sum_{i=j-k}^{j} d_i w_i^k(x)^\alpha\mathbf{G}_i^{k-1}(x) \\
&+ \sum_{i=j-k}^{j} d_i\left(1-w_{i+1}^k(x)\right){}^\alpha\mathbf{G}_{i+1}^{k-1}(x) \\
&= \sum_{i=j-k}^{j} d_i w_i^k(x)^\alpha\mathbf{G}_i^{k-1}(x) \\
&+ \sum_{i=j-k+1}^{j+1} d_{i-1}\left(1-w_i^k(x)\right){}^\alpha\mathbf{G}_i^{k-1}(x) \\
&= d_{j-k}w_{j-k}^k(x)^\alpha\mathbf{G}_{j-k}^{k-1}(x) + d_j\left(1-w_{j+1}^k(x)\right){}^\alpha\mathbf{G}_{j+1}^{k-1}(x) \\
&+ \sum_{i=j-k+1}^{j}\left[d_{i-1}\left(1-w_i^k(x)\right)+d_i w_i^k(x)\right]{}^\alpha\mathbf{G}_i^{k-1}(x) \\
G_\alpha(x) &= d_{j-k}w_{j-k}^k(x)^\alpha\mathbf{G}_{j-k}^{k-1}(x) + d_j\left(1-w_{j+1}^k(x)\right){}^\alpha\mathbf{G}_{j+1}^{k-1}(x) \\
&+ \sum_{i=j-k+1}^{j}\left[d_{i-1}\left(1-w_i^k(x)\right)+d_i w_i^k(x)\right]{}^\alpha\mathbf{G}_i^{k-1}(x) \\
&= \sum_{i=j-k+1}^{j}\left[d_{i-1}\left(1-w_i^k(x)\right)+d_i w_i^k(x)\right]{}^\alpha\mathbf{G}_i^{k-1}(x) \\
&= \sum_{i=j-k+1}^{j} d_i^1(x)^\alpha\mathbf{G}_i^{k-1}(x)
\end{aligned}
$$

with for all $j-k-1 \le i \le j$

$$
\begin{aligned}
d_i^1(x) &= d_{i-1}\left(1-w_i^k(x)\right)+d_i w_i^k(x) \\
&= d_{i-1}^0(x)\left(1-w_i^k(x)\right)+d_i^0(x)w_i^k(x)
\end{aligned}
$$

by setting $d_i^0(x) = d_i$ for all i; since

$$
\begin{aligned}
\mathbf{supp}\,{}^\alpha\mathbf{G}_{j-k}^{k-1} \cap [t_j\,,\,t_{j+1}) = \emptyset \\
\mathbf{supp}\,{}^\alpha\mathbf{G}_{j+1}^{k-1} \cap [t_j\,,\,t_{j+1}) = \emptyset
\end{aligned}
$$

We have established

$$
G_\alpha(x) = \sum_{i=j-k}^{j} d_i^0(x)^\alpha\mathbf{G}_i^k(x) = \sum_{i=j-k+1}^{j} d_i^1(x)^\alpha\mathbf{G}_i^{k-1}(x)
$$

Let us show by recurrence that for all $0 \le r \le k$ we have

$$
G_\alpha(x) = \sum_{i=j-k+r}^{j} d_i^r(x)^\alpha\mathbf{G}_i^{k-r}(x)
$$

with for all $r \le k$

$$
d_i^r(x) = d_{i-1}^{r-1}(x)\left(1-w_i^{k-r+1}(x)\right)+d_i^{r-1}(x)w_i^{k-r+1}(x)
$$

We assume that for all $1 \le r < k$ we have

$$
G_\alpha(x) = \sum_{i=j-k+r}^{j} d_i^r(x)^\alpha\mathbf{G}_i^{k-r}(x)
$$

with

$$
d_i^r(x) = d_{i-1}^{r-1}(x)\left(1-w_i^{k-r+1}(x)\right)+d_i^{r-1}(x)w_i^{k-r+1}(x)
$$

Then

$$\begin{aligned}
G_\alpha(x) &= \sum_{i=j-k+r}^{j} d_i^r(x)^\alpha\mathbf{G}_i^{k-r}(x)\\
&= \sum_{i=j-k+r}^{j} d_i^r(x)w_i^{k-r\,\alpha}\mathbf{G}_i^{k-r-1}(x)\\
&+ \sum_{i=j-k+r}^{j} d_i^r(x)\left(1-w_{i+1}^{k-r}\right){}^\alpha\mathbf{G}_{i+1}^{k-r-1}(x)\\
&= \sum_{i=j-k+r}^{j} d_i^r(x)w_i^{k-r\,\alpha}\mathbf{G}_i^{k-r-1}(x)\\
&+ \sum_{i=j-k+r+1}^{j+1} d_{i-1}^r(x)\left(1-w_i^{k-r}\right){}^\alpha\mathbf{G}_i^{k-r-1}(x)\\
&= d_{j-k+r}^r(x)w_{j-k+r}^{k-r}{}^\alpha\mathbf{G}_{j-k+r}^{k-r-1}(x) + \left(1-w_{j+1}^{k-r}\right)d_j^r(x)w_{j+1}^{k-r\,\alpha}\mathbf{G}_{j+1}^{k-r-1}(x)\\
&+ \sum_{i=j-k+r+1}^{j}\left[d_{i-1}^r(x)+\left(1-w_i^{k-r}\right)d_i^r(x)w_i^{k-r}\right]{}^\alpha\mathbf{G}_i^{k-r-1}(x)\\
&= \sum_{i=j-k+r+1}^{j}\left[d_{i-1}^r(x)+\left(1-w_i^{k-r}\right)d_i^r(x)w_i^{k-r}\right]{}^\alpha\mathbf{G}_i^{k-r-1}(x)\\
&= \sum_{i=j-k+r+1}^{j} d_i^{r+1}(x)^\alpha\mathbf{G}_i^{k-r-1}(x)
\end{aligned}$$

with

$$d_i^{r+1}(x) = d_{i-1}^r(x) + \left(1-w_i^{k-r}\right)d_i^r(x)w_i^{k-r}$$

since

$$\begin{aligned}
\mathbf{supp}\,{}^\alpha\mathbf{G}_{j-k+r}^{k-r-1} \cap [t_j\,,\,t_{j+1}) = \emptyset\\
\mathbf{supp}\,{}^\alpha\mathbf{G}_{j+1}^{k-r-1} \cap [t_j\,,\,t_{j+1}) = \emptyset
\end{aligned}$$

We have thus proved that for all $0 \le r \le k$ we have

$$G_\alpha(x) = \sum_{i=j-k+r}^{j} d_i^r(x)^\alpha\mathbf{G}_i^{k-r}(x)$$

with for all $r \le k$

$$d_i^r(x) = d_{i-1}^{r-1}(x)\left(1-w_i^{k-r+1}(x)\right) + d_i^{r-1}(x)w_i^{k-r+1}(x)$$

For $r = k$, we have for all $x \in [t_j\,,\,t_{j+1})$

$$G_\alpha(x) = \sum_{i=j}^{j} d_i^k(x)^\alpha\mathbf{G}_i^0(x) = d_j^k(x)^\alpha\mathbf{G}_j^0(x) = d_j^k(x)$$

This completes the proof.

