Well-posedness, wave breaking, Hölder continuity and periodic peakons for a nonlocal sine-μ-Camassa-Holm equation

Guoquan Qin[1,2], **Zhenya Yan**[2,3,*], **Boling Guo**[4]

[1] *National Center for Mathematics and Interdisciplinary Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

[2] *Key Laboratory of Mathematics Mechanization, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

[3] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

[4] *Institute of Applied Physics and Computational Mathematics, Beijing 100088, PR China*

Abstract: In this paper, we investigate the initial value problem of a nonlocal sine-type μ-Camassa-Holm (μCH) equation, which is the μ-version of the sine-type CH equation. We first discuss its local well-posedness in the framework of Besov spaces. Then a sufficient condition on the initial data is provided to ensure the occurance of the wave-breaking phenomenon. We finally prove the Hölder continuity of the data-to-solution map, and find the explicit formula of the global weak periodic peakon solution.

Keywords: Sine-μ-Camassa-Holm equation; Well-posedness; Blow-up; Wave breaking; Hölder continuity; Peakons

1 Introduction

In 2008, Khesin-Lenells-Misiolek [24] presented a new nonlocal equation (i.e., the μ-Camassa-Holm (μCH) equation)

$$m_t + um_x + 2u_x m = 0, \quad m = \mu(u) - u_{xx}, \tag{1.1}$$

where $\mu(u) = \int_{\mathbb{S}} u(t,x)\mathrm{d}x$ and $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. This equation can be viewed as an intermediate equation between the CH equation [4, 17, 29]

$$m_t + um_x + 2u_x m + \kappa u_x = 0, \quad m = u - u_{xx}, \quad \kappa \in \mathbb{R}. \tag{1.2}$$

and Hunter-Saxton(HS) equation (a short-wave limit to Eq. (1.2)) [23]

$$u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 = 0. \tag{1.3}$$

The CH equation can describe the propagation of axially symmetric waves in hyperelastic rods [12, 13], and possess a bi-Hamiltonian structure, infinitely many of conservation laws, weak peakon solutions ($ce^{-|x-ct|}(c > 0)$) [4]. In particular it is completely integrable and can be explicitly solved via the inverse scattering transform (IST) [3,4,6,8–11,26]. Constantin-Strauss [11] and Lenells [25] studied the orbital stability of the weak peakon solutions of CH equation.

Similar to the CH equation, the μ-CH equation (1.1) admits the Lax-pair and bi-Hamiltonian structure [24]. It can also describe a geodesic flow on diffeomorphism group of $\mathbb{S}$ with certain metric. Its integrability, well-posedness, blow-up and peakons have been investigated in [18, 20, 24]. Similar to the μ-CH equation (1.1), the integrable modified μ-CH equation [30], as a μ-version of the mCH equation, was presented in the form

$$m_t + \left(\left(2\mu(u)u - u_x^2\right) m\right)_x = 0, \quad m = \mu(u) - u_{xx}. \tag{1.4}$$

*Corresponding author. *Email address*: zyyan@mmrc.iss.ac.cn (Z. Yan).

Its local well-posedness in Besov spaces and the existence of peakon and multi-peakon solutions as well as the formation of singularities have been discussed in detail. Moreover, the dynamical stability of periodic peaked solitons of Eq. (1.4) was studied [28]. The non-uniform continuity of the solution map of Eq. (1.4) was established [34].

More recently, we came up with the sine-type generalization of the mCH equation (alias sine-mCH equation) [31]

$$m_t + [\sin(u^2 - u_x^2)m]_x, \quad m = u - u_{xx} \tag{1.5}$$

and sine-type generalization of the CH equation (alias sine-CH equation) [32]

$$m_t + \sin(u^2 - u_x^2)u_x m + [\sin(u^2 - u_x^2)um]_x = 0, \quad m = u - u_{xx} \tag{1.6}$$

and discussed their Cauchy problems. The studies of the above equations generalized the related research on the CH-type equations [1].

In this paper, we will investigate some μ-generalization of the sine-CH equation (1.6). Our research interest here is the Cauchy problem of the μ-version of the sine-type CH equation (1.6) (alias sine-μCH equation)

$$\begin{cases} m_t + \sin\big(2\mu(u)u - u_x^2\big)u_x m + [\sin\big(2\mu(u)u - u_x^2\big)um]_x = 0, \quad m = \mu(u) - u_{xx}, \\ u(t, x+1) = u(t, x), \\ m(0, x) = m_0(x), \end{cases} \tag{1.7}$$

where $u(t,x)$ denotes the fluid velocity and $\mu(u) = \int_{\mathbb{S}} u\mathrm{d}x$ represents the corresponding potential density. The main task here is to understand the effect of the nonlinear term $\sin(2\mu(u)u - u_x^2)$ on the breakdown mechanism of Eq. (1.7). One can check that there are two conserved quantities associated with Eq. (1.7), these are $\mu_0 = \int_{\mathbb{S}} u\mathrm{d}x$ and $\mu_1 = (\int_{\mathbb{S}} u_x^2 \mathrm{d}x)^{1/2}$. Eq. (1.7) can also be regarded as a sine-type extension of the μCH equation (1.1) and modified μCH equation (1.4). In particular, we have

- As $\sin(2\mu(u)u - u_x^2) = c \neq 0$, $c \in [-1, 1]$, the sine-μCH Eq. (1.7) with $u(x,t) = u(\tau, x)$, $\tau = ct$ reduces to the known μCH equation (1.1) with $t \to \tau$.
- As $2\mu(u)u - u_x^2 \to 0$, one has $\sin(2\mu(u)u - u_x^2) \sim 2\mu(u)u - u_x^2$ so that Eq. (1.7) just reduces to the new μ-version of the fourth-order CH equation

$$m_t + (2\mu(u)u - u_x^2)mu_x + [(2\mu(u)u - u_x^2)mu]_x = 0, \quad m = \mu(u) - u_{xx} \tag{1.8}$$

- As $0 < |2\mu(u)u - u_x^2| < 1$, we have

$$\sin(2\mu(u)u - u_x^2) \sim \sum_{k=1}^{N} \frac{(-1)^{k+1}}{(2k-1)!}\Big(2\mu(u)u - u_x^2\Big)^{2k-1} + O\left((2\mu(u)u - u_x^2)^{2N-1}\right),$$

  in which case the sine-μCH Eq. (1.7) becomes the higher-order μCH equation ($O\left((2\mu(u)u - u_x^2)^{2N-1}\right)$ is neglected)

$$m_t + mu_x \sum_{k=1}^{N} \frac{(-1)^{k+1}}{(2k-1)!}\Big(2\mu(u)u - u_x^2\Big)^{2k-1} + \partial_x\left(mu\sum_{k=1}^{N} \frac{(-1)^{k+1}}{(2k-1)!}\Big(2\mu(u)u - u_x^2\Big)^{2k-1}\right) = 0, \tag{1.9}$$

  where $m = \mu(u) - u_{xx}$.

The main contents, also the arrangement, of this paper are as follows. First, in the spirit of [14–16, 30], we will show the local well-posedness of strong solutions to Eq. (1.7) in the subcritical Besov spaces $B^s_{p,r}$, i.e., Theorem 2.1, and this will be done in Section 3. Here, the derivative index and the integrable index should satisfy $s > \max\{2+1/p, 5/2\}$. The main tool we will use to prove this result are the Besov space theory and the transport equations theory. Second, we will prove its local well-posedness in the critical Besov space $B^{5/2}_{2,1}$(Theorem 2.2) in Section 4, following the spirit of [35]. Then, in Section 5, we will be concerned with the blow-up criterion and the precise blow-up quantity of Eq. (1.7) by means of the Moser-type estimates in Sobolev spaces. Section 6 is devoted to putting forward a sufficient condition with regard to the initial data to ensure the occurance of the wave-breaking phenomenon by tracing the corresponding precise blow-up quantity along the characteristic. We will employ the energy method combined with some Sobolev inequalities and commutator estimates of Calderon-Coifman-Meyer type to establish Theorem 2.6–the Hölder continuity of the data-to-solution map in Section 7. The last section will provide the weak peakon solutions of the sine-μCH equation (1.7), i.e., Theorem 2.7.

