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Existence and nonexistence of nontrivial solutions for a critical biharmonic equations under the Steklov boundary conditions

Yutian Duan[1] | Qihan He*[1] | Zongyan Lv[2]

[1]College of Mathematics and Information Sciences, Guangxi Center for Mathematical Research, Guangxi University, Guangxi, China
[2]School of Mathematical Sciences, Beijing Normal University, Beijing, China

Correspondence
*Qihan He, College of Mathematics and Information Sciences, Guangxi Center for Mathematical Research, Guangxi University, Guangxi, China.
Email: heqihan277@gxu.edu.cn

Present Address
College of Mathematics and Information Sciences, Guangxi Center for Mathematical Research, Guangxi University, Guangxi, China

In this paper, we study the existence and nonexistence of nontrivial solutions to the following critical biharmonic problem with the Steklov boundary conditions

$$\begin{cases} \Delta^2 u = \lambda u + \mu \Delta u + |u|^{2^{**}-2}u & \text{in} \quad B, \\ u = \Delta u + k u_\nu = 0 & \text{on} \quad \partial B, \end{cases}$$

where $\lambda, \mu, k \in R$, $B \subset R^N (N \geq 5)$ is a unit ball, $2^{**} = \frac{2N}{N-4}$ denotes the critical Sobolev exponent for the embedding $H^2(B) \hookrightarrow L^{2^{**}}(B)$ and u_ν is the outer normal derivative of u on ∂B. Under some assumptions on λ, μ and k, we prove the existence of nontrivial solutions to the above biharmonic problem by the Mountain pass theorem and show the nonexistence of nontrivial solutions to it by the Pohozaev identity.

KEYWORDS:
Existence, Nonexistence, Nontrivial solutions, Critical biharmonic equation,

1 | INTRODUCTION

In the present paper, we study the existence and nonexistence of nontrivial solutions to the following critical biharmonic problem with the Steklov boundary conditions

$$\begin{cases} \Delta^2 u = \lambda u + \mu \Delta u + |u|^{2^{**}-2}u & \text{in} \quad B, \\ u = \Delta u + k u_\nu = 0 & \text{on} \quad \partial B, \end{cases} \tag{1}$$

where $\lambda, \mu, k \in R$, $B \subset R^N (N \geq 5)$ is a unit ball, $2^{**} = \frac{2N}{N-4}$ denotes the critical Sobolev exponent for the embedding $H^2(B) \hookrightarrow L^{2^{**}}(B)$ and u_ν is the outer normal derivative of u on ∂B.

When $k = 0$, (1) becomes the problem with the Navier boundary conditions as follows:

$$\begin{cases} \Delta^2 u = \lambda u + \mu \Delta u + u^{2^{**}-1} & \text{in} \quad B, \\ u = \Delta u = 0 & \text{on} \quad \partial B. \end{cases} \tag{2}$$

when $\mu = 0$ and $\lambda = 0$, the nonexistence of positive solution to (2) was proved by Mitidieri[1] and Vorst[2]. Many scholars considered for the case of $\mu = 0$ to (2) (see[3,4,5]). Pucci and Serrin[4] showed that the problem (2) admits a nontrivial radially symmetric solution for all $\lambda \in (0, \lambda_1)$ if and only if $N \geq 8$, where λ_1 is the first eigenvalue of Δ^2 with the homogeneous Dirichlet boundary conditions on B. While, they also proved that when $N \in \{5, 6, 7\}$, there exist $0 < \lambda_* \leq \lambda^* < \lambda_1$ such that the problem (2) admits a positive radially symmetric solution for all $\lambda \in (\lambda^*, \lambda_1)$ and no nontrivial radial solution for all $\lambda \in (0, \lambda_*]$.

On the other hand, if $k = \infty$, then (1) can be written as the following problem with the Dirichlet boundary conditions:

$$\begin{cases} \Delta^2 u = \lambda u + \mu \Delta u + |u|^{2^{**}-2}u & \text{in} \quad B, \\ u = u_\nu = 0 & \text{on} \quad \partial B. \end{cases} \tag{3}$$

Problem (3) with $\mu = 0$ in a general bounded smooth domain Ω was studied by Gu, Deng and Wang[6]. They showed that: (1) For $N \geq 8$, problem (3) possesses at least one nontrivial weak solutions if $\lambda \in (0, \delta_1(\Omega))$; (2) For $N \in \{5, 6, 7\}$ and $\Omega = B_R(0) \subset R^N$, there exist two positive constants $\lambda^{**}(N) < \lambda^*(N) < \delta_1(\Omega)$ such that problem (3) has at least one nontrivial weak solutions if $\lambda \in (\lambda^*(N), \delta_1(\Omega))$, and problem (3) has no nontrivial solutions if $\lambda < \lambda^{**}(N)$, where $\delta_1(\Omega)$ denotes the first eigenvalue of $-\Delta^2$ with homogeneous Dirichlet boundary condition on Ω. What has shown above implies that $N \in \{5, 6, 7\}$ are the critical dimensions of nontrivial solutions for (3).

Recently, the last two authors of this article[7] considered Problem (3) in a general bounded smooth domain Ω and proved that problem (3) possesses at least one nontrivial weak solution, provided one of the following assumptions holds: (1) $N \geq 5$, $\mu = 0$ and $\lambda \in (\lambda^*(N), \delta_1(\Omega))$; (2) $N \geq 6$, $\mu \in (-\beta(\Omega), 0)$ and $\lambda < \frac{(\mu+\beta(\Omega))\delta_1(\Omega)}{\beta(\Omega)}$; (3) $N = 5$, $(\lambda, \mu) \in A := \{(\lambda, \mu) | \lambda \in (-\infty, \delta_1(\Omega)), \max\{-\beta(\Omega), \frac{\beta(\Omega)}{\delta_1(\Omega)}\lambda - \beta(\Omega)\} < \mu\} \bigcap B := \{(\lambda, \mu) | \mu < 0.0317\lambda - 11.8681\}$, where $\beta(\Omega) := \inf\limits_{u \in H_0^2(\Omega)\backslash\{0\}} \frac{\|u\|_2^2}{\int_\Omega |\nabla u|^2 dx}$, $\delta_1(\Omega)$ denotes the first eigenvalue of $-\Delta^2$ with Dirichlet boundary condition on Ω and $\lambda^*(N)$ is a nonnegative constant depending only on N. At the same time, He and Lv also showed that there are no nontrivial solutions to (3) in $H_0^2(\Omega) \cap C^4(\Omega)$ for $\mu > \max\{0, \frac{2}{\lambda_1(\Omega)}\lambda\}$ if Ω is a starshaped domain.

In recent years, many researchers have studied the existence and nonexistence of solutions to the critical biharmonic problems (1). For example, Gazzda and Pierotti[8] studied the existence and uniqueness of solutions to the fourth-order nonlinear critical problems under the Steklov boundary conditions:

$$\begin{cases} \Delta^2 u = u^{2^{**}-1} & \text{in} \quad B, \\ u > 0 & \text{in} \quad B, \\ u = \Delta u + k u_\nu = 0 & \text{on} \quad \partial B, \end{cases} \tag{4}$$

where $k \in R$. The main results they showed are that if $k \geq -4$ or $k \leq -N$, then the problem (4) has no solutions, and if $-N < k < -4$, then (4) has a unique radially symmetric solution. In[9], Berchio and Gazzola investigated the existence and nonexistence results of positive solutions for linearly perturbed critical growth biharmonic problem with the Steklov boundary conditions:

$$\begin{cases} \Delta^2 u = \lambda u + u^{2^{**}-1} & \text{in} \quad B, \\ u > 0 & \text{in} \quad B, \\ u = \Delta u + k u_\nu = 0 & \text{on} \quad \partial B, \end{cases} \tag{5}$$

where $\lambda > 0$ and $k \in R$. They showed that: (1) if $N \in \{5, 6, 7\}$ and $-N < k \leq -4$ or $N \geq 8$ and $k > -N$, then there is a radial symmetric solution to (5) for $\lambda \in (0, \lambda_1(k))$; (2) For $N \geq 5$, if $k > -N$ and $\lambda \geq \lambda_1(k)$ or $k \leq -N$, then problem (5) admits no solutions, where $\lambda_1(k)$ is the first eigenvalue of the operator Δ^2 under Steklov boundary conditions.

Inspired by the results mentioned above, it is nature to think about what happen if we add another term $\mu\Delta u$ to the first equation of (5), and how the term $\mu\Delta u$ affects the existence, nonexistence and the critical dimensions of nontrivial solutions to (1). So we want to study problem (1).