□

5 Some illustrations of properties of the new class of rational B-spline curves

In this section, we will present a set of practical cases which depicts the established properties in previous sections. Here the aim is just to give some illustration view without being concerned with the issue of algorithm optimization. To this end, we have adopted **Scilab** scripts and sometimes **Maxima** scripts particularly for the formal expressions of B-spline basis listed in appendix.

We will first present the basis and then the B-spline curves.

5.1 The new class of rational B-spline basis

We emphasize on illustrations of first properties of the new class of B-spline basis.

We know that the B-spline basis are grouped in two categories regarding the fact that they are spanned by a periodic node vector or not and in each category, the node vector may be uniform or not. We shall go through all of these variations.

Case of periodic node vectors

We plan two illustrations. The first one explores the influence of the uniformity of node vector while the second one explores the non-uniformity.

Illustration 5.1. *We present here B-spline basis of degree* 0 *to* 3 *for the uniform periodic node vector* $U_0 = (0, 1, 2, 3, 4, 5, 6)$ *with* $\alpha \in \{-1, 2, 5, \infty\}$

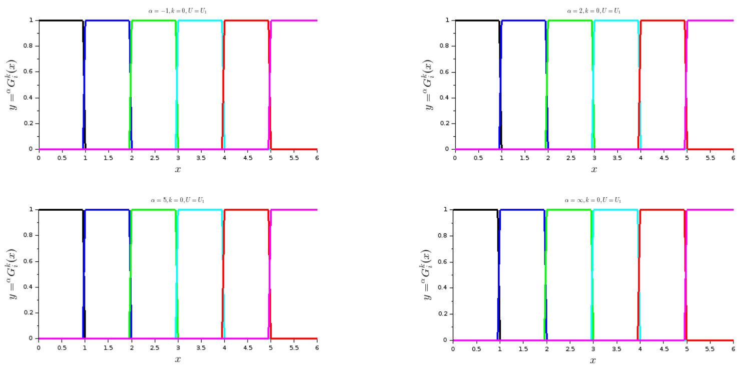

Figure 4: The B-spline basis ${}^{\alpha}\mathbf{G}_i^0$ of node vector U_0

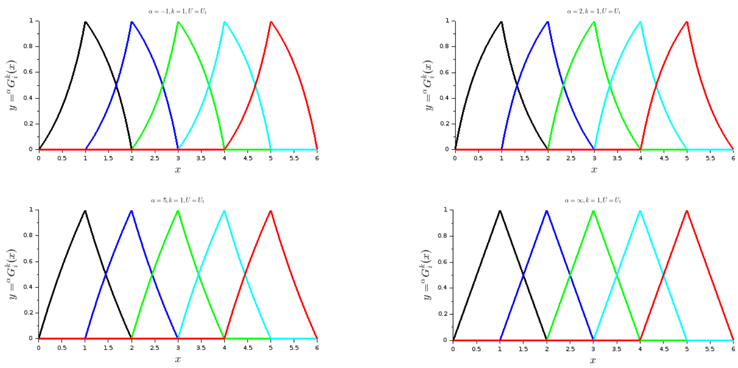

Figure 5: The B-spline basis ${}^{\alpha}\mathbf{G}_i^1$ of node vector U_0

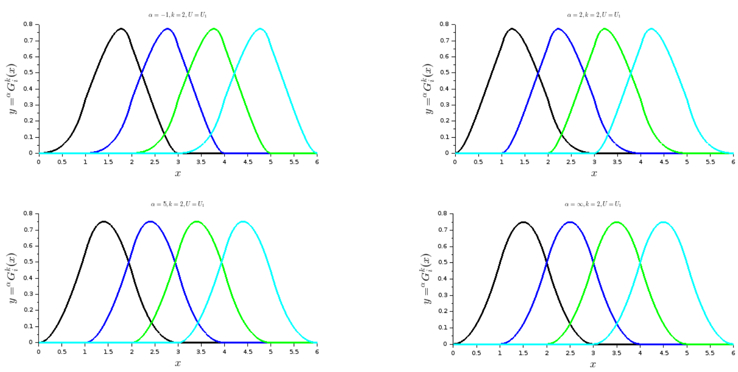

Figure 6: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_0

From the analysis of figures 4 to 7, we deduce that since U_0 is a uniform periodic node vector, an element of the basis $\left({}^{\alpha}\mathbf{G}_i^k\right)_{i=0}^{m-k-1}$ is obtained by simple translation of ${}^{\alpha}\mathbf{G}_0^k$ that is ${}^{\alpha}\mathbf{G}_i^k(x) = {}^{\alpha}\mathbf{G}_0^k(t_0 - t_i + x)$.

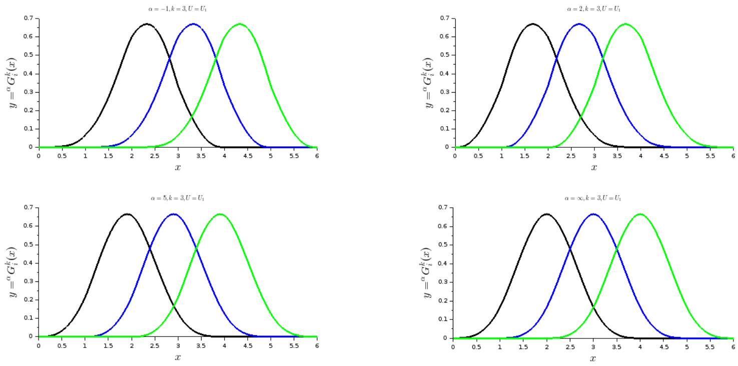

Figure 7: The B-spline basis ${}^{\alpha}\mathbf{G}_i^3$ of node vector U_0

We observe that $\mathbf{supp}\,{}^{\alpha}\mathbf{G}_i^k = [t_i\,,\,t_{i+k+1}]$ and also the effect of parameter α is crucial at the neighborhood of 0^- and 1^+. The figure 7 seems to show that α does not have any influence on $\left({}^{\alpha}\mathbf{G}_i^3\right)_{i=0}^{m-4}$ which corresponds to a context of node vector with no interior nodes.