Let $\mathcal{S}$ stand for the Schwartz space and $\mathcal{S}'$ represent the space of temperate distributions. Let $L^p(\mathbb{S})$ be the Lebesgue space equipped with the norm $\|\cdot\|_{L^p}$ for $1 \le p \le \infty$ and $H^s(\mathbb{S})$ be the Sobolev space equipped with the norm $\|\cdot\|_{H^s}$ for $s \in \mathbb{R}$. Since the local well-posedness for the Cauchy problem (1.7) will be proved in Besov-type space $B^s_{p,r}$ (**Appendix A**, also see [2, 5] for more details). Moreover, some lemmas of the transport equation theory are used (see [2, 14] for more details). Let

$$p(x) = \frac{1}{2}\left(x-\frac{1}{2}\right)^2 + \frac{23}{24} \tag{1.10}$$

be the Green function of the operator $(\mu-\partial_x^2)^{-1}$. Its derivative [27] can be assigned to zero at $x=0$, so one has

$$p_x(x) \stackrel{\text{def}}{=} \begin{cases} 0, & x=0, \\ x-\frac{1}{2}, & 0<x<1. \end{cases}$$

It is easy to see that the operator $\mu-\partial_x^2$ is an isomorphism between $B^s_{p,r}$ and $B^{s-2}_{p,r}$ with the inverse $v=(\mu-\partial_x^2)^{-1}w$ given explicitly by [24]

$$v(x) = \left(\frac{x^2-x}{2}+\frac{13}{12}\right)\mu(w) + \frac{2x-1}{2}\int_0^1\int_0^y w(s)\mathrm{d}s\mathrm{d}y - \int_0^x\int_0^y w(s)\mathrm{d}s\mathrm{d}y + \int_0^1\int_0^y\int_0^s w(r)\mathrm{d}r\mathrm{d}s\mathrm{d}y.$$

From [30], we know that

$$\|\mu(u)\|_{B^s_{p,r}} \leqslant c\|u\|_{B^q_{p,r}}, \quad s\in\mathbb{R},\ q>0,\ 1\le p,r\le\infty, \quad \|u\|^2_\mu \leqslant \|u\|^2_{H^1} \leqslant 3\|u\|^2_\mu,$$

where

$$\|u\|^2_\mu = \left(\left(\mu-\partial_x^2\right)u,u\right)_{L^2} = [\mu(u)]^2 + \int_{\mathbb{S}} u_x^2\mathrm{d}x, \quad \|u\|^2_{H^1} = \left(\left(1-\partial_x^2\right)u,u\right)_{L^2} = \int_{\mathbb{S}}\left(u^2+u_x^2\right)\mathrm{d}x.$$

2 Main results

Our first result is about the local well-posedness to the Cauchy problem (1.7) in subcritical Besov spaces.

Theorem 2.1. (**Local well-posedness in subcritical Besov spaces**) *Let $u_0 \in B^s_{p,r}$ with $1 \le p, r \le +\infty$, $s > \max\{2+1/p, 5/2\}$. Then there exists a time $T>0$ such that the Cauchy problem of the sine-μCH equation (1.7) possesses a unique solution $u \in E^s_{p,r}(T)$. Furthermore, the data-to-solution map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $B^s_{p,r}$ into $C\left([0,T]; B^{s'}_{p,r}\right) \cap C^1\left([0,T]; B^{s'-1}_{p,r}\right)$ for each $s'<s$ as $r=+\infty$ and $s'=s$ as $r<+\infty$.*

Setting $p=r=2$ in Theorem 2.1, one immediately deduces the following Corollary with respect to the local well-posedness of (1.7) in Sobolev space, which is a more convenience setting for us to show the blow-up results.

Corollary 2.1. *Let $s>5/2$ and $u_0\in H^s$. Then there exists a time $T>0$ such that the Cauchy problem (1.7) admits a unique strong solution $u\in C([0,T];H^s)\cap C^1\left([0,T];H^{s-1}\right)$. Furthermore, the data-to-solution map $u_0\mapsto u$ is continuous from a neighborhood of u_0 in H^s into $C([0,T];H^s)\cap C^1\left([0,T];H^{s-1}\right)$.*

The next Theorem states the local well-posedness of (1.7) in the critical Besov spaces $B^{5/2}_{2,1}(\mathbb{S})$.

Theorem 2.2. (**Local well-posedness in critical Besov spaces**) *Let the data $u_0\in B^{5/2}_{2,1}(\mathbb{S})$. Then there is some maximal time $T>0$ and a unique solution $u(t,x)$ of the Cauchy problem (1.7) such that*

$$u=u(t,\cdot)\in C([0,T];B^{5/2}_{2,1}(\mathbb{R}))\cap C^1([0,T];B^{3/2}_{2,1}(\mathbb{R})).$$

Furthermore, the data-to-solution mapping

$$u_0\longmapsto u(u_0,\cdot):B^{5/2}_{2,1}(\mathbb{R})\longmapsto C([0,T];B^{5/2}_{2,1}(\mathbb{R}))\cap C^1([0,T];B^{3/2}_{2,1}(\mathbb{R}))$$

is continuous.

The following Theorems are about the blow-up criterion and quantity.

Theorem 2.3. (**Blow-up criterion**) *Let $u_0\in H^s$ be given as in Corollary 2.1 and u be the corresponding solution to (1.7). Denote by T^* the maximal existence time, then*

$$T^*<\infty \quad \Rightarrow \quad \int_0^{T^*}\|m\|^3_{L^\infty}\,dt=\infty. \tag{2.1}$$

Theorem 2.4. (**Blow-up quantity**) *Let $u_0\in H^s(\mathbb{R})$ with $s>\frac{5}{2}$, and $T^*>0$ be the maximal existence time of the solution u to the Cauchy problem (1.7). Then u will blow up in finite time iff*

$$\liminf_{t\to T^*}\left(\inf_{x\in\mathbb{S}}\left(\cos(2\mu_0u-u_x^2)uu_xm+\sin(2\mu_0u-u_x^2)u_x\right)(t,x)\right)=-\infty. \tag{2.2}$$

Moreover, one has

$$\sup_{x\in\mathbb{S}}\Big(\cos(2\mu_0u-u_x^2)uu_xm+\sin(2\mu_0u-u_x^2)u_x\Big)(t,x)\le C\|u_0\|^2_{H^1}\sup_{x\in\mathbb{S}}m_0(x)+C\|u_0\|_{H^1} \tag{2.3}$$

for all $t\in[0,T^*)$ if $m_0(x)=\left(\mu-\partial_x^2\right)u_0\ge 0$ for all $x\in\mathbb{S}$, and $m_0(x_0)>0$ at some point $x_0\in\mathbb{S}$.