Before stating our results, we introduce some definitions and notations. The variational functional, corresponding to problem (1), can be defined by

$$\varphi(u) = \frac{1}{2}\int\limits_B (|\Delta u|^2 + \mu|\nabla u|^2 - \lambda u^2)dx + \frac{k}{2}\int\limits_{\partial B} |u_\nu|^2 d\omega - \frac{1}{2^{**}}\int\limits_B |u|^{2^{**}}dx, u \in H^2(B) \cap H_0^1(B).$$

It is easy to see that the functional $\varphi(u) \in C^2(H^2(B) \cap H_0^1(B), R)$. The weak solution of problem (1) can be defined as

$$\int\limits_B (\Delta u \Delta v + \mu \nabla u \nabla v)dx + k\int\limits_{\partial B} u_\nu v_\nu d\omega = \int\limits_B (\lambda u + |u|^{2^{**}-2}u)v dx, \quad \forall v \in C_0^\infty(B).$$

We call $\{u_n\}$ a $(PS)_c$ sequence, if

$$\varphi(u_n) \longrightarrow c \quad and \quad \varphi'(u_n) \longrightarrow 0, \quad as \quad n \longrightarrow \infty, \tag{6}$$

and say the functional φ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{u_n\}$ has a convergent subsequence. Hereafter we set

$$\lambda^*(N) := \begin{cases} 179.7135 & N = 5, \\ 133.0121 & N = 6, \\ 80.0706 & N = 7, \\ 0 & N \geq 8, \end{cases}$$

$$a_N := \begin{cases} 25.8611 & N = 5, \\ 22.3309 & N = 6, \\ 16.6477 & N = 7, \\ 0 & N \geq 8, \end{cases}$$

and

$$\beta(B) := \inf_{u \in H^2(B) \cap H_0^1(B) \setminus \{0\}} \frac{\int_B |\Delta u|^2 dx}{\int_B |\nabla u|^2 dx}. \tag{7}$$

If $u \in H^2(B) \cap H_0^1(B) \setminus \{0\}$, then $u \in H_0^1(B) \setminus \{0\}$. From the Poincaré inequality, we can see that $\int_B |\nabla u|^2 dx > 0$, which implies that $\beta(B)$ is well-defined. For any $k \geq -N$, we let $\lambda_1(k)$ to be the first eigenvalue of operator Δ^2 under the Steklov boundary conditions, namely

$$\lambda_1(k) := \inf_{u \in H^2(B) \cap H_0^1(B) \setminus \{0\}} \frac{\int_B |\Delta u|^2 dx + k \int_{\partial B} u_\nu^2 d\omega}{\int_B |u|^2 dx}. \tag{8}$$

Our results about the existence of nontrivial solutions to (1) can be stated as follows:

Theorem 1. Problem (1) has at least one nontrivial weak solution, provided one of the following assumptions holds:
(1) $N \geq 5, \mu = 0, k \in (-N, 0) \cup (0, +\infty)$ and $\lambda^*(N) + a_N k < \lambda < \lambda_1(k)$;
(2) $N = 5, k \in (-N, 0) \cup (0, +\infty)$, and $(\lambda, \mu) \in A_1 := \{(\lambda, \mu) \big| \lambda < \lambda_1(k), \mu > \min\{1, \frac{N+k}{N}\} \max\{-\beta(B), \frac{\beta(B)}{\lambda_1(k)}\lambda - \beta(B)\} \cap A_2 := \{(\lambda, \mu) \big| \mu \neq 0, \mu < 0.0396\lambda - 1.0238k - 7.1149\}$;
(3) $N \geq 6, k \in (-N, 0) \cup (0, +\infty), \lambda < \lambda_1(k)$ and $0 > \mu > \min\{1, \frac{N+k}{N}\} \max\{-\beta(B), \frac{\beta(B)}{\lambda_1(k)}\lambda - \beta(B)\}$.

The following Theorem is our another results on the nonexistence of nontrivial solution to (1):

Theorem 2. There are no nontrivial solutions of (1) in $H^2(B) \cap H_0^1(B) \cap C^4(\overline{B})$, when k, λ and μ satisfy one of the following two conditions:
(1) $(\lambda, \mu, k) \in \{(\lambda, \mu, k) \big| \mu < \min\{0, \frac{2}{\lambda_1(B)}\lambda, k^2 + (N-4)k\}\}$;
(2) $(\lambda, \mu, k) \in \{(\lambda, \mu, k) \big| k^2 + (N-4)k - \mu \leq 0, \mu > \max\{0, \frac{2}{\lambda_1(B)}\lambda\}\}$.

We will prove Theorem 1 by the Mountain pass theorem. To apply the Mountain pass theorem, we firstly need to introduce a equivalent norm of $H^2(B) \cap H_0^1(B)$ and show that the variational functional has the Mountain pass geometry structure, which implies that we can get a $(PS)_c$ sequence $\{u_n\}$ of φ. We can get the boundedness of $(PS)_c$ sequence $\{u_n\}$ easily. However, we can not obtain that the functional φ satisfies the $(PS)_c$ condition directly, since the embedding of $H^2(B) \hookrightarrow L^{2^{**}}(B)$ is not compact. Therefore, to get the compactness, we have to compare the Mountain pass level energy and the ground state energy of the limiting problem of (1)(See (34)). In the process of comparing the energies, we have to construct some special functions and introduce some new skills. As to the nonexistence, we mainly apply the Pohozaev identity to show it. At the same time, we need some variational theories and some meticulous calculations.

This paper is organized as follows: In Section 2, we firstly introduce a new norm $||.||_1$ of $H^2(B) \cap H_0^1(B)$ and show the equivalence of the norm $||.||_1$ and the standard norm of $H^2(B)$ in some specific condition, and secondly check the Mountain pass geometry structure and show the $(PS)_c$ condition under the assumption of $c < \frac{2}{N}S^{\frac{N}{4}}$. In Section 3, we construct some function $u_0 \in H^2(B) \cap H_0^1(B)$ such that $\sup_{t \geq 0} \varphi(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$ under suitable assumptions. We put the proofs of our results into Section 4.

2 | PRELIMINARIES

According to [10], we get the following inequality:

$$\int_B |\Delta u|^2 dx \geq N \int_{\partial B} |u_\nu|^2 d\omega, \quad u \in H^2(B) \cap H_0^1(B). \tag{9}$$

One can check that if $k \in (-N, 0) \cup (0, +\infty)$, $\lambda < \lambda_1(k)$ and $\mu > \min\{1, \frac{N+k}{N}\} \max\{-\beta(B), \frac{\beta(B)}{\lambda_1(k)}\lambda - \beta(B)\}$ and then

$$(u, v) := \int_B (\Delta u \Delta v + \mu \nabla u \nabla v - \lambda uv) dx + k \int_{\partial B} u_\nu v_\nu d\omega$$

is a inner product in $H^2(B) \cap H_0^1(B)$, which can induce a norm $\|u\|_1 = (\int_B (|\Delta u|^2 + \mu|\nabla u|^2 - \lambda|u|^2) dx + k \int_{\partial B} |u_\nu|^2 d\omega)^{\frac{1}{2}}$. Following from [11,12], we can see that the norm $\|\cdot\|_2$ is equivalent to the norm $\|\cdot\|_{H^2(B)}$, where $\|u\|_2 = (\int_B |\Delta u|^2 dx)^{\frac{1}{2}}$ and $\|\cdot\|_{H^2(B)}$ denotes the standard norm of $H^2(B)$. Therefore, if we can show that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$, then we obtain that $\|\cdot\|_1$ is a equivalent norm of $\|\cdot\|_{H^2(B)}$.

Let $D^{2,2}(R^N)$ denote the closure of $C_0^\infty(R^N)$ under the norm $\|\cdot\|_2$. We know that the best Sobolev constant for the embedding $D^{2,2}(R^N) \hookrightarrow L^{2^{**}}(R^N)$ can be characterized by

$$S = \inf\{\frac{\int_{R^N} |\Delta u|^2}{(\int_{R^N} |u|^{2^{**}})^{\frac{2}{2^{**}}}} : u \in D^{2,2}(R^N) \backslash \{0\}\}, \tag{10}$$

which can be attained by $lu_{\varepsilon,x_0}, \forall l \neq 0$, where

$$u_{\varepsilon,x_0} = \frac{[(N-4)(N-2)N(N+2)]^{\frac{(N-4)}{8}} \varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |x - x_0|^2)^{\frac{(N-4)}{2}}}, \quad \forall x_0 \in R^N, \forall \varepsilon > 0, \tag{11}$$

(see [13,14,15]). We also see that, up to translations and dilations, u_{ε,x_0} is the unique positive solution of

$$\begin{cases} \Delta^2 u = u^{2^{**}-1}, & x \in R^N \\ u \in H^2(R^N), & u > 0, \end{cases}$$

and

$$|\Delta u_{\varepsilon,x_0}|_2^2 = |u_{\varepsilon,x_0}|_{2^{**}}^{2^{**}} = S^{\frac{N}{4}}. \tag{12}$$

Lemma 1. If $k \in (-N, 0) \cup (0, +\infty)$, $\lambda < \lambda_1(k)$ and $\mu > \min\{1, \frac{N+k}{N}\} \max\{-\beta(B), \frac{\beta(B)}{\lambda_1(k)}\lambda - \beta(B)\}$, then the norm $\|u\|_1$ is equivalent to $\|u\|_2$ in $H^2(B) \cap H_0^1(B)$.