Illustration 5.2. *We present the influence of the non-uniformity of a periodic node vector by restricting ourselves on B-spline basis of degree* 2 *in the following cases:*

$U_1 = (0, 1, 2, 3, 3, 5, 6)$
$U_2 = (0, 1, 1, 2, 4, 5, 6)$
$U_3 = (0, 1, 1.5, 2, 3.5, 5, 6)$

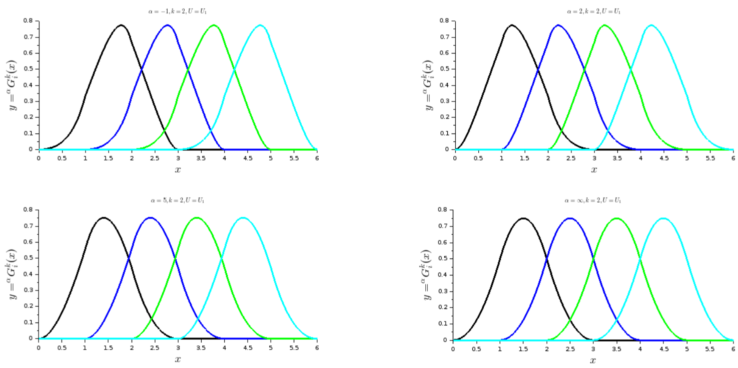

Figure 8: Les Bases B-splines ${}^{\alpha}\mathbf{G}_i^2$ de vecteur nœud U_0

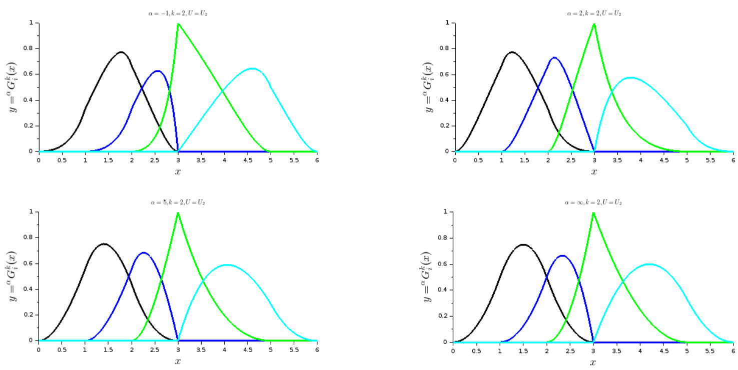

Figure 9: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_1

The non-uniformity may come from the presence of a multiple node, it is the case of node vectors U_1 and U_2. It may be also due to the step of variable between nodes as in U_3.

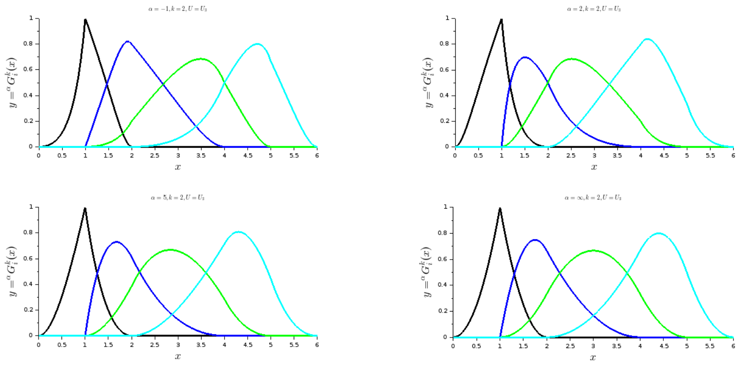

Figure 10: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_2

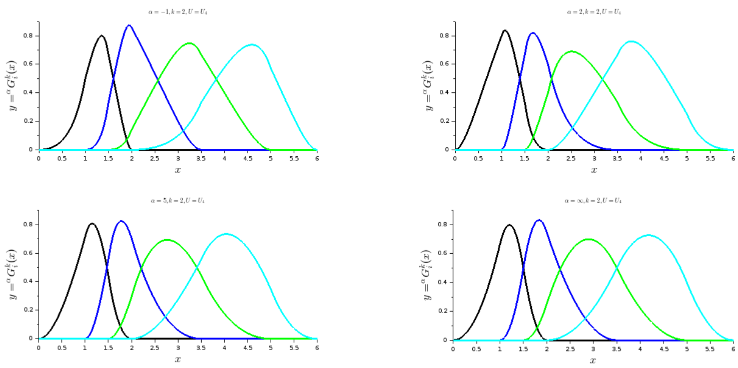

Figure 11: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_3

The figures 9 to 11 show that in all the cases we have $\mathbf{supp}\,{}^{\alpha}\mathbf{G}_i^2 = [t_i\,,\,t_{i+3}]$ and the effect of the parameter α remains important at the neighborhood of 0^- and 1^+. We observe a large diversity among the elements of the basis concerning the regularity.

The two illustrations of this subsection seem to confirm the conjecture 3.1 related to the existence of a unique maximum for ${}^{\alpha}\mathbf{G}_i^k$ when $k > 0$.

Case of open node vectors

This subsection is also based on two test cases which give light on the basis of degree 2 generated by open node vectors for $\alpha \in \{-1, 2, 5, \infty\}$.

The first test case dealts with five node vectors having two multiple interior nodes or not.

In the second test case we also have five node vectors but having three interior nodes where the multiplicity may reach 3.

Illustration 5.3. *We explore the case of B-spline basis of degree* 2 *associated with an open node vector in the following cases:*

$U_4 = (0, 0, 0, 1, 2, 3, 3, 3)$
$U_5 = (0, 0, 0, 0.4, 2.6, 3, 3, 3)$
$U_6 = (0, 0, 0, 1.8, 2.2, 3, 3, 3)$
$U_7 = (0, 0, 0, 1, 1, 3, 3, 3)$
$U_8 = (0, 0, 0, 2, 2, 3, 3, 3)$

The figures 12 to 16 illustrate abundantly the properties of the proposition 3.1 especially those of values at extreme nodes.

The figures 12 and 13 depict the behaviors of basis generated respectively by U_4 and U_5 which are symmetric node vectors. One can observe that for all $x \in [t_0\,,\,t_7]$, we have

$$\begin{aligned}
{}^{-1}\mathbf{G}_i^2(t_0 + t_7 - x) &= {}^{2}\mathbf{G}_{4-i}^2(x)\\
{}^{2}\mathbf{G}_i^2(t_0 + t_7 - x) &= {}^{-1}\mathbf{G}_{4-i}^2(x)\\
{}^{\infty}\mathbf{G}_i^2(t_0 + t_7 - x) &= {}^{\infty}\mathbf{G}_{4-i}^2(x)
\end{aligned}$$

For the non-uniform open node vector U_6, U_7 and U_8 we observe a large diversity of behaviors of generated basis.