Having established Theorem 2.3, we will prove the following wave-breaking result:

Theorem 2.5. (**Wave-breaking**) *Suppose that $m_0\in H^s(\mathbb{S})$ with $s>\frac{1}{2}$ and $m_0(x)\ge 0$ for all $x\in\mathbb{R}$, and $m_0(x_0)>0$ for some $x_0\in\mathbb{R}$. Let $T^*>0$ be the maximal existence time of strong solution m to the Cauchy problem (1.7). Let $M(t,x)$ be defined by (6.1), $\widehat{M}(t)=M(t,q(t,x_0))$ and $\widehat{m}(t)=m(t,q(t,x_0))$ with $q(t,x)$ being defined in (5.12). Also, assume $|\mu\left(\sin(2\mu_0u-u_x^2)uu_x\right)|+|2\mu\left(\cos(2\mu_0u-u_x^2)mu_x^2\right)|\le C_*$ for some constant $C_*>0$ independent of t and $\widehat{m}(t)\ge\varepsilon$ for some small constant $\varepsilon>0$. If*

$$\widehat{M}(0)<0 \quad \text{and} \quad \frac{C_2}{2}\xi^2+\frac{\widehat{M}(0)\xi+1}{\widehat{m}(0)}<0, \tag{2.4}$$

where $C_2 = C_1(1+1/\varepsilon)$ *with* C_1 *defined by (6.7) and* $\xi = -\frac{\widehat{M}(0)}{C_2\widehat{m}(0)}$, *then the solution* m *will blow up at a time* $T^* \in (0,\xi)$. *Furthermore, as* $T^* = t_- = -\frac{\bar{M}(0)}{C_2\widehat{m}(0)} - \frac{1}{2}\sqrt{\left(\frac{2\widehat{M}(0)}{C_2\widehat{m}(0)}\right)^2 - \frac{2}{C_2\widehat{m}(0)}}$, *we can evaluate the blow-up rate as*

$$\liminf_{t\to(T^*)^-}\left((T^*-t)\inf_{x\in\mathbb{S}} M(t,x)\right) \le -\frac{1}{2}, \tag{2.5}$$

Corollary 2.1 implies the continuity of the data-to-solution map $u(0)\in H^s \to u(t)\in H^s$ for the Cauchy problem (1.7). We next show the Hölder continuity of this map in H^s under a weaker topology H^r, i.e.,

Theorem 2.6. (**Hölder continuity**) *Let* $0 \le r < s$ *with* $s > 5/2$, *then the data-to-solution map for the Cauchy problem (1.7) is Hölder continuous in* H^s *under the* H^r *norm. More precisely, for initial data* u_0, v_0 *with* $\|u_0\|_{H^s} \le \rho$ *and* $\|v_0\|_{H^s} \le \rho$, *the corresponding solutions* u, v *of Eq. (1.7) satisfy*

$$\|u-v\|_{C([0,T];H^r)} \le C\,\|u_0-v_0\|_{H^r}^{\beta}, \tag{2.6}$$

with the constant $C = C(s,r,\rho)$ *and the exponent* β *given by*

$$\beta = \begin{cases} 1, & (s,r)\in D_1 \\ (2s-3)/(s-r), & (s,r)\in D_2, \\ (s-r)/2, & (s,r)\in D_3, \\ s-r, & (s,r)\in D_4, \end{cases} \tag{2.7}$$

where the regions D_1, D_2, D_3 *and* D_4 *in the* (s,r)*-plane are defined by*

$$\begin{cases} D_1 = \{(s,r)\,|\,0\le r\le 3/2,\ 3-s\le r\le s-2\}\cup\{(s,r)\,|\,3/2<r\le s-1\}, \\ D_2 = \{(s,r)\,|\,5/2<s<3,\ 0\le r\le -s+3\}, \\ D_3 = \{(s,r)\,|\,5/2<s,\ s-2\le r\le 3/2\}, \\ D_4 = \{(s,r)\,|\,5/2<s,\ s-1\le r<s\}. \end{cases}$$

To establish the explicit formula of the peakon solution, we first give the definition of weak solution associated to (1.7).

Definition 2.1. *Given initial data* $u_0 \in W^{1,3}$, *the function* $u \in L^\infty\left([0,T), W^{1,3}\right)$ *is said to be a weak solution to (1.7) with initial data* u_0 *if it satisfies the following identity:*

$$\begin{aligned} &\int_0^T\int_{\mathbb{S}}\Big[-u\varphi_t + \sin(2\mu(u)u-u_x^2)uu_x\varphi \\ &\quad - p_x * [-2\mu(u)\sin(2\mu(u)u-u_x^2)u - \cos(2\mu(u)u-u_x^2)(2\mu(u)uu_x^2 - 2/3\partial_x(uu_x^3) + 2/3u_x^4)]\varphi \\ &\quad - p * [-1/2\sin(2\mu(u)u-u_x^2)\partial_x(u_x^2) + \mu(u)\cos(2\mu(u)u-u_x^2)(\mu(u)\partial_x(u^2) - \partial_x(uu_x^2) - u_x^3) \\ &\quad + 1/2\cos(2\mu(u)u-u_x^2)\partial_x(u_x^4)]\varphi - \mu\left(\sin(2\mu(u)u-u_x^2)uu_x\right)\varphi\Big]\mathrm{d}x\mathrm{d}t \\ &+ \int_{\mathbb{S}} u_0(x)\varphi(0,x)\mathrm{d}x = 0 \end{aligned}$$

for any smooth test function $\varphi(t,x)\in C_c^\infty([0,T)\times\mathbb{S})$. *If* u *is a weak solution on* $[0,T)$ *for every* $T>0$, *then it is called a global weak solution.*

Then our final result reads

Theorem 2.7. *Eq. (1.7) possesses a global weak peakon solution, in the sense of Definition 2.1, of the form*

$$u_c(t,x) = a\left[\frac{1}{2}\left(\xi - [\xi] - \frac{1}{2}\right)^2 + \frac{23}{24}\right], \tag{2.8}$$

where $\xi = x - ct$ with $\xi \in [-1/2, 1/2]$ and $c = \frac{13}{12}a\sin(\frac{23}{12}a^2)$. Notice that $u_c(t,x)$ can be extended periodically to the whole real line.

3 Local well-posedness in $B^s_{p,r}$ with $s > \max\{2+1/p, 5/2\}$

This section will give the proof of the local well-posedness result in subcritical Besov spaces $B^s_{p,r}$ with the derivative index satisfying $s > \max\{2+1/p, 5/2\}$, namely, Theorem 2.1. The proof is completed based on the properties [2, 5, 14].

Proof. First, the classical Friedrichs regularization approach is used to construct the approximate solutions of (1.7). Let $m^{(l+1)}$ solve the following linear transport equation inductively

$$\begin{cases} \partial_t m^{(l+1)} + \sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}\partial_x m^{(l+1)} \\ \qquad = -2\cos[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}u_x^{(l)}(m^{(l)})^2 - 2\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u_x^{(l)}(m^{(l)}), \\ m^{(l+1)}_{t=0} = m_0^{(l+1)}(x) = S_{l+1}m_0, \end{cases} \tag{3.1}$$

where $m^{(0)} := 0,\ l = 0, 1, 2, ...$

Suppose $m^{(l)} \in L^\infty([0,T]; B^{s-2}_{p,r})$ with $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$ and consequently $B^{s-2}_{p,r}$ is an algebra. So the right-hand side of Eq. (3.1) is in $L^\infty\left([0,T]; B^{s-2}_{p,r}\right)$. Hence, Eq. (3.1) possesses a global solution $m^{(l+1)} \in E^{s-2}_{p,r}$ for all positive T and the high regularity of u.