Proof. According to (9), (7) and (8), we have

$$\int_{\partial B} |u_\nu|^2 d\omega \leq \frac{1}{N} \int_B |\Delta u|^2 dx, \quad \forall u \in H^2(B) \cap H_0^1(B), \tag{13}$$

$$\beta(B) \int_B |\nabla u|^2 dx \leq \int_B |\Delta u|^2 dx, \quad \forall u \in H^2(B) \cap H_0^1(B), \tag{14}$$

and

$$\lambda_1(k) \int_B |u|^2 dx \leq \int_B |\Delta u|^2 dx + k \int_{\partial B} u_\nu^2 d\omega, \quad \forall u \in H^2(B) \cap H_0^1(B). \tag{15}$$

By (14) and (15), we obtain

$$\begin{aligned} \|u\|_1^2 &= \int_B (|\Delta u|^2 + \mu|\nabla u|^2 - \lambda|u|^2) dx + k \int_{\partial B} |u_\nu|^2 d\omega \\ &\leq \int_B |\Delta u|^2 dx + \frac{|\mu|}{\beta(B)} \int_B |\Delta u|^2 dx + \frac{|\lambda|}{\lambda_1(k)} (\int_B |\Delta u|^2 dx + k \int_{\partial B} u_\nu^2 d\omega) + k \int_{\partial B} |u_\nu|^2 d\omega \\ &\leq (1 + \frac{|\mu|}{\beta(B)} + \frac{|\lambda|}{\lambda_1(k)}) \int_B |\Delta u|^2 dx + (\frac{|\lambda k|}{\lambda_1(k)} + |k|) \int_{\partial B} |u_\nu|^2 d\omega \\ &\leq (1 + \frac{|\mu|}{\beta(B)} + \frac{|\lambda|}{\lambda_1(k)}) \int_B |\Delta u|^2 dx + \frac{\frac{|\lambda k|}{\lambda_1(k)} + |k|}{N} \int_B |\Delta u|^2 dx \\ &\leq C \int_B |\Delta u|^2 dx = C\|u\|_2^2. \end{aligned} \tag{16}$$

Next, we will prove $\|u\|_1^2 \geq C\|u\|_2^2$:

(i) When $k>0$, $\lambda\le 0$ and $\mu>-\beta(B)$, it is easy to know that $c_1:=1+\min\{0,\frac{\mu}{\beta(B)}\}>0$. We can deduce that

$$\|u\|_1^2\ge\int\limits_B|\Delta u|^2dx+\mu\int\limits_B|\nabla u|^2dx\ge c_1\int\limits_B|\Delta u|^2dx=c_1\|u\|_2^2. \tag{17}$$

(ii) When $-N<k<0$, $\lambda\le 0$ and $\mu\ge 0$, we can deduce that $c_2:=1+\frac{k}{N}>0$ and

$$\|u\|_1^2\ge\int\limits_B|\Delta u|^2dx+k\int\limits_{\partial B}u_\nu^2d\omega\ge(1+\frac{k}{N})\int\limits_B|\Delta u|^2dx=c_2\|u\|_2^2. \tag{18}$$

(iii) When $-N<k<0$, $\lambda\le 0$ and $-\frac{(N+k)\beta(B)}{N}<\mu<0$, we have $c_3:=1+\frac{k}{N}+\frac{\mu}{\beta(B)}>0$ and

$$\begin{aligned}\|u\|_1^2&\ge\textstyle\int_B|\Delta u|^2dx+\mu\int_B|\nabla u|^2dx+k\int_{\partial B}|u_\nu|^2d\omega\\&\ge(1+\tfrac{\mu}{\beta(B)}+\tfrac{k}{N})\textstyle\int_B|\Delta u|^2dx=c_3\|u\|_2^2.\end{aligned} \tag{19}$$

(iv) When $k>0, 0<\lambda<\lambda_1(k)$ and $\mu\ge 0$, we can deduced that $c_4:=1-\frac{\lambda}{\lambda_1(k)}>0$ and

$$\begin{aligned}\|u\|_1^2&\ge\textstyle\int_B|\Delta u|^2dx-\lambda\int_B|u|^2dx+k\int_{\partial B}u_\nu^2d\omega\\&\ge(1-\tfrac{\lambda}{\lambda_1(k)})\textstyle\int_B|\Delta u|^2dx+k(1-\tfrac{\lambda}{\lambda_1(k)})\int_{\partial B}u_\nu^2d\omega\\&\ge(1-\tfrac{\lambda}{\lambda_1(k)})\textstyle\int_B|\Delta u|^2dx=c_4\|u\|_2^2.\end{aligned} \tag{20}$$

(v) When $-N<k<0, 0<\lambda<\lambda_1(k)$ and $\mu\ge 0$, we can deduced that $c_5:=(1+\frac{k}{N})(1-\frac{\lambda}{\lambda_1(k)})>0$ and

$$\begin{aligned}\|u\|_1^2&\ge\textstyle\int_B|\Delta u|^2dx-\lambda\int_B|u|^2dx+k\int_{\partial B}u_\nu^2d\omega\\&\ge(1-\tfrac{\lambda}{\lambda_1(k)})\textstyle\int_B|\Delta u|^2dx+k(1-\tfrac{\lambda}{\lambda_1(k)})\int_{\partial B}u_\nu^2d\omega\\&\ge(1+\tfrac{k}{N})(1-\tfrac{\lambda}{\lambda_1(k)})\textstyle\int_B|\Delta u|^2dx=c_5\|u\|_2^2.\end{aligned} \tag{21}$$

(vi) When $k>0, 0<\lambda<\lambda_1(k)$ and $\frac{\beta(B)}{\lambda_1(k)}\lambda-\beta(B)<\mu<0$, we can deduced that $c_6:=1+\frac{\mu}{\beta(B)}-\frac{\lambda}{\lambda_1(k)}>0$ and

$$\begin{aligned}\|u\|_1^2&=\textstyle\int_B(|\Delta u|^2+\mu|\nabla u|^2-\lambda|u|^2)dx+k\int_{\partial B}u_\nu^2d\omega\\&\ge(1+\tfrac{\mu}{\beta(B)}-\tfrac{\lambda}{\lambda_1(k)})\textstyle\int_B|\Delta u|^2dx+k(1-\tfrac{\lambda}{\lambda_1(k)})\int_{\partial B}u_\nu^2d\omega\\&\ge(1+\tfrac{\mu}{\beta(B)}-\tfrac{\lambda}{\lambda_1(k)})\textstyle\int_B|\Delta u|^2dx=c_6\|u\|_2^2.\end{aligned} \tag{22}$$

(vii) When $-N<k<0, 0<\lambda<\lambda_1(k)$ and $\frac{N+k}{N}(\frac{\beta(B)}{\lambda_1(k)}\lambda-\beta(B))<\mu<0$, it is easy to see that $c_7:=\frac{\mu}{\beta(B)}+(1+\frac{k}{N})(1-\frac{\lambda}{\lambda_1(k)})>0$. We can deduced that

$$\begin{aligned}\|u\|_1^2&=\textstyle\int_B(|\Delta u|^2+\mu|\nabla u|^2-\lambda|u|^2)dx+k\int_{\partial B}|u_\nu|^2d\omega\\&\ge(1+\tfrac{\mu}{\beta(B)}-\tfrac{\lambda}{\lambda_1(k)})\textstyle\int_B|\Delta u|^2dx+k(1-\tfrac{\lambda}{\lambda_1(k)})\int_{\partial B}u_\nu^2dx\\&\ge(1+\tfrac{\mu}{\beta(B)}-\tfrac{\lambda}{\lambda_1(k)})\textstyle\int_B|\Delta u|^2dx+\tfrac{k}{N}(1-\tfrac{\lambda}{\lambda_1(k)})\int_B|\Delta u|^2dx\\&\ge(\tfrac{\mu}{\beta(B)}+(1+\tfrac{k}{N})(1-\tfrac{\lambda}{\lambda_1(k)}))\textstyle\int_B|\Delta u|^2dx=c_7\|u\|_2^2.\end{aligned} \tag{23}$$

To sum up, it is obvious that the Lemma is true. □

Lemma 2. The functional $\varphi(u)$ has Mountain pass geometry structure:

(i) there exist two constants $\alpha,\rho>0$ such that $\varphi(v)\ge\alpha$ for all $\|v\|_1=\rho$;

(ii) there exists $\omega\in H^2(B)\cap H_0^1(B)$ such that $\varphi(\omega)<0$ and $\|\omega\|_1>\rho$.

Proof. According to the fact that the embedding $H^2(B)\hookrightarrow L^{2^{**}}(B)$ is continuous, we know

$$\begin{aligned}\varphi(u)&=\frac{1}{2}\textstyle\int_B(|\Delta u|^2+\mu|\nabla u|^2-\lambda u^2)dx+\dfrac{k}{2}\int_{\partial B}|u_\nu|^2d\omega-\dfrac{1}{2^{**}}\int_B|u|^{2^{**}}dx\\&=\frac{1}{2}\|u\|_1^2-\frac{1}{2^{**}}|u|_{2^{**}}^{2^{**}}\\&\ge\frac{1}{2}\|u\|_1^2-C\|u\|_1^{2^{**}},\end{aligned}$$

which implies that we can find $\alpha>0$ and $\rho>0$ such that $\varphi(v)\ge\alpha>0$ for all $\|v\|_1=\rho$ with ρ small enough.