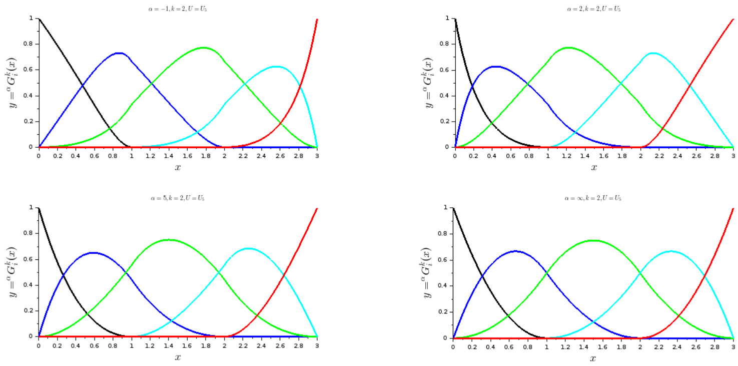

Figure 12: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_4

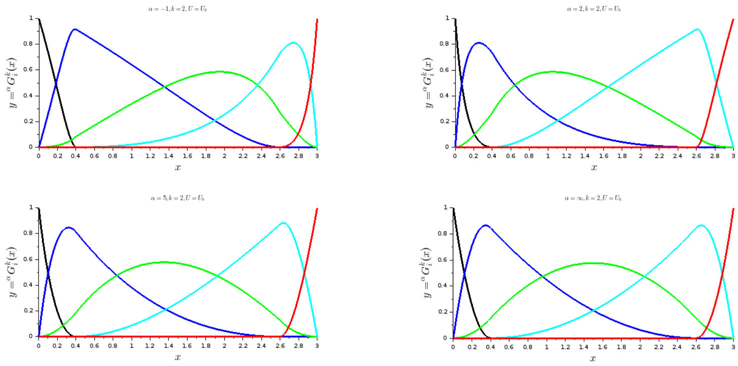

Figure 13: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_5

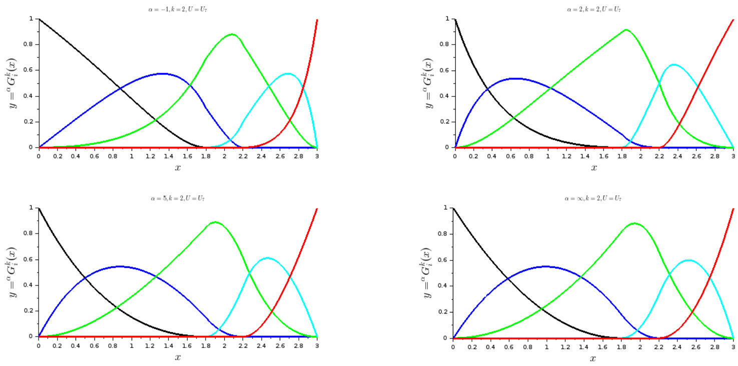

Figure 14: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$of node vector U_6

Illustration 5.4. *The B-spline basis of degree* 2 *we are illustrating explore the existing relation between the regularity and the multiplicity of an interior node of an open node vector in the following cases:*

$U_9 = (0, 0, 0, 3/4, 6/4, 9/4, 3, 3, 3)$
$U_{10} = (0, 0, 0, 3/4, 3/4, 9/4, 3, 3, 3)$
$U_{11} = (0, 0, 0, 3/4, 3/4, 3/4, 3, 3, 3)$
$U_{12} = (0, 0, 0, 3/4, 9/4, 9/4, 3, 3, 3)$
$U_{13} = (0, 0, 0, 9/4, 9/4, 9/4, 3, 3, 3)$

The node vector U_9 is uniform with interior nodes of multiplicity 1 *and we observe in figure 17 that the generated basis confirms the behaviors we already observed with U_4. We can state their regularity of $\mathcal{C}^0$ as well as the left and right differentiability at any interior node as provided in proposition 3.11.*

Each of the node vectors U_{10} and U_{12} has one interior node with multiplicity 2*. The analysis of figures 18 and 20 shows that the associated basis ${}^{\alpha}\mathbf{G}_i^2$ are at least of $\mathcal{C}^0$ with the existence of a left and right derivatives at any interior node even at a double node confirming the results in proposition 3.11.*

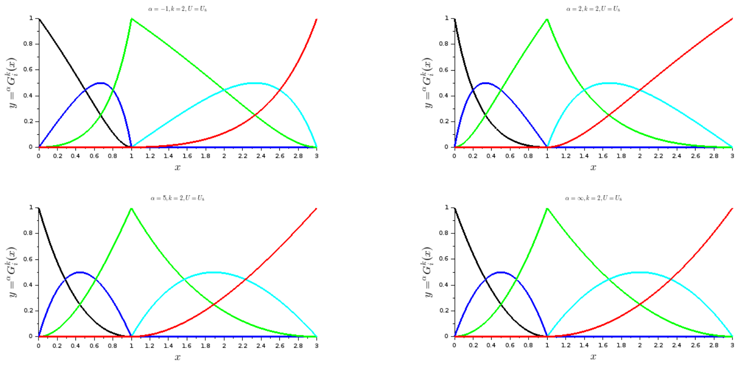

Figure 15: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_7

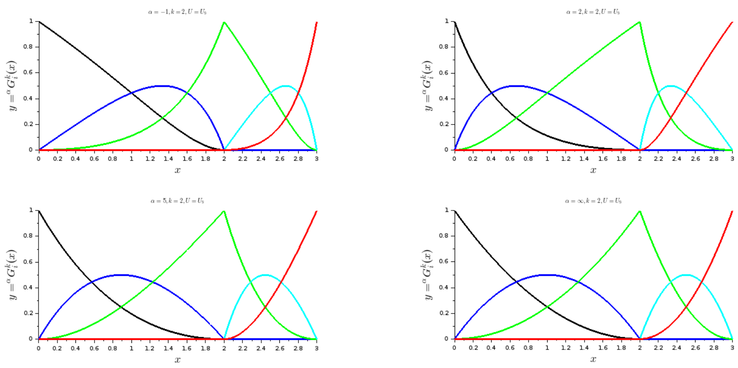

Figure 16: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_8

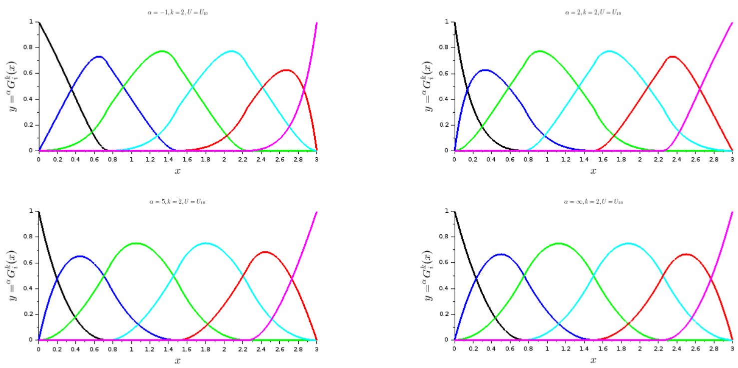

Figure 17: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_9

Each of the node vectors U_{11} and U_{13} has one interior triple node $t_3 = t_4 = t_5$. We must expect a first type of discontinuity for the elements ${}^{\alpha}\mathbf{G}_2^2$ and ${}^{\alpha}\mathbf{G}_3^2$ of the associated basis as **supp** ${}^{\alpha}\mathbf{G}_2^2 = [t_2\, , \, t_5]$ *and* **supp** ${}^{\alpha}\mathbf{G}_3^2 = [t_3\, , \, t_6]$. *The other elements of the basis keep the regularity of $\mathcal{C}^0$ with the existence of a left and right derivatives at any interior node. This is confirmed by the analysis of figures 19 and 21.*

Remarque 5.1. *Either the node vector is periodic or open, uinform or not, we observe in all the cases that ${}^{\infty}\mathbf{G}_i^2 \approx {}^{5}\mathbf{G}_i^2$ and the conjecture 3.1 is verified.*

5.2 The new class of rational B-spline curves

Let us have a look on some examples showing the behavior of new B-spline curves under the effect of various parameter appearing in their definition.