Based on the property of the transport equation [2, 14] it follows from Eq. (3.1) that one has

$$\begin{aligned} \|m^{(l+1)}(t)\|_{B^{s-2}_{p,r}} \leq{}& \exp\left(C\int_0^t \|\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}\|_{B^{s-2}_{p,r}}\mathrm{d}\tau\right)\|m_0\|_{B^{s-2}_{p,r}} \\ &+C\int_0^t \exp\left(C\int_\tau^t \|\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}\|_{B^{s-2}_{p,r}}\mathrm{d}\tau'\right) \\ &\times\Big[\|2\cos[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}u_x^{(l)}(m^{(l)})^2\|_{B^{s-2}_{p,r}} \\ &\quad+\|2\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u_x^{(l)}(m^{(l)})\|_{B^{s-2}_{p,r}}\Big]\mathrm{d}\tau, \qquad l = 0, 1, 2, \cdots. \end{aligned} \tag{3.2}$$

According to the product law in Besov spaces, one finds

$$\begin{aligned} &\|\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}\|_{B^{s-2}_{p,r}} \leq C\|u^{(l)}\|_{B^{s-2}_{p,r}}\|2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2\|_{B^{s-2}_{p,r}} \leq C\|u^{(l)}\|^3_{B^{s-2}_{p,r}}, \\ &\|2\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u_x^{(l)}(m^{(l)})\|_{B^{s-2}_{p,r}} \leq C\|u_x^{(l)}m^{(l)}\|_{B^{s-2}_{p,r}}\|2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2\|_{B^{s-2}_{p,r}} \leq C\|u^{(l)}\|^4_{B^s_{p,r}}, \\ &\|2\cos[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}u_x^{(l)}(m^{(l)})^2\|_{B^{s-2}_{p,r}} \leq C\|u^{(l)}u_x^{(l)}(m^{(l)})^2\|_{B^{s-2}_{p,r}} \leq C\|u^{(l)}\|^4_{B^s_{p,r}}. \end{aligned} \tag{3.3}$$

Plugging (3.3) into (3.1) leads to

$$\|u^{(l+1)}(t)\|_{B^s_{p,r}} \leq e^{\int_0^t C\|u^{(l)}\|^3_{B^s_{p,r}}\mathrm{d}\tau}\|u_0\|_{B^s_{p,r}} + C\int_0^t e^{\int_\tau^t C\|u^{(l)}\|^3_{B^s_{p,r}}\mathrm{d}\tau'}\|u^{(l)}\|^4_{B^s_{p,r}}\mathrm{d}\tau. \tag{3.4}$$

Now, we need to find the uniform bound of the solution sequence $\{u^{(l)}(t)\}$. Suppose $\|u^{(l)}(t)\|_{B^s_{p,r}} \le a(t)$. Substituting this into (3.4) generates

$$\|u^{(l+1)}(t)\|_{B^s_{p,r}} \le e^{\int_0^t Ca^3 \mathrm{d}\tau}\|u_0\|_{B^s_{p,r}} + C\int_0^t e^{\int_\tau^t Ca^3\mathrm{d}\tau'} a^4 \mathrm{d}\tau. \tag{3.5}$$

Then we take the equality sign to obtain

$$\dot{a}(t) = 2Ca^4(t), \quad a(0) = \|u_0\|_{B^s_{p,r}},$$

which admits the solution

$$a(t) = \|u_0\|_{B^s_{p,r}}\left(1 - 6Ct\|u_0\|^3_{B^s_{p,r}}\right)^{-1/3}.$$

Accordingly, one can draw the conclusion that the solution sequence $\{u^{(l)}\}_{l=1}^\infty$ of Eq. (3.1) is uniformly bounded in $C([0,T]; B^s_{p,r})$ with $T < \frac{1}{6C\|u_0\|^3_{B^s_{p,r}}}$.

Next, we will prove that $\{m^{(l+1)}\}_{l=1}^\infty$ is a Cauchy sequence in $C([0,T]; B^{s-3}_{p,r})$. In fact, one can derive from Eq. (3.1) that

$$\begin{aligned}
&\partial_t[m^{(l+i+1)} - m^{(l+1)}] + \sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u^{(l+i)}\partial_x[m^{(l+i+1)} - m^{(l+1)}] \\
&= \left\{\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)} - \sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u^{(l+i)}\right\}\partial_x m^{(l+1)} \\
&-2\left\{\sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u_x^{(l+i)}m^{(l+i)} - \sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u_x^{(l)}m^{(l)}\right\} \\
&-2\left\{\cos[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u^{(l+i)}u_x^{(l+i)}(m^{(l+i)})^2 - \cos[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}u_x^{(l)}(m^{(l)})^2\right\} := g.
\end{aligned} \tag{3.6}$$

As a result, we have

$$\begin{aligned}
\|m^{(l+i+1)} - m^{(l+1)}\|_{B^{s-3}_{p,r}} &\le \exp\left[C\int_0^t \|u^{(l+i)}\sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]\|_{B^{s-3}_{p,r}}\mathrm{d}\tau\right] \\
&\times\left\{\int_0^t \exp\left[-C\int_0^\tau \|u^{(l+i)}\sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]\|_{B^{s-3}_{p,r}}\mathrm{d}\tau'\right]\|g\|_{B^{s-3}_{p,r}}\mathrm{d}\tau\right. \\
&\left.+\|m^{(l+i+1)} - m^{(l+1)}\|_{B^{s-3}_{p,r}}\right\}.
\end{aligned} \tag{3.7}$$

We will evaluate $\|g\|_{B^{s-3}_{p,r}}$ step by step. One obtains

$$\begin{aligned}
&\left\|\left\{\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)} - \sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u^{(l+i)}\right\}\partial_x m^{(l+1)}\right\|_{B^{s-3}_{p,r}} \\
&\le C\|\partial_x m^{(l+1)}\|_{B^{s-3}_{p,r}}\|\sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u^{(l+i)} - \sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)}\|_{B^{s-2}_{p,r}} \\
&\le C\|u^{(l+1)}\|_{B^s_{p,r}}\left\{\left\|2\cos\left(\frac{2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2 + 2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2}{2}\right)\right.\right. \\
&\quad\left.\left.\times\sin\left(\frac{2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2 - 2\mu(u^{(l)})u^{(l)} + (u_x^{(l)})^2}{2}\right)u^{(l+i)}\right\|_{B^{s-2}_{p,r}} + \|u^{(l+i)} - u^{(l)}\|_{B^{s-2}_{p,r}}\right\} \\
&\le C\|u^{(l+1)}\|_{B^s_{p,r}}\{\|\mu(u^{(l+i)} - u^{(l)})u^{(l+i)} + u^{(l)}(u^{(l+i)} - u^{(l)})\|_{B^{s-2}_{p,r}}\|u^{(l+i)}\|_{B^{s-2}_{p,r}}
\end{aligned}$$

$$
\begin{aligned}
&+\|(u_x^{(l+i)})^2-(u_x^{(l)})^2\|_{B^{s-2}_{p,r}}\|u^{(l+i)}\|_{B^{s-2}_{p,r}}+\|u^{(l+i)}-u^{(l)}\|_{B^{s-2}_{p,r}}\} \\
\leq{}& C\|u^{(l+1)}\|_{B^{s}_{p,r}}\|u^{(l)}-u^{(l+i)}\|_{B^{s-1}_{p,r}}(\|u^{(l)}\|^2_{B^{s}_{p,r}}+\|u^{(l+i)}\|^2_{B^{s}_{p,r}}+1).
\end{aligned}\tag{3.8}
$$

Similarly, we have

$$
\begin{aligned}
&\Big\|-2\Big\{\sin[2\mu(u^{(l+i)})u^{(l+i)}-(u_x^{(l+i)})^2]u_x^{(l+i)}m^{(l+i)}-\sin[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2]u_x^{(l)}m^{(l)}\Big\}\Big\|_{B^{s-3}_{p,r}} \\
\leq{}& C\Big\|\Big\{\sin[2\mu(u^{(l+i)})u^{(l+i)}-(u_x^{(l+i)})^2]-\sin[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2]\Big\}u_x^{(l+i)}m^{(l+i)}\Big\|_{B^{s-3}_{p,r}} \\
&+C\Big\|\sin[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2](u_x^{(l+i)}-u_x^{(l)})m^{(l+i)}\Big\|_{B^{s-3}_{p,r}} \\
&+C\Big\|\sin[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2]u_x^{(l)}(m^{(l+i)}-m^{(l)})\Big\|_{B^{s-3}_{p,r}} \\
\leq{}& C\|m^{(l+i)}\|_{B^{s-3}_{p,r}}\|u_x^{(l+i)}\|_{B^{s-2}_{p,r}}\|u^{(l+i)}-u^{(l)}\|_{B^{s-1}_{p,r}}(\|u^{(l+i)}\|_{B^{s-1}_{p,r}}+\|u^{(l)}\|_{B^{s-1}_{p,r}}) \\
&+C\|m^{(l+i)}\|_{B^{s-3}_{p,r}}\|u_x^{(l+i)}-u_x^{(l)}\|_{B^{s-2}_{p,r}}+C\|m^{(l+i)}-m^{(l)}\|_{B^{s-3}_{p,r}}\|u_x^{(l)}\|_{B^{s-2}_{p,r}} \\
\leq{}& C\|u^{(l+i)}-u^{(l)}\|_{B^{s-1}_{p,r}}(\|u^{(l+i)}\|^3_{B^{s}_{p,r}}+\|u^{(l)}\|^3_{B^{s}_{p,r}}+\|u^{(l+i)}\|_{B^{s}_{p,r}}+\|u^{(l)}\|_{B^{s}_{p,r}})
\end{aligned}\tag{3.9}
$$