On the other hand, for any fixed $u\in H^2(B)\cap H_0^1(B)\backslash\{0\}$, we have

$$\varphi(tu)=\frac{t^2}{2}\int\limits_B(|\Delta u|^2+\mu|\nabla u|^2-\lambda u^2)dx+\frac{t^2}{2}k\int\limits_{\partial B}|u_\nu|^2d\omega-\frac{t^{2^{**}}}{2^{**}}\int\limits_B|u|^{2^{**}}dx\to-\infty,$$

as $t \to +\infty$. So we can take $\omega = t_0 u$ with t_0 sufficiently large such that the conclusion (ii) is true. □

Lemma 3. Any $(PS)_c$ sequence $\{u_m\}$ of the functional φ is bounded in $H^2(B) \cap H_0^1(B)$.

Proof. According to the definition of $(PS)_c$ sequence, we have that, for m large enough,

$$\begin{aligned}\varphi(u_m) &= \frac{1}{2}\int_B(|\Delta u_m|^2 + \mu|\nabla u_m|^2 - \lambda|u_m|^2)dx + \frac{k}{2}\int_{\partial B}|\frac{\partial u_m}{\partial \nu}|^2 d\omega - \frac{1}{2^{**}}\int_B |u_m|^{2^{**}}dx \\ &= c + o_m(1),\end{aligned}$$

and

$$\begin{aligned}&| < \varphi'(u_m), u_m > | \\ &= |\int_B(|\Delta u_m|^2 + \mu|\nabla u_m|^2 - \lambda|u_m|^2)dx + k\int_{\partial B}|\frac{\partial u_m}{\partial \nu}|^2 d\omega - \int_B |u_m|^{2^{**}}dx| \\ &= o_m(1)\|u_m\|_1,\end{aligned}$$

which implies that

$$c + 1 + \|u_m\|_1 \geq \varphi(u_m) - \frac{1}{2^{**}} < \varphi'(u_m), u_m >= \frac{2}{N}\|u_m\|_1^2.$$

Therefore, $\{u_m\}$ is bounded in $H^2(B) \cap H_0^1(B)$. □

Lemma 4. If $c < \frac{2}{N}S^{\frac{N}{4}}$, then $\varphi(u)$ satisfies the $(PS)_c$ condition, where S is as in (10)

Proof. Let $\{u_m\}$ be a $(PS)_c$ sequence of φ. By Lemma 3, we see that $\{u_m\}$ is bounded in $H^2(B) \cap H_0^1(B)$. So $\{\nabla u_m\}$ is also bounded in $H^1(B)$. By the imbeddings of $H^1(B) \hookrightarrow L^2(\partial B)$ and $H^2(B) \cap H_0^1(B) \hookrightarrow L^2(B)$ are compact. There exist a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and a $u \in H^2(B) \cap H_0^1(B)$

$$\begin{aligned}&u_m \rightharpoonup u \quad weakly \ \ in \ \ H^2(B) \cap H_0^1(B), \\ &(u_m)_\nu \to u_\nu \quad strongly \ \ in \ \ L^2(\partial B), \\ &u_m \to u \quad strongly \ \ in \ \ L^2(B), \\ &u_m \to u \quad a.e. \ \ on \ \ B.\end{aligned} \tag{24}$$

By the definition of $(PS)_c$ sequence, we have, for any $\phi \in C_0^\infty$,

$$< \varphi'(u), \phi >= \lim_{m \to +\infty} < \varphi'(u_m), \phi >= 0,$$

which implies that u is a weak solution of

$$\Delta^2 u = \lambda u + \mu \Delta u + |u|^{2^{**}-2}u,$$

and

$$\int_B(|\Delta u|^2 + \mu|\nabla u|^2 - \lambda u^2)dx + k\int_{\partial B}|u_\nu|^2 d\omega - \int_B |u|^{2^{**}}dx = 0. \tag{25}$$

Then

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_B(|\Delta u|^2 + \mu|\nabla u|^2 - \lambda u^2)dx + \frac{k}{2}\int_{\partial B}|u_\nu|^2 d\omega - \frac{1}{2^{**}}\int_B |u|^{2^{**}}dx \\ &= \frac{1}{2}\int_B |u|^{2^{**}}dx - \frac{1}{2^{**}}\int_B |u|^{2^{**}}dx \\ &= \frac{2}{N}\int_B |u|^{2^{**}}dx \geq 0.\end{aligned} \tag{26}$$

On the other hand, according to the definition of $(PS)_c$ sequence again, we have

$$\int_B (|\Delta u_m|^2 + \mu|\nabla u_m|^2 - \lambda u_m^2)dx + k\int_{\partial B}|\frac{\partial u_m}{\partial \nu}|^2 d\omega - \int_B |u_m|^{2^{**}}dx = o_m(1), \tag{27}$$

and

$$\frac{1}{2}\int_B (|\Delta u_m|^2 + \mu|\nabla u_m|^2 - \lambda u_m^2)dx + \frac{k}{2}\int_{\partial B}|\frac{\partial u_m}{\partial \nu}|^2 d\omega - \frac{1}{2^{**}}\int_B |u_m|^{2^{**}}dx = c + o_m(1). \tag{28}$$

Let $v_m = u_m - u$. It follows from Brézis-Lieb Lemma [16] that

$$\int_B |\Delta u_m|^2 dx = \int_B |\Delta u|^2 dx + \int_B |\Delta v_m|^2 dx + o_m(1), \tag{29}$$

and

$$\int_B |u_m|^{2^{**}}dx = \int_B |u|^{2^{**}}dx + \int_B |v_m|^{2^{**}}dx + o_m(1). \tag{30}$$

Since (24) and the embedding of $H^2(B) \cap H_0^1(B) \hookrightarrow H_0^1(B)$ is compact, we have

$$\begin{cases} \int_B |\nabla u_m|^2 dx = \int_B |\nabla u|^2 dx + o_m(1), \\ \int_{\partial B} |\frac{\partial u_m}{\partial \nu}|^2 d\omega = \int_{\partial B} u_\nu^2 d\omega + o_m(1), \\ \int_B u_m^2 dx = \int_B u^2 dx + o_m(1). \end{cases} \tag{31}$$

Thus, following from (25), (27), (28), (29), (30) and (31), we obtain that

$$\int_B |\Delta v_m|^2 dx - \int_B |v_m|^{2^{**}} dx = o_m(1), \tag{32}$$

and

$$\varphi(u) + \frac{1}{2} \int_B |\Delta v_m|^2 dx - \frac{1}{2^{**}} \int_B |v_m|^{2^{**}} dx = c + o_m(1). \tag{33}$$

We may suppose

$$\int_B |\Delta v_m|^2 dx \longrightarrow a, \quad as \quad m \longrightarrow \infty.$$

Following from (32), we have

$$\int_B |v_m|^{2^{**}} dx \longrightarrow a, \quad as \quad m \longrightarrow \infty.$$

According to the definition of S, we obtain

$$|\Delta u|_2^2 \geq S|u|_{2^{**}}^2, \quad \forall u \in H^2(B) \cap H_0^1(B),$$

which implies that

$$a + o_m(1) = \int_B |\Delta v_m|^2 dx \geq S(\int_B |v_m|^{2^{**}} dx)^{\frac{2}{2^{**}}} = S a^{\frac{N-4}{N}} + o_m(1).$$

If $a > 0$, then $a \geq S^{\frac{N}{4}}$ and

$$\varphi(u) = c - (\frac{1}{2} a - \frac{1}{2^{**}} a) = c - \frac{2}{N} a \leq c - \frac{2}{N} S^{\frac{N}{4}} < 0,$$

which contradicts to (26). Thus $a = 0$, which gives that

$$\int_B |\Delta v_m|^2 dx \longrightarrow 0, \quad as \quad m \longrightarrow \infty.$$

So $u_m \longrightarrow u$ strongly in $H^2(B)$. □

Lemma 5. If $k \in (-N, 0) \cup (0, +\infty)$, $\mu > \min\{1, \frac{N+k}{N}\} \max\{-\beta(B), \frac{\beta(B)}{\lambda_1(k)} \lambda - \beta(B)\}$, $\lambda < \lambda_1(k)$ and there exists a function $u_0 \in H^2(B) \cap H_0^1(B) \backslash \{0\}$ such that

$$\sup_{t \geq 0} \varphi(t u_0) < \frac{2}{N} S^{\frac{N}{4}}, \tag{34}$$

then problem (1) has at least one nontrivial solution.