Amongst some parameters we can refer to index α, the degree k, the node vector U and the control polygon Π.

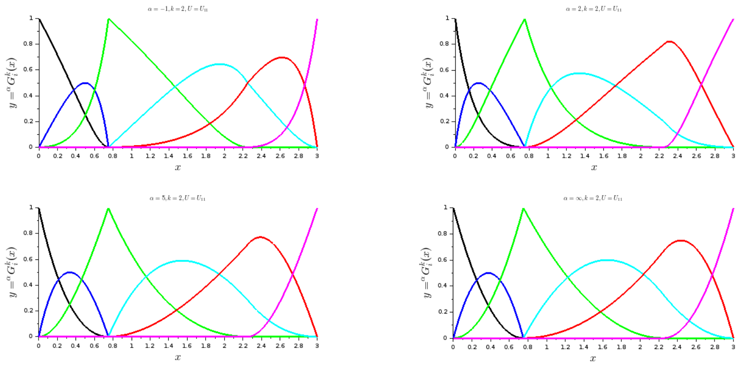

Figure 18: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_{10}

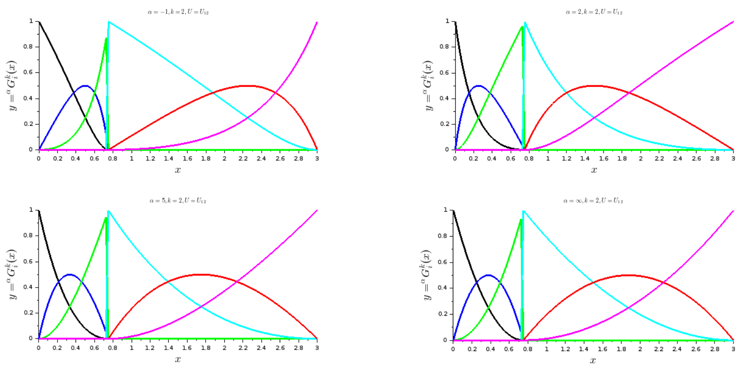

Figure 19: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_{11}

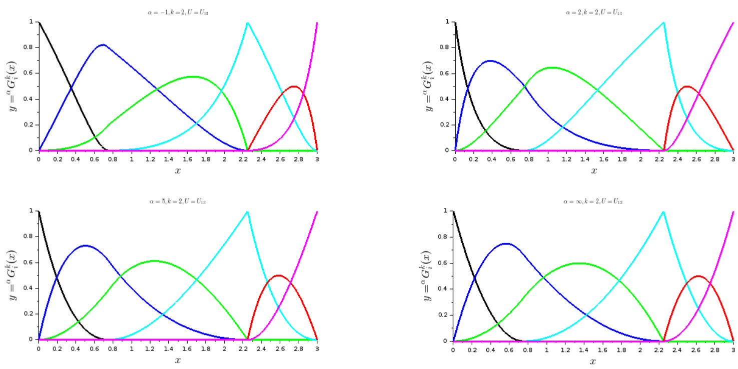

Figure 20: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_{12}

Illustration 5.5. *Let begin with the new parameter which is the index α. We fix the degree to 3 on the uniform and open node vector U and the control polygon Π as follows:*

$U = (0, 0, 0, 0, 1, 2, 3, 4, 5, 5, 5, 5)$

$\Pi = \{(0\,,\,2), (1.5\,,\,5), (2.5\,,\,4), (3\,,\,1), (5\,,\,4), (7\,,\,1), (8\,,\,4), (10\,,\,4)\}$

We will go through $\alpha \in \{-\infty, -4, -1/2, -1/5, -1/7\}$, as well as its conjugated $1-\alpha$.

A quick analysis of figure 22 reveals:

1. *For $\alpha \leq -4$ and $\alpha \geq 5$, the B-spline curve G_α of degree k and index α is a good approximationof the standard polynomial B-spline curve G_∞ generated by the same control polygon Π.*

2. *When α tends to 0^- or to 1^+, the curve G_α is really separated from the standard curve G_∞. The effect seems more viewed at the neighborhood of 0 but the question is still to be tackled later on.*

3. *We reach a conclusion that the B-spline curves family becomes more interesting.*

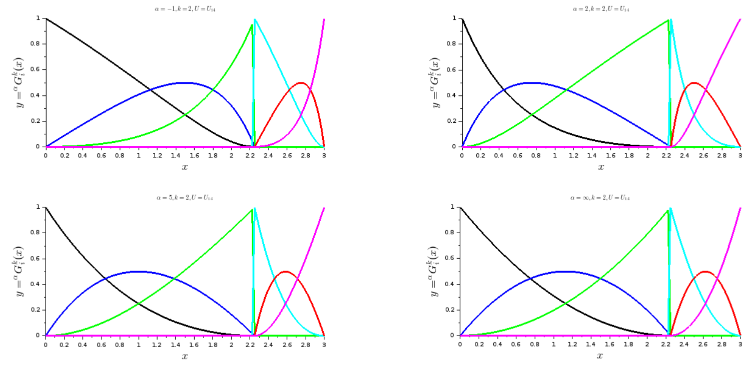

Figure 21: The B-spline basis ${}^{\alpha}\mathbf{G}_i^2$ of node vector U_{13}

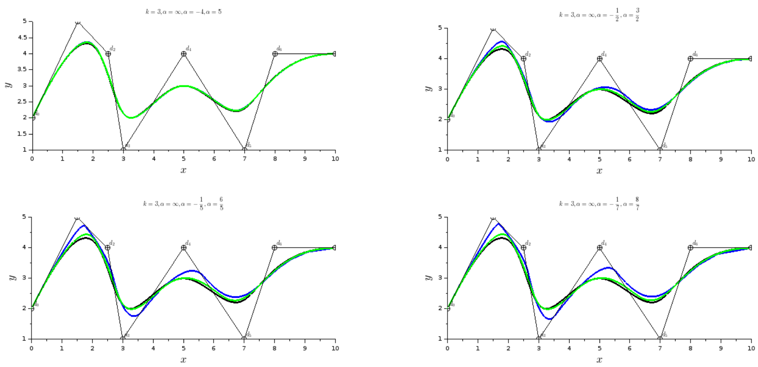

Figure 22: Influence of α to $k = 3$, U uniform and open with fixed Π

Illustration 5.6. *The second important parameter is the degree k of the basis which generates the B-spline curve. We will observe its influence on two examples discribed by the following data where the control polygon Π_i has been fixed with a uniform and open node vector $U_{i,k}$ giving the degree k as follows:*