and

$$
\begin{aligned}
&\|\cos[2\mu(u^{(l+i)})u^{(l+i)}-(u_x^{(l+i)})^2]u^{(l+i)}u_x^{(l+i)}(m^{(l+i)})^2-\cos[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2]u^{(l)}u_x^{(l)}(m^{(l)})^2\|_{B^{s-3}_{p,r}} \\
\leq{}& \|\{\cos[2\mu(u^{(l+i)})u^{(l+i)}-(u_x^{(l+i)})^2]-\cos[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2]\}u^{(l+i)}u_x^{(l+i)}(m^{(l+i)})^2\|_{B^{s-3}_{p,r}} \\
&+\|\cos[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2](u^{(l+i)}-u^{(l)})u_x^{(l+i)}(m^{(l+i)})^2\|_{B^{s-3}_{p,r}} \\
&+\|\cos[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2]u^{(l)}[u_x^{(l+i)}-u_x^{(l)}](m^{(l+i)})^2\|_{B^{s-3}_{p,r}} \\
&+\|\cos[2\mu(u^{(l)})u^{(l)}-(u_x^{(l)})^2]u^{(l)}u_x^{(l)}((m^{(l+i)})^2-(m^{(l+i)})^2)\|_{B^{s-3}_{p,r}} \\
\leq{}& \|m^{(l+i)}\|_{B^{s-3}_{p,r}}\|m^{(l+i)}\|_{B^{s-2}_{p,r}}\|u_x^{(l+i)}\|_{B^{s-2}_{p,r}}\|u^{(l+i)}\|_{B^{s-2}_{p,r}} \\
&\times\Big\|2\sin\left(\frac{2\mu(u^{(l+i)})u^{(l+i)}-(u_x^{(l+i)})^2+(u^{(l)})^2-(u_x^{(l)})^2}{2}\right) \\
&\qquad\times\sin\left(\frac{2\mu(u^{(l+i)})u^{(l+i)}-(u_x^{(l+i)})^2-2\mu(u^{(l)})u^{(l)}+(u_x^{(l)})^2}{2}\right)\Big\|_{B^{s-2}_{p,r}} \\
&+\|m^{(l+i)}\|_{B^{s-3}_{p,r}}\|m^{(l+i)}\|_{B^{s-2}_{p,r}}\|u_x^{(l+i)}\|_{B^{s-2}_{p,r}}\|u^{(l+i)}-u^{(l)}\|_{B^{s-2}_{p,r}} \\
&+\|m^{(l+i)}\|_{B^{s-3}_{p,r}}\|m^{(l+i)}\|_{B^{s-2}_{p,r}}\|u^{(l)}\|_{B^{s-2}_{p,r}}\|u_x^{(l+i)}-u_x^{(l)}\|_{B^{s-2}_{p,r}} \\
&+\|m^{(l+i)}-m^{(l)}\|_{B^{s-3}_{p,r}}\|m^{(l+i)}+m^{(l)}\|_{B^{s-2}_{p,r}}\|u_x^{(l)}\|_{B^{s-2}_{p,r}}\|u^{(l)}\|_{B^{s-2}_{p,r}} \\
\leq{}& C\|u^{(l+i)}\|^4_{B^{s}_{p,r}}(\|\mu(u^{(l+i)}-u^{(l)})u^{(l+i)}+u^{(l)}(u^{(l+i)}-u^{(l)})\|_{B^{s-2}_{p,r}}+\|(u_x^{(l+i)})^2-(u_x^{(l)})^2\|_{B^{s-2}_{p,r}}) \\
&+C\|u^{(l+i)}\|^3_{B^{s}_{p,r}}\|u^{(l+i)}-u^{(l)}\|_{B^{s-1}_{p,r}}+C(\|u^{(l+i)}\|_{B^{s}_{p,r}}+\|u^{(l)}\|_{B^{s}_{p,r}})\|u^{(l)}\|^2_{B^{s}_{p,r}}\|u^{(l+i)}-u^{(l)}\|_{B^{s-1}_{p,r}} \\
\leq{}& C\|u^{(l+i)}\|^4_{B^{s}_{p,r}}(\|u^{(l+i)}\|_{B^{s}_{p,r}}+\|u^{(l)}\|_{B^{s}_{p,r}})\|u^{(l+i)}-u^{(l)}\|_{B^{s-1}_{p,r}} \\
&+C(\|u^{(l+i)}\|^3_{B^{s}_{p,r}}+\|u^{(l)}\|^3_{B^{s}_{p,r}})\|u^{(l+i)}-u^{(l)}\|_{B^{s-1}_{p,r}}.
\end{aligned}\tag{3.10}
$$

On the other hand, notice that

$$
\|m_0^{(l+i+1)}-m_0^{(l+1)}\|_{B^{s-3}_{p,r}}=\|S_{l+i+1}m_0-S_{l+1}m_0\|_{B^{s-3}_{p,r}}=\|\textstyle\sum_{q=l+1}^{l+l}\Delta_q m_0\|_{B^{s-3}_{p,r}}\leq C2^{-l}\|m_0\|_{B^{s-3}_{p,r}}\tag{3.11}
$$

and $\{m^{(l)}\}$ is bounded in $C([0,T];B^{s-2}_{p,r})$, one derives from (3.7)-(3.11) that

$$\|m^{(l+i+1)}-m^{(l+1)}\|_{B^{s-3}_{p,r}} \le C_T\left(2^{-l}+\int_0^t \|m^{(l+l)}-m^{(l)}\|_{B^{s-3}_{p,r}}\mathrm{d}\tau\right).$$

Consequently, there holds

$$\|m^{(l+i+1)}-m^{(l+1)}\|_{C([0,T];B^{s-3}_{p,r})} \le \frac{C_T}{2^l}\sum_{k=0}^{l}\frac{(2TC_T)^k}{k!}+\frac{(TC_T)^{l+1}}{(l+1)!}\|m^{(l)}-m^{(0)}\|_{C([0,T];B^{s-3}_{p,r})}.$$

Since $\{m^{(l)}\}$ is uniformly bounded in $C([0,T];B^{s-2}_{p,r})$, one can find a new constant C'_T so that

$$\|m^{(l+i+1)}-m^{(l+1)}\|_{C(0,T;B^{s-3}_{p,r})} \le \frac{C'_T}{2^n}.$$

Therefore, $\{m^{(n)}\}$ is a Cauchy sequence in $C([0,T];B^{s-3}_{p,r})$ and converges to some limit function $m \in C([0,T];B^{s-3}_{p,r})$.

To show the existence of the solution of Eq. (1.7), we would like to verify that the obtained limit function m solves Eq. (1.7) in the sense of distribution, and one step further belongs to $E^s_{p,r}$.