Proof. According to (34), we have

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \varphi(\gamma(t)) \leq \sup_{t \geq 0} \varphi(t u_0) < \frac{2}{N} S^{\frac{N}{4}},$$

where $\Gamma := \{\gamma \in C([0,1], H^2(B) \cap H_0^1(B)) : \gamma(0) = 0, \varphi(\gamma(1)) < 0\}$. Thus, by Lemmas 2, 3, 4 and the Mountain pass theorem, we can see that c is a critical value of φ. Therefore there exists a function $\omega \in H^2(B) \cap H_0^1(B) \backslash \{0\}$ such that $\varphi(\omega) = c, \varphi'(\omega) = 0$, which implies that ω is a nontrivial weak solution of problem (1). □

3 | VERIFICATION OF (34)

When $N = 5, 6, 7$, let

$$\chi_\varepsilon(x) = \psi(x) u_{\varepsilon,0}(x), \tag{35}$$

where $\psi(x)$ is some given function with

$$\psi(x) = \psi(|x|) \in C^2(\overline{B}, R), \psi(0) = 1, \psi(1) = 0. \tag{36}$$

By direct computation, we have

$$|\frac{\partial \chi_\varepsilon}{\partial \nu}|^2 = (\psi' u_{\varepsilon,0} + \psi u'_{\varepsilon,0})^2 = (\psi')^2 u^2_{\varepsilon,0} + \psi^2 (u'_{\varepsilon,0})^2 + 2\psi\psi' u_{\varepsilon,0} u'_{\varepsilon,0},$$

and

$$|\frac{\partial \chi_\varepsilon}{\partial \nu}|^2\Big|_{r=1} = (\psi'(1))^2 u^2_{\varepsilon,0}(1).$$

So

$$\int\limits_{\partial B} |\frac{\partial \chi_\varepsilon}{\partial \nu}|^2 d\omega = |\frac{\partial \chi_\varepsilon}{\partial \nu}|^2\Big|_{r=1} |\partial B| = C_N (\psi'(1))^2 \frac{\varepsilon^{N-4}}{(\varepsilon^2+1)^{N-4}}, \tag{37}$$

where $C_N := [(N-4)(N-2)N(N-2)]^{\frac{N-4}{4}} \omega_N$ and ω_N denotes the surface measure of the unit ball in R^N.

Since $\varphi(0) = 0$ and $\lim\limits_{t\to+\infty} \varphi(t\chi_\varepsilon) = -\infty$, we have that there exists a $t_\varepsilon \in (0, +\infty)$ such that

$$\varphi(t_\varepsilon \chi_\varepsilon) = \sup_{t\geq 0} \varphi(t\chi_\varepsilon), \tag{38}$$

and

$$t_\varepsilon^{\frac{8}{N-4}} = \frac{\int_B (|\Delta\chi_\varepsilon|^2 + \mu|\nabla\chi_\varepsilon|^2 - \lambda|\chi_\varepsilon|^2)dx + k\int_{\partial B} |\frac{\partial\chi_\varepsilon}{\partial\nu}|^2 d\omega}{\int_B |\chi_\varepsilon|^{2^{**}} dx}. \tag{39}$$

Let $g(t) = \frac{1}{2}t^2 - \frac{1}{2^{**}} t^{2^{**}}$. By direct calculation, we can see that $g(t) \leq g(1) = \frac{2}{N}$ for all $t > 0$.

Lemma 6. If $N = 5$ and $12.2924 + 1.7277\mu - 0.0684\lambda + 1.7689k < 0$, then there exists a function $u_0 \in H^2 \cap H^1_0(\Omega)\backslash\{0\}$ such that $\sup\limits_{t\geq 0} \varphi(tu_0) < \frac{2}{N} S^{\frac{N}{4}}$.

Proof. Similar to the proof of Lemma 3.3 in [7], if $\psi(r)$ satisfies (36), $|\psi^2(r) - 1| \leq Cr^{1+\delta}$ and $|\psi^{10}(r) - 1| \leq Cr^{1+\delta}$, then we have, as $\varepsilon \to 0^+$,

$$\begin{aligned}\int_B |\Delta\chi_\varepsilon|^2 dx = & (105)^{\frac{1}{4}} \omega_5 \varepsilon [\int_0^1 (\psi'')^2 r^2 dr + 6\int_0^1 (\psi')^2 dr \\ & + 2(\psi'(1))^2] + S^{\frac{N}{4}} + O(\varepsilon^{1+\delta}),\end{aligned} \tag{40}$$

$$\int\limits_B |\nabla\chi_\varepsilon|^2 dx = (105)^{\frac{1}{4}} \omega_5 \varepsilon [\int\limits_0^1 (\psi')^2 r^2 dr + 2\int\limits_0^1 \psi^2 dr] + O(\varepsilon^2), \tag{41}$$

$$\int\limits_B |\chi_\varepsilon|^2 dx = (105)^{\frac{1}{4}} \omega_5 \varepsilon \int\limits_0^1 \psi^2 r^2 dr + O(\varepsilon^2), \tag{42}$$

and

$$\int\limits_B |\chi_\varepsilon|^{2^{**}} dx = S^{\frac{N}{4}} + O(\varepsilon^{1+\delta}), \tag{43}$$

where $\delta \in (0, 1)$.

Here we choose $\psi(x) = 1 - |x|^{1.33} \sin(\frac{\pi}{2}|x|^{1.12})$. It is easy to check that $\psi(x)$ satisfies (36), $|\psi^2(r) - 1| \leq Cr^{1+\delta}$ and $|\psi^{10}(r) - 1| \leq Cr^{1+\delta}$. By (40), (41), (42) and the matlab, we can obtain that

$$\begin{aligned}&\int_B |\Delta\chi_\varepsilon|^2 dx = 12.2924 * (105)^{\frac{1}{4}} \omega_5 \varepsilon + S^{\frac{N}{4}} + O(\varepsilon^{1+\delta}), \\ &\int_B |\nabla\chi_\varepsilon|^2 dx = 1.7277 * (105)^{\frac{1}{4}} \omega_5 \varepsilon + O(\varepsilon^2), \\ &\int_B |\chi_\varepsilon|^2 dx = 0.0684 * (105)^{\frac{1}{4}} \omega_5 \varepsilon + O(\varepsilon^2).\end{aligned} \tag{44}$$

From (37), direct computation implies that

$$\int_{\partial B} |\frac{\partial \chi_\varepsilon}{\partial \nu}|^2 d\omega = (105)^{\frac{1}{4}}\omega_5\varepsilon\frac{(\psi'(1))^2}{\varepsilon^2+1} = 1.7689 * (105)^{\frac{1}{4}}\omega_5\varepsilon + \mathrm{O}(\varepsilon^3) \tag{45}$$

since $\frac{1}{\varepsilon^2+1} = 1-\varepsilon^2+o(\varepsilon^2)$.

So, from (39) and (43)–(45), we have

$$t_\varepsilon^{\frac{8}{N-4}} = \frac{S^{\frac{N}{4}}+\mathrm{O}(\varepsilon)}{S^{\frac{N}{4}}+\mathrm{O}(\varepsilon^{1+\delta})} \in [\frac{1}{2},2] \quad for\ \ \varepsilon\ \ small\ \ enough.$$

and as $\varepsilon \to 0^+$,

$$\begin{aligned}\sup_{t\geq 0}\varphi(t\chi_\varepsilon) &= \varphi(t_\varepsilon\chi_\varepsilon)\\ &= (\frac{t_\varepsilon^2}{2}-\frac{t_\varepsilon^{2^{**}}}{2^{**}})S^{\frac{N}{4}} + \frac{t_\varepsilon^2}{2}105^{\frac{1}{4}}\omega_5\varepsilon(12.2924+1.7277\mu-0.0684\lambda\\ &\quad + 1.7689k+\mathrm{O}(\varepsilon^2)) + \mathrm{O}(\varepsilon^{1+\delta})\\ &< \frac{2}{N}S^{\frac{N}{4}},\end{aligned} \tag{46}$$

if $12.2924+1.7277\mu-0.0684\lambda+1.7689k<0$. Therefore, we can choose $u_0 = \chi_\varepsilon \in H^2(B)\cap H_0^1(B)$ (ε small enough) such that $\sup_{t\geq 0}\varphi(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$. □

Lemma 7. If $N=6$, then there exists a function $u_0 \in H^2(B)\cap H_0^1(B)\backslash\{0\}$ such that $\sup_{t\geq 0}\varphi(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$, provided one of the following assumptions holds:
(i) $\mu = 0,\ 27.5335 - 0.2070\lambda + 4.6225k < 0$;
(ii) $\mu < 0,\ \lambda, k \in R$.