1. ***Example 1***

   $\Pi_1 = \{(0\,,\,0),(3\,,\,9),(6\,,\,3),(9\,,\,6)\}$
   $U_{1,1} = (0, 0, 1, 2, 3, 3)$
   $U_{1,2} = (0, 0, 0, 1.5, 3, 3, 3)$
   $U_{1,3} = (0, 0, 0, 0, 3, 3, 3, 3)$

2. ***Example 2***

   $\Pi_2 = \{(1\,,\,3),(0\,,\,5),(5\,,\,5),(3\,,\,0),(8\,,\,0),(7\,,\,3)\}$
   $U_{2,1} = (0, 0, 1, 2, 3, 4, 5, 5)$
   $U_{2,2} = (0, 0, 0, 5/4, 5/2, 15/4, 5, 5, 5)$
   $U_{2,3} = (0, 0, 0, 0, 5/3, 10/3, 5, 5, 5, 5)$
   $U_{2,4} = (0, 0, 0, 0, 0, 5/2, 5, 5, 5, 5, 5)$
   $U_{2,5} = (0, 0, 0, 0, 0, 0, 5, 5, 5, 5, 5, 5)$

The figure 23 summarizes example 1 and show on one hand that independently from α, the degree $k = 1$ yields the control polygon Π. On the other hand, $k = 3$ corresponds to a node vector without any interior node and the obtained B-spline curve G_α is independent from α. Only the degree $k = 2$ between the extremes undergo the influence of index α with some highlight when α tends to 0.

The results of example 2 shown in figure 24 confirm above observations.

The degree $k = 1$ yields the control polygon Π_2 and the degree $k = 5$ which corresponds to a node vector with no interior node does not have any influence under α. For the intermediate degrees k the index α has an incresing influence when α tends to 0.

Illustration 5.7. *Now we intend to look at the influence of control polygon Π on the local behavior of a B-spline curve. We fix the degree to 3 on the uniform and open node vector U by varing only one point of the control polygon as follows:*

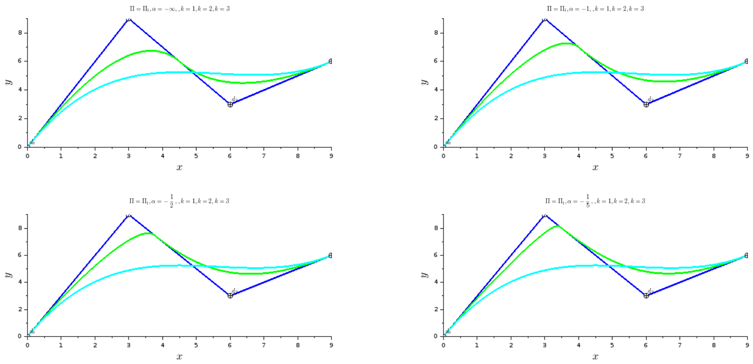

Figure 23: Influence of degree k, U uniform and open at α with fixed Π

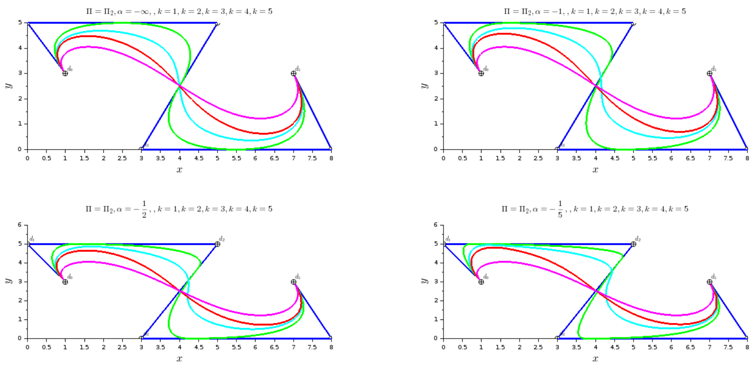

Figure 24: Influence of degree k, U uniform and open at α with fixed Π

$U = (0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4)$
$\Pi_1 = \{(0\,,\,4), (5\,,\,4), (5\,,\,8), (11\,,\,7.5), (6\,,\,2), (12\,,\,0), (2\,,\,0)\}$
$\Pi_2 = \{(0\,,\,4), (5\,,\,4), (5\,,\,8), (11\,,\,7.5), (9\,,\,3), (12\,,\,0), (2\,,\,0)\}$
$\Pi_3 = \{(0\,,\,4), (5\,,\,4), (5\,,\,8), (11\,,\,7.5), (12\,,\,4), (12\,,\,0), (2\,,\,0)\}$
We take $\alpha \in \{-\infty, -4, -1/2, -1/5, -1/7\}$, as well as its conjugated $1 - \alpha$.

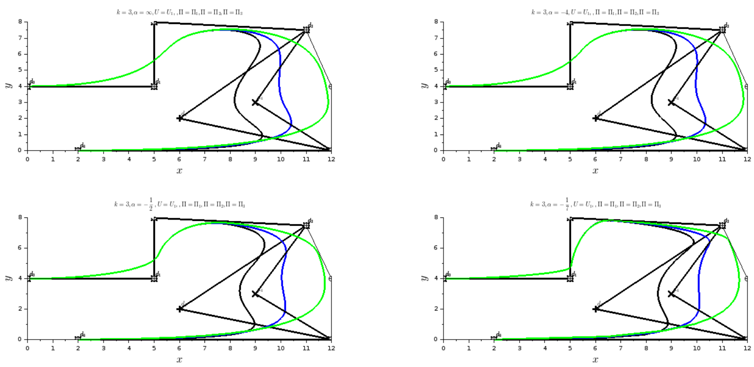

Figure 25: Influence of the variation of a point of Π at $k = 3$, U uniform and open and $\alpha \in \{-\infty, -4, -1/2, -1/5, -1/7\}$

Figures 25 and 26 let us to state that each curve G_α is made up of three segments where the second one is under the motion of the fifth endpoint of the control polygon Π. As we have noted so far, the influence of α is not so remarkable for $\alpha \leq -4$ and $\alpha \geq 5$ as one can note in polynomial case that is to say $G_\alpha \approx G_\infty$.

In the deformation region of the curve G_α at the neighborhood of a segment $[d_i\,,\,d_{i+1}]$ of control polygon Π_j, the deformation moves towards the point d_i when $\alpha \in (-1\,,\,0)$ and towards the point d_{i+1} when $\alpha \in (1\,,\,2)$ as shown in figures 25 and 26 respectively. In all cases, the curve G_α belongs to the convex envelop of the control polygon Π_j.