Firstly, the uniform boundedness of $\{m^{(l)}\}$ in $L^\infty(0,T;B^{s-2}_{p,r})$ give rise to $m\in L^\infty([0,T];B^{s-2}_{p,r})$. Secondly, we find that $\{m^{(l)}\}$ converges to m in $C([0,T];B^{s'}_{p,r})$ for all $s'-s<-2$, which follows from

$$\|m_l-m\|_{B^{s'}_{p,r}} \le \begin{cases} C\|m_l-m\|_{B^{s-3}_{p,r}}, & s'-s\le -3,\\ C\|m_l-m\|^{\theta}_{B^{s-3}_{p,r}}(\|m_l\|_{B^{s-2}_{p,r}}+\|m\|_{B^{s-2}_{p,r}})^{1-\theta},\ \theta=s-s'-2, & -3<s'-s\le -2.\end{cases}$$

This claim enables one to take the limit in Eq. (3.1) to find that the limit function m indeed solves Eq.(1.7).

Moreover, Eq. (1.7) can be rewritten as the following transport equation

$$\partial_t m+\sin(2\mu(u)u-u_x^2)u\partial_x m = -2u_x m[\sin(2\mu(u)u-u_x^2)+\cos(2\mu(u)u-u_x^2)um]. \tag{3.12}$$

Since $m\in L^\infty(0,T;B^{s-2}_{p,r})$, thus the right-hand side of equation (3.12) also belongs to $L^\infty(0,T;B^{s-2}_{p,r})$ by means of the product law in Besov spaces and the Sobolev embedding. Consequently, one has $m\in C([0,T);B^{s-2}_{p,r})$ as $r<\infty$ or $m\in C_w([0,T);B^{s-2}_{p,r})$ as $r=\infty$. On the other hand, the Moser-type estimates can deduce that $[\sin(u^2-u_x^2)]\partial_x m$ is bounded in $L^\infty(0,T;B^{s-3}_{p,r})$ and consequently one deduces $\partial_t m\in C([0,T);B^{s-3}_{p,r})$ as $r<\infty$ in light of the high regularity of u and equation (1.7). Therefore, $m\in E^{s-2}_{p,r}$.

Furthermore, the continuity of the solution m in $E^{s-2}_{p,r}(T)$ can be shown by using the result that a sequence of viscosity approximate solutions $\{u_\epsilon\}_{\epsilon>0}$ for (1.7)converges uniformly in $C([0,T];B^{s-2}_{p,r})\cap C^1([0,T];B^{s-3}_{p,r})$.

We next proof the uniqueness. Let $m=\mu(u)-u_{xx}$ and $n=\mu(v)-v_{xx}$ both be solutions of Eq. (1.7). Then one has

$$\begin{aligned}&\partial_t(m-n)+\sin(2\mu(u)u-u_x^2)u\partial_x(m-n)\\ &=-[\sin(2\mu(u)u-u_x^2)u-\sin(2\mu(v)v-v_x^2)v]\partial_x n-2[\sin(2\mu(u)u-u_x^2)u_x m-\sin(2\mu(v)v-v_x^2)v_x n]\\ &-2[\cos(2\mu(u)u-u_x^2)u_x u m^2-\cos(2\mu(v)v-v_x^2)v_x v n^2]\coloneqq f.\end{aligned} \tag{3.13}$$

As a result, one has

$$\|m-n\|_{B^{s-3}_{p,r}} \le \|m_0-n_0\|_{B^{s-3}_{p,r}}+C\int_0^t\left(\|m-n\|_{B^{s-3}_{p,r}}\|u\sin(2\mu(u)u-u_x^2)\|_{B^{s-3}_{p,r}}+\|f\|_{B^{s-3}_{p,r}}\right)\mathrm{d}\tau. \tag{3.14}$$

Again, according to the product law in Besov spaces and the embedding relation, we find

$$\|u\sin(2\mu(u)u-u_x^2)\|_{B^{s-3}_{p,r}} \le C\|u\|_{B^s_{p,r}}. \tag{3.15}$$

We use the Moser-type estimates to generate

$$\begin{aligned}
&\|\cos(2\mu(u)u-u_x^2)u_xum^2-\cos(2\mu(v)v-v_x^2)v_xvn^2\|_{B^{s-3}_{p,r}}\\
&\le C\|\{\cos(2\mu(u)u-u_x^2)-\cos(2\mu(v)v-v_x^2)\}u_xum^2\|_{B^{s-3}_{p,r}}+C\|\cos(2\mu(v)v-v_x^2)(u_x-v_x)um^2\|_{B^{s-3}_{p,r}}\\
&\quad +C\|\cos(2\mu(v)v-v_x^2)v_x(u-v)m^2\|_{B^{s-3}_{p,r}}+C\|\cos(2\mu(v)v-v_x^2)v_xv(m^2-n^2)\|_{B^{s-3}_{p,r}}\\
&\le C\left\|\sin\left(\frac{2\mu(u)u-u_x^2+2\mu(v)v-v_x^2}{2}\right)\sin\left(\frac{2\mu(u)u-u_x^2-2\mu(v)v+v_x^2}{2}\right)uu_xm^2\right\|_{B^{s-3}_{p,r}}\\
&\quad +\|m\|_{B^{s-3}_{p,r}}\|m\|_{B^{s-2}_{p,r}}\|u\|_{B^{s-2}_{p,r}}\|u_x-v_x\|_{B^{s-2}_{p,r}}+\|m\|_{B^{s-3}_{p,r}}\|m\|_{B^{s-2}_{p,r}}\|u-v\|_{B^{s-2}_{p,r}}\|v_x\|_{B^{s-2}_{p,r}}\\
&\quad +\|v_x\|_{B^{s-2}_{p,r}}\|v\|_{B^{s-2}_{p,r}}\|m-n\|_{B^{s-3}_{p,r}}\|m+n\|_{B^{s-2}_{p,r}}\\
&\le C(\|u\|^5_{B^s_{p,r}}+\|v\|^5_{B^s_{p,r}}+\|u\|^3_{B^s_{p,r}}+\|v\|^3_{B^s_{p,r}})\|u-v\|_{B^{s-1}_{p,r}},
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\|-2[\sin(2\mu(u)u-u_x^2)u_xm-\sin(2\mu(v)v-v_x^2)v_xn]\|_{B^{s-3}_{p,r}}\\
&\le C\|[\sin(2\mu(u)u-u_x^2)-\sin(2\mu(v)v-v_x^2)]u_xm\|_{B^{s-3}_{p,r}}+C\|\sin(2\mu(v)v-v_x^2)(u_x-v_x)m\|_{B^{s-3}_{p,r}}\\
&\quad +C\|\sin(2\mu(v)v-v_x^2)v_x(m-n)\|_{B^{s-3}_{p,r}}\\
&\le C\left(\|u\|^3_{B^s_{p,r}}+\|v\|^3_{B^s_{p,r}}+\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}}\right)\|u-v\|_{B^{s-1}_{p,r}},
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\|\{\sin(2\mu(u)u-u_x^2)u-\sin(2\mu(v)v-v_x^2)v\}n_x\|_{B^{s-3}_{p,r}}\\
&\le C\|n_x\|_{B^{s-3}_{p,r}}\Big\{\|\cos[(2\mu(u)u-u_x^2+2\mu(v)v-v_x^2)/2]\sin[(2\mu(u)u-u_x^2-2\mu(v)v+v_x^2)/2]u\|_{B^{s-2}_{p,r}}\\
&\qquad\qquad +\|u-v\|_{B^{s-2}_{p,r}}\Big\}\\
&\le C\|u-v\|_{B^{s-1}_{p,r}}\left(\|u\|^3_{B^s_{p,r}}+\|v\|^3_{B^s_{p,r}}+\|v\|_{B^s_{p,r}}\right).
\end{aligned} \tag{3.18}$$