Proof. Similar to the proof of Lemma 3.5 of[7], if ψ satisfies (36) and $\frac{(\psi')^2}{r} \leq C$, $|\psi^2(r)-1| \leq Cr^{2+\delta}$, $|\psi^6(r)-1| \leq Cr^{2+\delta}$, then we have, as $\varepsilon \to 0^+$,

$$\begin{aligned}\textstyle\int_B |\Delta\chi_\varepsilon|^2 dx = {}& (384)^{\frac{1}{2}}\omega_6\varepsilon^2[9\textstyle\int_0^1 \frac{(\psi')^2}{r}dr + (\psi'(1))^2 - (\psi'(0))^2\\ &+ \textstyle\int_0^1(\psi'')^2 rdr] + S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{2+\delta}),\end{aligned} \tag{47}$$

$$\textstyle\int_B |\nabla\chi_\varepsilon|^2 dx = (384)^{\frac{1}{2}}\omega_6\varepsilon^2[\int_0^1(\psi')^2 rdr + 2 + 4\int_0^1 \frac{r^7}{(\varepsilon^2+r^2)^4}dr + 4\int_0^1 \frac{\psi^2-1}{r}dr] + \mathrm{O}(\varepsilon^3), \tag{48}$$

$$\int_B |\chi_\varepsilon|^2 dx = (384)^{\frac{1}{2}}\omega_6\varepsilon^2\int_0^1 \psi^2 rdr + \mathrm{O}(\varepsilon^3), \tag{49}$$

and

$$\int_B |\chi_\varepsilon|^{2^{**}} dx = S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{2+\delta}), \tag{50}$$

where $\delta \in (0,1)$. Let $\psi(x) = 1-|x|^{2.15}\sin(\frac{\pi}{2}|x|^{1.17})$. By matlab, (47) and (49), we can see that, as $\varepsilon \to 0^+$,

$$\begin{aligned}&\textstyle\int_B |\Delta\chi_\varepsilon|^2 dx = 27.5335 * (384)^{\frac{1}{2}}\omega_6\varepsilon^2 + S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{2+\delta}),\\ &\textstyle\int_B |\chi_\varepsilon|^2 dx = 0.2070 * (384)^{\frac{1}{2}}\omega_6\varepsilon^2 + \mathrm{O}(\varepsilon^3).\end{aligned} \tag{51}$$

From (37) and (48), by direct computation, we obtain that

$$\begin{aligned}&\textstyle\int_{\partial B} |\frac{\partial\chi_\varepsilon}{\partial\nu}|^2 d\omega = 4.6225 * (384)^{\frac{1}{2}}\omega_6\varepsilon^2 + \mathrm{O}(\varepsilon^4),\\ &\textstyle\int_B |\nabla\chi_\varepsilon|^2 dx = -2\ln\varepsilon^2 * (384)^{\frac{1}{2}}\omega_6\varepsilon^2 + \mathrm{O}(\varepsilon^2),\end{aligned} \tag{52}$$

where we have used the fact that $\frac{1}{(\varepsilon^2+1)^2} = 1-2\varepsilon^2+o(\varepsilon^2)$. So it follows from (39), (50), (51) and (52) that

$$t_\varepsilon^{\frac{8}{N-4}} = \frac{S^{\frac{N}{4}}+\mathrm{O}(\varepsilon^2\ln\varepsilon^2)}{S^{\frac{N}{4}}+\mathrm{O}(\varepsilon^{2+\delta})} \in [\frac{1}{2},2] \quad for\ \ \varepsilon\ \ small\ \ enough.$$

Thus, if $\mu = 0$ and $27.5335 - 0.2070\lambda + 4.6225k < 0$ or $\mu < 0$ and $\lambda, k \in R$, then we can see that, as $\varepsilon \to 0^+$,

$$\begin{aligned}
&\sup_{t\geq 0} \varphi(t\chi_\varepsilon) \\
&= \varphi(t_\varepsilon \chi_\varepsilon) \\
&= (\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2^{**}}}{2^{**}})S^{\frac{N}{4}} + \frac{t_\varepsilon^2}{2} 384^{\frac{1}{2}}\omega_6 \varepsilon^2[27.5335 + \mu(-2\ln\varepsilon^2 + \mathrm{O}(1)) \\
&- 0.2070\lambda + 4.6225k] + \mathrm{O}(\varepsilon^{2+\delta}) \\
&< \frac{2}{N}S^{\frac{N}{4}},
\end{aligned} \tag{53}$$

which implies that we can choose $u_0 = \chi_\varepsilon \in H^2(B) \cap H_0^1(B)$ (ε small enough) such that $\sup_{t\geq 0} \varphi(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$. $\square$

Lemma 8. If $N = 7$, then there exists a function $u_0 \in H^2(B) \cap H_0^1(B)\backslash\{0\}$ such that $\sup_{t\geq 0} \varphi(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$, provided one of the following assumptions holds:
(i) $\mu = 0,\ 56.5859 - 0.7067\lambda + 11.7649k < 0$;
(ii) $\mu < 0,\ \lambda, k \in R$.

Proof. Similar to the proof of Leamma 3.7 of[7], if ψ satisfies (36), $\frac{(\psi')^2}{r^2} \leq C$, $|\psi^2(r) - 1| \leq Cr^{3+\delta}$ and $||\psi(r)|^{\frac{14}{3}} - 1| \leq Cr^{3+\delta}$, then we have, as $\varepsilon \to 0^+$,

$$\int_B |\Delta\chi_\varepsilon|^2 dx = (945)^{\frac{3}{4}}\omega_7\varepsilon^3[\int_0^1 (\psi'')^2 dr + 12\int_0^1 \frac{(\psi')^2}{r^2} dr] + S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{3+\delta}), \tag{54}$$

$$\int_B |\nabla\chi_\varepsilon|^2 dx = (945)^{\frac{3}{4}}\omega_7\varepsilon^3[\int_0^1 (\psi')^2 dr - 6\int_0^1 \frac{\psi\psi'}{r} dr + 9\int_0^1 \frac{r^8}{(\varepsilon^2+r^2)^5} dr + 9\int_0^1 \frac{\psi^2-1}{r^2} dr] + \mathrm{O}(\varepsilon^4), \tag{55}$$

$$\int_B |\chi_\varepsilon|^2 dx = (945)^{\frac{3}{4}}\omega_7\varepsilon^3 \int_0^1 \psi^2 dr + O(\varepsilon^4), \tag{56}$$

and

$$\int_B |\chi_\varepsilon|^{2^{**}} dx = S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{3+\delta}), \tag{57}$$

where $\delta \in (0, 1)$. Let $\psi(x) = 1 - |x|^{3.43} sin(\frac{\pi}{2}|x|^{1.2})$. By matlab, (54) and (56), we can see that, as $\varepsilon \to 0^+$,

$$\begin{aligned}
\int_B |\Delta\chi_\varepsilon|^2 dx &= 56.5859 * (945)^{\frac{3}{4}}\omega_7\varepsilon^3 + S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{3+\delta}), \\
\int_B |\chi_\varepsilon|^2 dx &= 0.7067 * (945)^{\frac{3}{4}}\omega_7\varepsilon^3 + \mathrm{O}(\varepsilon^4).
\end{aligned} \tag{58}$$

According to (37) and (55), by direct computation, we have

$$\begin{aligned}
\int_{\partial B} |\frac{\partial\chi_\varepsilon}{\partial\nu}|^2 d\omega &= 11.7649 * (945)^{\frac{3}{4}}\omega_7\varepsilon^3 + \mathrm{O}(\varepsilon^5), \\
\int_B |\nabla\chi_\varepsilon|^2 dx &= -\frac{9}{2}\ln(\varepsilon^2) * (945)^{\frac{3}{4}}\omega_7\varepsilon^3 + \mathrm{O}(\varepsilon^3),
\end{aligned} \tag{59}$$

where we have used the fact that $\frac{1}{(\varepsilon^2 + 1)^3} = 1 - 3\varepsilon^2 + o(\varepsilon^2)$. So it follows from (39), (57), (58) and (59) that

$$t_\varepsilon^{\frac{8}{N-4}} = \frac{S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^3 \ln\varepsilon^2)}{S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{3+\delta})} \in [\frac{1}{2}, 2] \quad for\ \ \varepsilon\ \ small\ \ enough.$$

Therefore, when $\mu = 0$ and $56.5859 - 0.7067\lambda + 11.7649k < 0$ or $\mu < 0$ and $\lambda, k \in R$, we have, as $\varepsilon \to 0^+$,

$$\begin{aligned}\sup_{t\geq 0} \varphi(t\chi_\varepsilon) &\leq (\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2^{**}}}{2^{**}})S^{\frac{N}{4}} + \frac{t_\varepsilon^2}{2} 945^{\frac{3}{4}}\omega_7\varepsilon^3[56.5859 + \mu(-\frac{9}{2}\ln\varepsilon^2 + \mathrm{O}(1)) \\ &\quad - 0.7067\lambda + 11.7649k] + \mathrm{O}(\varepsilon^{3+\delta}) \\ &< \frac{2}{N}S^{\frac{N}{4}},\end{aligned} \tag{60}$$

which implies that we can find a $u_0 = \chi_\varepsilon \in H^2(B) \cap H_0^1(B)$ (ε small enough) such that $\sup\limits_{t\geq 0} \varphi(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$. □

Lemma 9. If $N \geq 8$, then there exists a function $u_0 \in H^2(B) \cap H_0^1(B)\backslash\{0\}$ such that $\sup\limits_{t\geq 0} \varphi(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$, provided one of the following assumptions holds:
(i) $\mu < 0,\ \lambda, k \in R$;
(ii) $\mu = 0,\ \lambda > 0,\ k \in R$.