Remarque 5.2. *Through the figure 22 of illustration 5.5 and figures 23 and 24 of illustration 5.6 as well as figures 25 and*

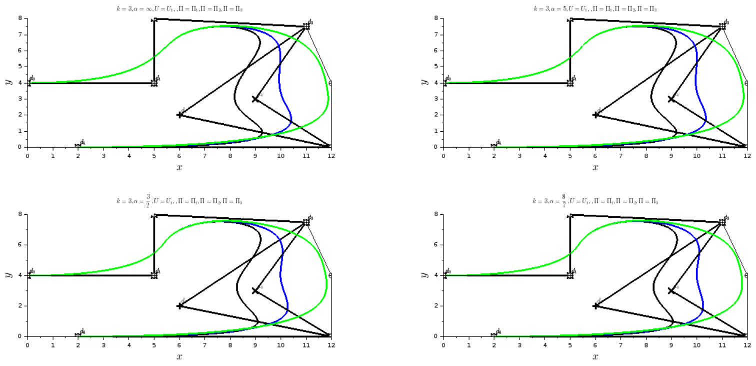

Figure 26: Influence of the variation of a point of Π at $k = 3$, U uniform and open and $\alpha \in \{\infty, 5, 3/2, 6/5, 8/7\}$

26 of illustration 5.7, we realize that the property of convex envelop is widely verified.

Illustration 5.8. *In this test case, we will explore the property of symmetry proved in proposition 4.3 through seven contexts where we restrict ourselves to an axis of symmetry parallel to the coordinate axes which does not reduce generality. The data are as follow:*

1. *Axial symmetry of* Π *with axis parallel to Oy with no multiple point*

$$\Pi_1 = \left\{ \begin{array}{l} (4\,,\,0), (0\,,\,11), (6\,,\,14), \\ (10\,,\,14), (16\,,\,11), (12\,,\,0) \end{array} \right\}$$
$$U_1 = (0, 0, 0, 0, 1, 2, 3, 3, 3, 3)$$

2. *Axial symmetry of* Π *with axis parallel to Oy with one double point*

$$\Pi_2 = \left\{ \begin{array}{l} (4\,,\,0), (0\,,\,11), (8\,,\,14), \\ (8\,,\,14), (16\,,\,11), (12\,,\,0) \end{array} \right\}$$
$$U_2 = (0, 0, 0, 0, 1, 2, 3, 3, 3, 3)$$

3. *Axial symmetry of* Π *with axis parallel to Oy with double point and double node*

$$\Pi_3 = \left\{ \begin{array}{l} (4\,,\,0), (0\,,\,11), (8\,,\,14), \\ (8\,,\,14), (16\,,\,11), (12\,,\,0) \end{array} \right\}$$
$$U_3 = (0, 0, 0, 0, 2, 2, 4, 4, 4, 4)$$

4. *Axial symmetry of* Π *with axis parallel to Ox with no multiple point*

$$\Pi_4 = \left\{ \begin{array}{l} (0\,,\,5), (0\,,\,4), (1\,,\,4), \\ (2\,,\,4), (2\,,\,6), (4\,,\,6), (5\,,\,5), \\ (5\,,\,1), (4\,,\,0), (2\,,\,0), \\ (2\,,\,2), (1\,,\,2), (0\,,\,2), (0\,,\,1) \end{array} \right\}$$
$$U_4 = (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 11, 11, 11)$$

5. *Axial symmetry of* Π *with axis parallel to Ox with double point*

$$\Pi_5 = \left\{ \begin{array}{l} (0\,,\,5), (0\,,\,4), (1\,,\,4), \\ (2\,,\,4), (2\,,\,6), (4\,,\,6), (5\,,\,3), \\ (5\,,\,3), (4\,,\,0), (2\,,\,0), \\ (2\,,\,2), (1\,,\,2), (0\,,\,2), (0\,,\,1) \end{array} \right\}$$
$$U_5 = (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 11, 11, 11)$$

6. *Axial symmetry of* Π *with axis parallel to Ox with double point and double node*

$$\Pi_6 = \left\{ \begin{array}{l} (0\,,\,5), (0\,,\,4), (1\,,\,4), \\ (2\,,\,4), (2\,,\,6), (4\,,\,6), (5\,,\,3), \\ (5\,,\,3), (4\,,\,0), (2\,,\,0), \\ (2\,,\,2), (1\,,\,2), (0\,,\,2), (0\,,\,1) \end{array} \right\}$$
$$U_6 = (0, 0, 0, 0, 1, 2, 3, 4, 5, 5, 6, 7, 8, 9, 10, 10, 10, 10)$$

7. *Double axial symmetry of* Π *with one double point*

$$\Pi_7 = \left\{ \begin{array}{l} (0\,,\,2),(0\,,\,3),(1\,,\,4), \\ (3\,,\,4),(5\,,\,4),(6\,,\,3), \\ (6\,,\,2),(6\,,\,1),(5\,,\,0), \\ (3\,,\,0),(1\,,\,0),(0\,,\,1),(0\,,\,2) \end{array} \right\}$$
$$U_7 = (0,0,0,0,1,2,3,4,5,6,7,8,9,10,10,10,10)$$

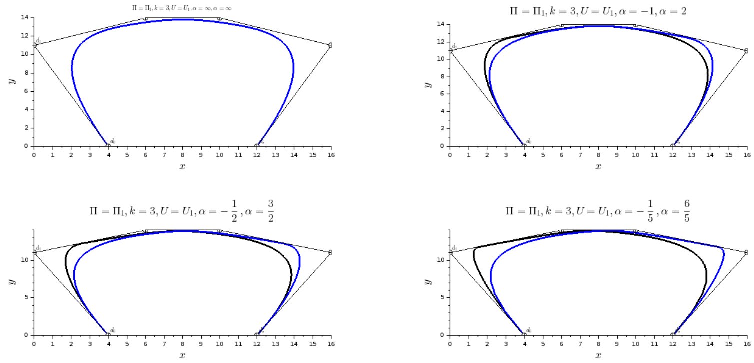

Figure 27: G_α curves of degree $k = 3$, U_1 uniform and open, Π_1 symmetric with no multiple point and $\alpha \in \{\infty, -1, -1/2, -1/5\}$

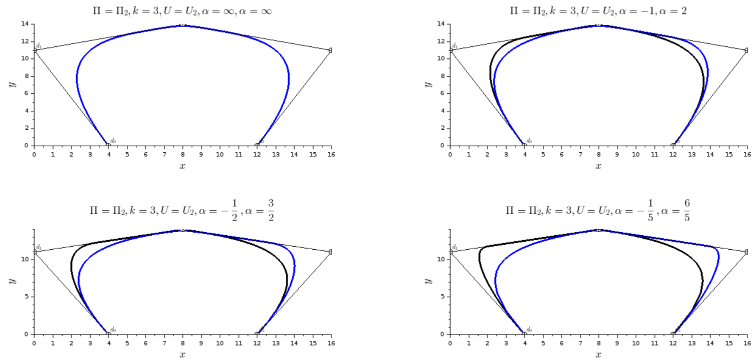

Figure 28: G_α curves of degree $k = 3$, U_2 uniform and open, Π_2 symmetric with double point and $\alpha \in \{\infty, -1, -1/2, -1/5\}$