Finally, plugging (3.15)-(3.18) into (3.14) leads to

$$\|m-n\|_{B^{s-1}_{p,r}} \le \|m_0-n_0\|_{B^{s-1}_{p,r}}+C\int_0^t\|m-n\|_{B^{s-1}_{p,r}}\left(\|u\|_{B^s_{p,r}}+\|u\|^5_{B^s_{p,r}}+\|v\|_{B^s_{p,r}}+\|v\|^5_{B^s_{p,r}}\right)\mathrm{d}\tau. \tag{3.19}$$

Then we find $m=n$ or $u=v$ with the aid of the Gronwall inequality. This completes the proof of Theorem 2.1. □

4 Local well-posedness in $B^{5/2}_{2,1}$

We now show the local well-posedness of Eq. (1.7) in the critical Besov spaces $B^{5/2}_{2,1}$ following the spirit of [35]. The proof of the existence part will be handled first. We construct the smooth approximate sequence $\{m^{(l)}\}_{l=0}^{\infty}$ as in Section 3. Suppose that $m^{(l)}\in L^\infty([0,T];B^{\frac{3}{2}}_{2,1})$. Since $B^{\frac{3}{2}}_{2,1}$ is an algebra, one can check that the right-hand side of Eq. (3.1) belongs to $L^\infty([0,T];B^{\frac{3}{2}}_{2,1})$, which indicates that $m^{(l+1)}\in L^\infty([0,T];B^{\frac{5}{2}}_{2,1})$. Using the similar method as the case $s>\max\{2+\frac{1}{p},\frac{5}{2}\}$ in Section 3, we can find a time $T>0$ depending on the initial data such that, for all $l\in\mathbb{N}$,

$$\|u^{(l)}(t)\|_{B^{5/2}_{2,1}} \le \|u_0\|_{B^{5/2}_{2,1}}\left(1-6Ct\|u_0\|^3_{B^{5/2}_{2,1}}\right)^{-1/3}, \quad \forall t\in[0,T], \tag{4.1}$$

where the constant C is independent of n and T. Accordingly, $\{u^{(l)}\}_{l\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^{\frac{5}{2}}_{2,1})$. Employing Eq. (3.1), we easily know that $\{u^{(l)}\}_{l\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^{\frac{5}{2}}_{2,1})\cap C^1([0,T];B^{\frac{3}{2}}_{2,1})$.

To show $\{u^{(l)}\}_{l\in\mathbb{N}}$ is Cauchy in $C([0,T];B^{\frac{3}{2}}_{2,\infty})$, we first denote the right-hand side of (3.1) by $F(u^{(l)},u^{(l)}_x,m^{(l)})$ and find that the difference $m^{(l+i+1)}-m^{(l+1)}$ satisfies

$$\left[\partial_t+\sin[2\mu(u^{(l+i)})u^{(l+i)}-(\partial_x u^{(l+i)})^2]u^{(l+i)}\partial_x\right]\left(m^{(l+i+1)}-m^{(l+1)}\right)=J(t,x) \tag{4.2}$$

with

$$\begin{aligned}J(t,x)=&F\left(u^{(l+i)},\partial_x u^{(l+i)},m^{(l+i)}\right)-F\left(u^{(l)},\partial_x u^{(l)},m^{(l)}\right)\\&+\left\{\sin[2\mu(u^{(l)})u^{(l)}-(u^{(l)}_x)^2]u^{(l)}-\sin[2\mu(u^{(l+i)})u^{(l+i)}-(u^{(l+i)}_x)^2]u^{(l+i)}\right\}\partial_x m^{(l+1)}.\end{aligned}$$

According to the equality $\left(\mu-\partial_x^2\right)(f\cdot\partial_x g)=f\partial_x\left((\mu-\partial_x^2)g\right)+\mu\left(f\partial_x g\right)-\partial_x^2 f\partial_x g-2\partial_x f\partial_x^2 g$, one can find from (4.2) that

$$\left[\partial_t+\sin[2\mu(u^{(l+i)})u^{(l+i)}-(\partial_x u^{(l+i)})^2]\cdot u^{(l+i)}\partial_x\right]\left(u^{(l+i+1)}-u^{(l+1)}\right)=(\mu-\partial_x^2)^{-1}\sum_{n=1}^{6}A_n(t,x) \tag{4.3}$$

with

$$\begin{aligned}
A_1(t,x)&=-\partial_x^2\Big\{\sin[2\mu(u^{(l+i)})u^{(l+i)}-(u^{(l+i)}_x)^2]u^{(l+i)}\Big\}\partial_x(u^{(l+i+1)}-u^{(l+1)}),\\
A_2(t,x)&=-2\partial_x\Big\{\sin[2\mu(u^{(l+i)})u^{(l+i)}-(u^{(l+i)}_x)^2]u^{(l+i)}\Big\}\partial_{xx}(u^{(l+i+1)}-u^{(l+1)}),\\
A_3(t,x)&=2\Big\{\sin[2\mu(u^{(l)})u^{(l)}-(u^{(l)}_x)^2]u^{(l)}_x m^{(l)}-\sin[2\mu(u^{(l+i)})u^{(l+i)}-(u^{(l+i)}_x)^2]u^{(l+i)}_x m^{(l+i)}\Big\},\\
A_4(t,x)&=2\Big\{\cos[2\mu(u^{(l)})u^{(l)}-(u^{(l)}_x)^2]u^{(l)}u^{(l)}_x(m^{(l)})^2-\cos[2\mu(u^{(l+i)})u^{(l+i)}-(u^{(l+i)}_x)^2]u^{(l+i)}u^{(l+i)}_x(m^{(l+i)})^2\Big\},\\
A_5(t,x)&=\Big\{\sin[2\mu(u^{(l)})u^{(l)}-(u^{(l)}_x)^2]u^{(l)}-\sin[2\mu(u^{(l+i)})u^{(l+i)}-(u^{(l+i)}_x)^2]u^{(l+i)}\Big\}\partial_x m^{(l+1)},\\
A_6(t,x)&=\mu\left(\sin(2\mu(u^{(l+i)})u^{(l+i)}-(u^{(l+i)}_x)^2)u^{(l+i)}\partial_x(u^{(l+i+1)}-u^{(l+1)})\right).
\end{aligned}$$

Lemma 2.5 in [35] then yields

$$\begin{aligned}&\left\|\left(u^{(l+i+1)}-u^{(l+1)}\right)(t)\right\|_{B^{\frac{3}{2}}_{2,\infty}}\\&\leq e^{CW_{l+i}(t)}\left\|u_0^{(l+i+1)}-u_0^{(l+1)}\right\|_{B^{\frac{3}{2}}_{2,\infty}}+C\int_0^t e^{CW_{l+i}(t)-CW_{l+i}(\tau)}\left\|\sum_{n=1}^{6}A_n(\tau,\cdot)\right\|_{B^{-\frac{1}{2}}_{2,\infty}}\mathrm{d}\tau,\end{aligned} \tag{4.4}$$

where $W_{l+i}(t)=\int_0^t\left\|u^{(l+i)}(\tau')\right\|_{B^{\frac{5}{2}}_{2,1}}\mathrm{d}\tau'$.