Proof. Let $\psi \in C_0^\infty(R^N, [0, 1])$ be a cut-off function such that

$$\psi(|x|) = \begin{cases} 1 & |x| \leq \rho \\ (0,1) & \rho < |x| < 2\rho \\ 0 & |x| \geq 2\rho, \end{cases} \tag{61}$$

and set

$$\chi_\varepsilon(x) = \psi(x)u_{\varepsilon,0}(x), \tag{62}$$

where $\rho \in (0, \frac{1}{4})$ is any fixed constant. Similar to Lemma 3.1 of[7], we have

$$\begin{aligned}|\Delta\chi_\varepsilon|_2^2 &= S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{N-4}), \\ |\nabla\chi_\varepsilon|_2^2 &= C_N K_1\varepsilon^2 + \mathrm{O}(\varepsilon^{N-4}), \\ |\chi_\varepsilon|_{2^{**}}^{2^{**}} &= S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^N),\end{aligned}$$

and

$$|\chi_\varepsilon|_2^2 = \begin{cases} c_N K_2\varepsilon^4 + \mathrm{O}(\varepsilon^{N-4}), & \text{for} \quad N > 8, \\ -\frac{1}{2}c_8\omega_8\varepsilon^4\ln\varepsilon^2 + \mathrm{O}(\varepsilon^4), & \text{for} \quad N = 8, \end{cases}$$

where $c_N = (N(N-4)(N^2-4))^{\frac{N-4}{4}}$, $C_N = c_N(N-4)^2$, $K_1 = \int_{R^N} \frac{|z|^2}{(1+|z|^2)^{N-4}}dz$ and $K_2 = \int_{R^N} \frac{1}{(1+|z|^2)^{N-4}}dz$. Since

$$|\frac{\partial\chi_\varepsilon}{\partial\nu}|^2 = (\psi' u_{\varepsilon,0} + \psi u_{\varepsilon,0}')^2 = (\psi')^2 u_{\varepsilon,0}^2 + \psi^2(u_{\varepsilon,0}')^2 + 2\psi\psi' u_{\varepsilon,0}u_{\varepsilon,0}', \quad x \in \partial B,$$

we have

$$\int\limits_{\partial B} |\frac{\partial\chi_\varepsilon}{\partial\nu}|^2 d\omega = |\frac{\partial\chi_\varepsilon}{\partial\nu}|^2\Big|_{\partial B}|\partial B| = 0. \tag{63}$$

Then, from (39), we have

$$t_\varepsilon^{\frac{8}{N-4}} = \frac{S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^2)}{S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^N)} \in [\frac{1}{2}, 2] \quad for\ \ \varepsilon\ \ small\ \ enough.$$

(i) If $N > 8$, then we have, for $\varepsilon > 0$ small enough,

$$\begin{aligned}\varphi(t_\varepsilon\chi_\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_B(|\Delta\chi_\varepsilon|^2+\mu|\nabla\chi_\varepsilon|^2-\lambda|\chi_\varepsilon|^2)dx+\frac{k_\varepsilon^2}{2}\int_{\partial B}|\frac{\partial\chi_\varepsilon}{\partial\nu}|^2d\omega\\ &-\frac{t_\varepsilon^{2^{**}}}{2^{**}}\int_B|\chi_\varepsilon|^{2^{**}}dx\\ &=(\frac{t_\varepsilon^2}{2}-\frac{t_\varepsilon^{2^{**}}}{2^{**}})S^{\frac{N}{4}}+\frac{t_\varepsilon^2}{2}(\mu|\nabla\chi_\varepsilon|_2^2-\lambda|\chi_\varepsilon|_2^2)+\mathrm{O}(\varepsilon^{N-4})\\ &\le\frac{2}{N}S^{\frac{N}{4}}+\frac{t_\varepsilon^2}{2}\left(\mu C_NK_1\varepsilon^2-\lambda c_NK_2\varepsilon^4+\mathrm{O}(\varepsilon^{N-4})\right)\\ &=\frac{2}{N}S^{\frac{N}{4}}+\frac{t_\varepsilon^2}{2}\varepsilon^4\big(\mu C_NK_1\frac{1}{\varepsilon^2}-\lambda c_NK_2+\mathrm{O}(\varepsilon^{N-8})\big)\\ &<\frac{2}{N}S^{\frac{N}{4}},\end{aligned}\tag{64}$$

where we have used the facts that $t_\varepsilon^{\frac{8}{N-4}}\in[\frac{1}{2},2]$, $\mu C_NK_1\frac{1}{\varepsilon^2}-\lambda c_NK_2\to-\infty$ as $\varepsilon\to0^+$ for any $\mu<0$ and any $\lambda\in R$, and $-\lambda c_NK_2+\mathrm{O}(\varepsilon^{N-8})<0$ as $\varepsilon\to0^+$ for $\mu=0$ and any $\lambda>0$.

(ii) It is easy to check that if $\mu<0$ and $\lambda\in R$, then $\mu C_NK_1+\frac{\lambda}{2}c_8\omega_8\varepsilon^2\ln\varepsilon^2+\mathrm{O}(\varepsilon^2)<0$ as $\varepsilon\to0^+$, and if $\lambda>0$, then $\frac{\lambda}{2}c_8\omega_8\varepsilon^2\ln\varepsilon^2+\mathrm{O}(\varepsilon^2)<0$ as $\varepsilon\to0^+$. Therefore, when $N=8$ and (i) or (ii) holds, we obtain that for $\varepsilon>0$ small enough,

$$\begin{aligned}\varphi(t_\varepsilon\chi_\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_B(|\Delta\chi_\varepsilon|^2+\mu|\nabla\chi_\varepsilon|^2-\lambda|\chi_\varepsilon|^2)dx+\frac{kt_\varepsilon^2}{2}\int_{\partial B}|\frac{\partial\chi_\varepsilon}{\partial\nu}|^2d\omega\\ &-\frac{t_\varepsilon^{2^{**}}}{2^{**}}\int_B|\chi_\varepsilon|^{2^{**}}dx\\ &\le\frac{2}{N}S^{\frac{N}{4}}+\frac{t_\varepsilon^2}{2}\varepsilon^2\left(\mu C_NK_1+\frac{\lambda}{2}c_8\omega_8\varepsilon^2\ln\varepsilon^2+\mathrm{O}(\varepsilon^2)\right)\\ &<\frac{2}{N}S^{\frac{N}{4}},\end{aligned}\tag{65}$$

which, combining (38) and (64), implies the conclusion is true.

We complete the proof. □

4 | THE PROOFS OF OUR RESULTS

The proof of Theorem 1: According to Lemmas 5, 6, 7, 8 and 9, we can see that under the assumptions of Theorem 1, problem (1) has at least one nontrivial solution. This completes the proof.

The proof of Theorem 2: Assume that $u\in H^2(B)\cap H_0^1(B)\cap C^4(\overline{B})$ is a nontrivial solution of (1). According to[17?], we have

$$\begin{aligned}&\int_B(\Delta^2u)x\cdot\nabla udx-\frac{N}{2}\int_B(\Delta u)^2dx-(N-2)\int_B\nabla\Delta u\cdot\nabla udx\\ &=-\frac{1}{2}\int_{\partial B}(\Delta u)^2x\cdot\nu dS+\int_{\partial B}[(\Delta u)_\nu(x\cdot\nabla u)+u_\nu(x\cdot\nabla\Delta u)-\nabla\Delta u\cdot\nabla u(x\cdot\nu)]dS.\end{aligned}\tag{66}$$

Since $u|_{\partial B}=0$, we have $\nabla u=-|\nabla u|\nu,\ \ x\in\partial B$ (the sign here has no effect on the calculation below, so we may write it as the negative sign). So, on ∂B,

$$\begin{aligned}\nabla\Delta u\cdot\nabla u(x\cdot\nu)&=\nabla\Delta u\cdot(-|\nabla u|)\nu(x\cdot\nu)\\ &=(\Delta u)_\nu(x\cdot(-|\nabla u|\nu))\\ &=(\Delta u)_\nu(x\cdot\nabla u),\end{aligned}\tag{67}$$

which implies that (66) can be written as

$$\begin{aligned}&\int_B(\Delta^2u)x\cdot\nabla udx-\frac{N}{2}\int_B(\Delta u)^2dx-(N-2)\int_B\nabla\Delta u\cdot\nabla udx\\ &=-\frac{1}{2}\int_{\partial B}(\Delta u)^2x\cdot\nu dS+\int_{\partial B}u_\nu(x\cdot\nabla\Delta u)dS.\end{aligned}\tag{68}$$