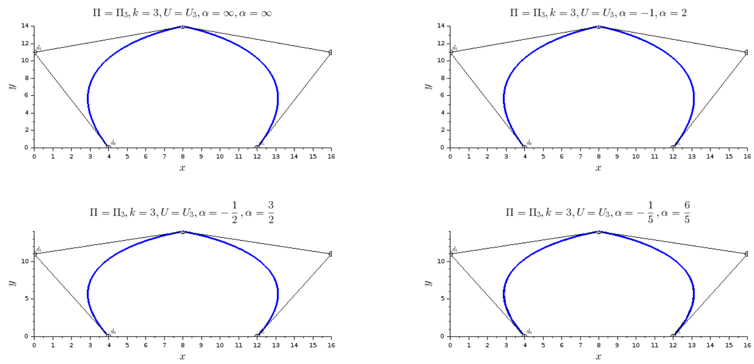

Figure 29: G_α curves of degree $k = 3$, U_3 symmetric and open with double node, Π_3 symmetric with double point and $\alpha \in \{\infty, -1, -1/2, -1/5\}$

Based on figures from 27 to 33, it can be drawn that the curves G_α and $G_{1-\alpha}$ are symmetric with respect to the perpendicular bisector of extreme points of the control polygon Π. As stated above, the effect àf index α is very remarkable for $\alpha \in (-1\,,\,0) \cup (1\,,\,2)$.

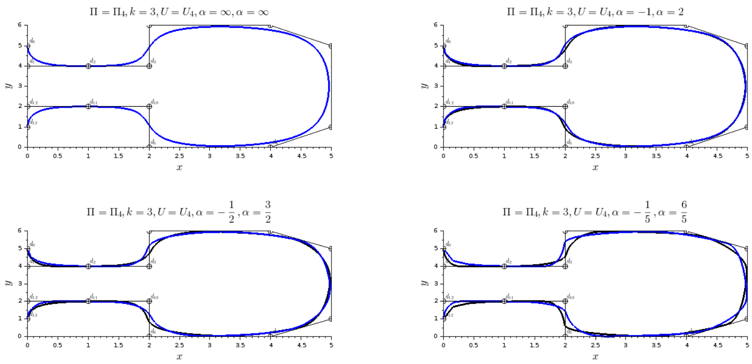

Figure 30: G_α curves of degree $k = 3$, U_4 uniform and open, Π_4 symmetric with no multiple point and $\alpha \in \{\infty, -1, -1/2, -1/5\}$

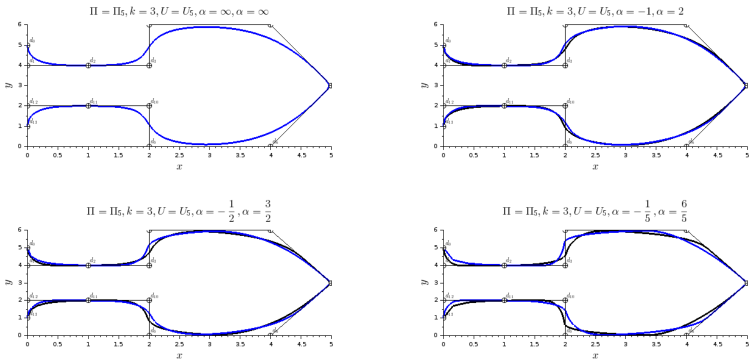

Figure 31: G_α curves of degree $k = 3$, U_5 uniform and open, Π_5 symmetric with double point and $\alpha \in \{\infty, -1, -1/2, -1/5\}$

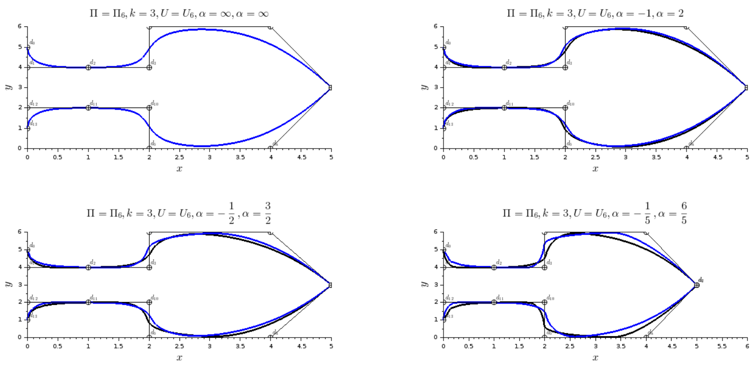

Figure 32: G_α curves of degree $k = 3$, U_6 symmetric and open with double node, Π_6 symmetric with double point and $\alpha \in \{\infty, -1, -1/2, -1/5\}$

The multiplicity of a node acts on the geometrical regularity of curves G_α and $G_{1-\alpha}$. In the presence of a double control point, the curves G_α and $G_{1-\alpha}$ adhere to this point.

The figure 29 shows however a singular case which we will light upon later on since α seems to have no influence on it.

6 Conclusion

The class of parametrization we developed allows us to construct a family of rational B-spline basis depending on a parameter α which generalizes all including polynomial B-spline basis. This new family of B-spline basis possesses all the classical fundamental properties such as positivity, unit partition property and linear independence. Some symmetry property has been established. We have proved that the family of B-spline curves we obtained

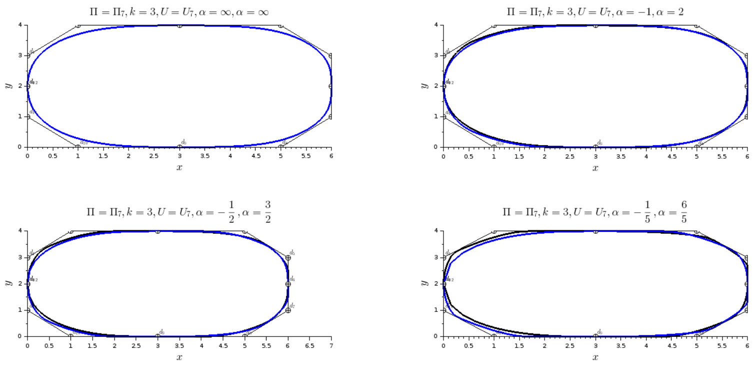

Figure 33: G_α curves of degree $k = 3$, U_7 uniform and open, Π_7 symmetric with double point and $\alpha \in \{\infty, -1, -1/2, -1/5\}$

is larger than the polynomial B-spline curves one and globally extend their properties. Illustrations are given to explain more the properties we proved with the desire of the extension to practical computation algorithms of curves (deBoor algorithm) in future work. It is left with the exploration in more details of the effect of this new parametrization on Bernstein functions and the resulting Bezier curves.

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