By direct computation, we have

$$
\begin{aligned}
&\mu \int_B \Delta u(x\cdot\nabla u)dx \\
&= \mu \sum_{i,j=1}^{N} \int_B u_{x_i x_i} x_j u_{x_j} dx \\
&= -\mu \sum_{i,j=1}^{N} \int_B u_{x_i}(x_j u_{x_j})_{x_i} dx + \mu \sum_{i,j=1}^{N} \int_{\partial B} x_j u_{x_j} u_{x_i} \nu^i dS \\
&= -\mu \sum_{i=1}^{N} \int_B u_{x_i} u_{x_i} dx - \mu \sum_{i,j=1}^{N} \int_B x_j u_{x_i} u_{x_j x_i} dx + \mu \int_{\partial B} (x\cdot\nabla u) u_\nu dS \\
&= -\mu \int_B |\nabla u|^2 dx - \sum_{j=1}^{N} \mu \int_B \left(\frac{|\nabla u|^2}{2}\right)_{x_j} x_j dx + \mu \int_{\partial B} (x\cdot\nabla u) u_\nu dS \\
&= -\mu \int_B |\nabla u|^2 dx + \frac{N}{2}\mu \int_B |\nabla u|^2 dx - \sum_{j=1}^{N} \mu \int_{\partial B} \frac{|\nabla u|^2}{2} x_j \nu^j dS + \mu \int_{\partial B} (x\cdot\nabla u) u_\nu dS \\
&= \frac{N-2}{2}\mu \int_B |\nabla u|^2 dx - \frac{1}{2}\mu \int_{\partial B} |\nabla u|^2 (x\cdot\nu) dS + \mu \int_{\partial B} (x\cdot\nabla u) u_\nu dS.
\end{aligned}
\tag{69}
$$

which, combining (69) and divergence formula, implies that

$$
\begin{aligned}
&\int_B (\Delta^2 u) x\cdot\nabla u dx - \frac{N}{2}\int_B (\Delta u)^2 dx - (N-2)\int_B \nabla\Delta u\cdot\nabla u dx \\
&= \int_B (\lambda u + \mu\Delta u + |u|^{2^{**}-2}u)(x\cdot\nabla u)dx - \frac{N}{2}\Big[\int_B (\lambda u^2 + |u|^{2^{**}} - \mu|\nabla u|^2)dx \\
&- k\int_{\partial B} u_\nu^2 dS\Big] + (N-2)\int_B (\lambda u + \mu\Delta u + |u|^{2^{**}-2}u)u dx \\
&= \lambda\int_B u(x\cdot\nabla u)dx + \mu\int_B \Delta u(x\cdot\nabla u)dx + \int_B |u|^{2^{**}-2}u(x\cdot\nabla u)dx \\
&+ \frac{N-4}{2}\int_B (\lambda u^2 + |u|^{2^{**}} - \mu|\nabla u|^2)dx + \frac{kN}{2}\int_{\partial B} u_\nu^2 dS \\
&= \frac{\lambda}{2}\int_B x\cdot\nabla(u^2)dx + \mu\int_B \Delta u(x\cdot\nabla u)dx + \frac{1}{2^{**}}\int_B x\cdot\nabla(|u|^{2^{**}})dx \\
&+ \frac{N-4}{2}\int_B (\lambda u^2 + |u|^{2^{**}} - \mu|\nabla u|^2)dx + \frac{kN}{2}\int_{\partial B} u_\nu^2 dS \\
&= \frac{\lambda}{2}\int_B div(u^2 x)dx - 2\lambda\int_B u^2 dx + \mu\int_B |\nabla u|^2 dx - \frac{1}{2}\mu\int_{\partial B} |\nabla u|^2(x\cdot\nu)dS \\
&+ \mu\int_{\partial B} (x\cdot\nabla u)u_\nu dS + \frac{1}{2^{**}}\int_B div(|u|^{2^{**}}x)dx + \frac{kN}{2}\int_{\partial B} u_\nu^2 dS \\
&= \frac{\lambda}{2}\int_{\partial B} u^2 x\cdot\nu dS - 2\lambda\int_B u^2 dx + \mu\int_B |\nabla u|^2 dx - \frac{1}{2}\mu\int_{\partial B} |\nabla u|^2(x\cdot\nu)dS \\
&+ \mu\int_{\partial B} (x\cdot\nabla u)u_\nu dS + \frac{1}{2^{**}}\int_{\partial B} |u|^{2^{**}} x\cdot\nu dS + \frac{kN}{2}\int_{\partial B} u_\nu^2 dS
\end{aligned}
\tag{70}
$$

$$= -2\lambda \int_B u^2 dx + \mu \int_B |\nabla u|^2 dx - \frac{1}{2}\mu \int_{\partial B} |\nabla u|^2 (x\cdot\nu) dS + \mu \int_{\partial B} (x\cdot\nabla u) u_\nu dS + \frac{kN}{2}\int_{\partial B} u_\nu^2 dS.$$

So, following from (68) and (70), we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\partial B} |\Delta u|^2 (x\cdot\nu) - \frac{1}{2}\mu \int_{\partial B} |\nabla u|^2 (x\cdot\nu) \\ &= 2\lambda \int_B u^2 - \mu\int_B |\nabla u|^2 - \mu \int_{\partial B} (x\cdot\nabla u)u_\nu - \frac{kN}{2}\int_{\partial B} u_\nu^2 + \int_{\partial B} u_\nu (x\cdot\nabla\Delta u). \end{aligned} \tag{71}$$

On the other hand, since $x = \nu$ on ∂B, we have

$$\begin{aligned} \int_{\partial B} u_\nu (x\cdot\nabla\Delta u) &= \int_{\partial B} u_\nu (\Delta u)_\nu = -k\int_{\partial B} u_\nu (u_\nu)_\nu = -k \int_{\partial B} u_\nu (\nabla u\cdot\nu)_\nu \\ &= -k\int_{\partial B} u_\nu \nabla(\nabla u\cdot\nu)\cdot\nu = -k\int_{\partial B} u_\nu (u_{\nu\nu} + u_\nu) \\ &= -k \int_{\partial B} u_\nu (\Delta u - (N-1)u_\nu + u_\nu) \\ &= -k\int_{\partial B} u_\nu(-ku_\nu - (N-1)u_\nu + u_\nu) \\ &= k(k+N-2)\int_{\partial B} u_\nu^2, \end{aligned} \tag{72}$$

where we have used that

$$\begin{aligned} \nabla(\nabla u\cdot\nu)\cdot\nu &= \sum_{j=1}^N (\nabla u\cdot x)_{x_j} x_j \\ &= \sum_{i,j=1}^N (u_{x_i x_j} x_i + u_{x_i}\delta_{ij}) x_j \\ &= \sum_{i,j=1}^N u_{x_i x_j} x_i x_j + \sum_{i=1}^n u_{x_i} x_i \\ &= \nu\cdot\nabla^2 u\cdot\nu + u_\nu \\ &= u_{\nu\nu} + u_\nu. \end{aligned}$$

Since $u|_{\partial B} = 0$, we have

$$\int_{\partial B} |u_\nu|^2 = \int_{\partial B} |(\nabla u\cdot\nu)|^2 = \int_{\partial B} |(|\nabla u|(-\nu)\cdot\nu)|^2 = \int_{\partial B} |\nabla u|^2. \tag{73}$$

Thus, (71) can be written as

$$\frac{1}{2}(k^2 + kN - 4k - \mu)\int_{\partial B} u_\nu^2 = \mu\int_B |\nabla u|^2 - 2\lambda \int_B u^2. \tag{74}$$

When $k^2 + (N-4)k - \mu > 0$, the left hand side of (74) is greater than or equal to zero, i.e,

$$\mu \int_B |\nabla u|^2 - 2\lambda\int_B u^2 \geq 0. \tag{75}$$

(i) When $\lambda \geq 0$ and $\mu < 0$, it is obvious that $\mu\int_B |\nabla u|^2 - 2\lambda\int_B u^2 \leq 0$. So we can see that $\int_B |\nabla u|^2 = 0$, which, together with $u \in H_0^1(B)\cap C^4(\bar{B})$, implies that $u \equiv 0$.

(ii) When $\mu < \frac{2}{\lambda_1(B)}\lambda < 0$, by the Poincaré inequality, we obtain

$$\mu\int_B |\nabla u|^2 - 2\lambda\int_B u^2 \leq (\mu - \frac{2\lambda}{\lambda_1(B)})\int_B |\nabla u|^2 \leq 0, \tag{76}$$

which, together with (75), implies that $\int_B |\nabla u|^2 = 0$. So $u \equiv 0$.

Therefore, If $\mu < \min\{0, \frac{2}{\lambda_1(B)}\lambda\}$ and $k^2 + (N-4)k - \mu > 0$, then $u \equiv 0$.
Similarly, we can show that If $k^2 + (N-4)k - \mu \le 0$ and $\mu > \max\{0, \frac{2}{\delta_1(B)}\lambda\}$, then $u \equiv 0$.
We complete the proof.

5 | CONCLUSIONS

In this paper, we prove the existence of nontrivial solutions to the problem (1) by the Mountain pass theorem and show the nonexistence of nontrivial solutions to it by the Pohozaev identity. To apply the Mountain pass theorem, we firstly introduce a new norm $||.||_1$ of $H^2(B) \cap H_0^1(B)$ and show the equivalence of the norm $||.||_1$ and the standard norm of $H^2(B)$ in some specific condition. Secondly, we show that the variational functional has the Mountain pass geometry structure, which implies that we can get a $(PS)_c$ sequence $\{u_n\}$ of φ. We can get the boundedness of $(PS)_c$ sequence $\{u_n\}$ easily. However, we can not obtain that the functional φ satisfies the $(PS)_c$ condition directly, since the embedding of $H^2(B) \hookrightarrow L^{2^{**}}(B)$ is not compact. Therefore, to get the compactness, we have to compare the Mountain pass level energy and the ground state energy of the limiting problem of (1)(See (34)). In the process of comparing the energies, we have to construct some special functions and introduce some new skills. As to the nonexistence, we mainly apply the Pohozaev identity to show it. At the same time, we need some variational theories and some meticulous calculations.

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Conflict of interest

The authors have no conflict of interests regarding the publication of this paper.

Author contributions

All the authors have equal contributions in this article.

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