According to (4.1) and the uniformly boundedness of $\{u^{(l)}\}$ in $E^{5/2}_{2,1}(T)$, there holds

$$\begin{aligned}&W_{l+i}(t)\leq\frac{1}{2C\|u_0\|^3_{B^{5/2}_{2,1}}}\left[1-\left(1-6C\|u_0\|^3_{B^{5/2}_{2,1}}t\right)^{2/3}\right]\leq C,\\&\left\|u^{(l+i)}\right\|_{B^{\frac{5}{2}}_{2,1}}+\left\|u^{(l)}\right\|_{B^{\frac{5}{2}}_{2,1}}\leq 1+2\|u_0\|_{B^{\frac{5}{2}}_{2,1}}:=K,\\&\|S_{n+l+1}u_0-S_{n+1}u_0\|_{B^{\frac{3}{2}}_{2,\infty}}\leq C2^{-n}\|u_0\|_{B^{\frac{5}{2}}_{2,1}}.\end{aligned}$$

We use Lemma 2.6 in [35] to evaluate A_i $(i = 1, 2, 3, 4, 5, 6)$. Since $B^{\frac{1}{2}}_{2,1}$ is an algebra, we find

$$\begin{aligned}
\|A_1\|_{B^{-\frac{1}{2}}_{2,\infty}} &\le C\|\partial_x^2\{\sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u^{(l+i)}\}\|_{B^{-\frac{1}{2}}_{2,\infty}}\|\partial_x(u^{(l+i+1)} - u^{(l+1)})\|_{B^{\frac{1}{2}}_{2,1}}\\
&\le C\|u^{(l+i)}\|_{B^{\frac{3}{2}}_{2,1}}\|u^{(l+i+1)} - u^{(l+1)}\|_{B^{\frac{3}{2}}_{2,1}}\\
&\le CK\|u^{(l+i+1)} - u^{(l+1)}\|_{B^{\frac{3}{2}}_{2,1}}.
\end{aligned}$$

Similarly, there holds

$$\begin{aligned}
\|A_2\|_{B^{-\frac{1}{2}}_{2,\infty}} &\le C\|\partial_x^2(u^{(l+i+1)} - u^{(l+1)})\|_{B^{-\frac{1}{2}}_{2,\infty}}\|\partial_x\{\sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u^{(l+i)}\}\|_{B^{\frac{1}{2}}_{2,1}}\\
&\le C\|u^{(l+i)}\|_{B^{\frac{3}{2}}_{2,1}}\|u^{(l+i+1)} - u^{(l+1)}\|_{B^{\frac{3}{2}}_{2,1}}\\
&\le CK\|u^{(l+i+1)} - u^{(l+1)}\|_{B^{\frac{3}{2}}_{2,1}},
\end{aligned}$$

$$\begin{aligned}
\|A_3\|_{B^{-\frac{1}{2}}_{2,\infty}} \le\ & C\left\{\|m^{(l)}u_x^{(l)}[\mu(u^{(l)} - u^{(l+i)})u^{(l)} + (u^{(l)} - u^{(l+i)})\mu(u^{(l+i)}) - (u_x^{(l)} - u_x^{(l+i)})(u_x^{(l)} + u_x^{(l+i)})]\|_{B^{-\frac{1}{2}}_{2,\infty}}\right\}\\
&+ C\|m^{(l)}(u_x^{(l)} - u_x^{(l+i)})\|_{B^{-\frac{1}{2}}_{2,\infty}} + C\|u_x^{(l+i)}(m^{(l)} - m^{(l+i)})\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
\le\ & C\|m^{(l)}\|_{B^{-\frac{1}{2}}_{2,\infty}}\|u_x^{(l)}\|_{B^{\frac{1}{2}}_{2,1}}(\|u^{(l)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(l+i)}\|_{B^{\frac{3}{2}}_{2,1}})\|u^{(l)} - u^{(l+i)}\|_{B^{\frac{3}{2}}_{2,1}}\\
&+ C\|m^{(l)}\|_{B^{-\frac{1}{2}}_{2,\infty}}\|u^{(l)} - u^{(l+i)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u_x^{(l+i)}\|_{B^{\frac{1}{2}}_{2,1}}\|m^{(l)} - m^{(l+i)}\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
\le\ & CK^3\|u^{(l+i+1)} - u^{(l+1)}\|_{B^{\frac{3}{2}}_{2,1}},
\end{aligned}$$

$$\begin{aligned}
\|A_4\|_{B^{-\frac{1}{2}}_{2,\infty}} \le\ & C\Big\{\|(m^{(l)})^2u_x^{(l)}u^{(l)}[\mu(u^{(l)} - u^{(l+i)})u^{(l)} + (u^{(l)} - u^{(l+i)})\mu(u^{(l+i)})\\
&\qquad - (u_x^{(l)} - u_x^{(l+i)})(u_x^{(l)} + u_x^{(l+i)})]\|_{B^{-\frac{1}{2}}_{2,\infty}}\Big\}\\
&+ C\|(m^{(l)})^2u_x^{(l)}(u_x^{(l)} - u_x^{(l+i)})\|_{B^{-\frac{1}{2}}_{2,\infty}} + C\|(m^{(l)})^2u^{(l+i)}(u_x^{(l)} - u_x^{(l+i)})\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
&+ C\|u^{(l+i)}u_x^{(l+i)}(m^{(l)} - m^{(l+i)})(m^{(l)} + m^{(l+i)})\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
\le\ & C\|m^{(l)}\|_{B^{-\frac{1}{2}}_{2,\infty}}\|m^{(l)}\|_{B^{\frac{1}{2}}_{2,1}}\|u_x^{(l)}\|_{B^{\frac{1}{2}}_{2,1}}\|u^{(l)}\|_{B^{\frac{1}{2}}_{2,1}}(\|u^{(l)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(l+i)}\|_{B^{\frac{3}{2}}_{2,1}})\|u^{(l)} - u^{(l+i)}\|_{B^{\frac{3}{2}}_{2,1}}\\
&+ C\|m^{(l)}\|_{B^{-\frac{1}{2}}_{2,\infty}}\|m^{(l)}\|_{B^{\frac{1}{2}}_{2,1}}\|u_x^{(l)}\|_{B^{\frac{1}{2}}_{2,1}}\|u^{(l)} - u^{(l+i)}\|_{B^{\frac{3}{2}}_{2,1}}\\
&+ C\|m^{(l)}\|_{B^{-\frac{1}{2}}_{2,\infty}}\|m^{(l)}\|_{B^{\frac{1}{2}}_{2,1}}\|u^{(l+i)}\|_{B^{\frac{1}{2}}_{2,1}}\|u_x^{(l)} - u_x^{(l+i)}\|_{B^{\frac{1}{2}}_{2,1}}\\
&+ C\|u_x^{(l+i)}\|_{B^{\frac{1}{2}}_{2,1}}\|u^{(l+i)}\|_{B^{\frac{1}{2}}_{2,1}}\|m^{(l)} + m^{(l+i)}\|_{B^{\frac{1}{2}}_{2,1}}\|m^{(l)} - m^{(l+i)}\|_{B^{-\frac{1}{2}}_{2,\infty}}\\
\le\ & CK^5\|u^{(l+i+1)} - u^{(l+1)}\|_{B^{\frac{3}{2}}_{2,1}},
\end{aligned}$$

$$\begin{aligned}
\|A_5\|_{B^{-\frac{1}{2}}_{2,\infty}} &\le C\|\sin[2\mu(u^{(l)})u^{(l)} - (u_x^{(l)})^2]u^{(l)} - \sin[2\mu(u^{(l+i)})u^{(l+i)} - (u_x^{(l+i)})^2]u^{(l+i)}\|_{B^{\frac{1}{2}}_{2,1}}\|\partial_x m^{(l+1)}\|_{B^{-\frac{1}{2}}_{2,1}}\\
&\le CK^2\|u^{(l+i+1)} - u^{(l+1)}\|_{B^{\frac{3}{2}}_{2,1}},
\end{aligned}$$

$$\|A_6\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\|u^{(l+i)}\partial_x(u^{(l+i+1)} - u^{(l+1)})\|_{B^{\frac{1}{2}}_{2,1}} \le CK\|u^{(l+i+1)} - u^{(l+1)}\|_{B^{\frac{3}{2}}_{2,1}}.$$

Having these estimates in hand, one can employ the similar arguments as those in Section 4 of [35] to complete the proof of the existence part. So we just omit the remaining details here.

We next verify the uniqueness. Let $u, v \in E^{\frac{5}{2}}_{2,1}(T)$ be two solutions of the Cauchy problem (1.7) with the initial data $u_0, v_0 \in B^{\frac{5}{2}}_{2,1}$, respectively. Set $w = u - v$, $w_0 = u_0 - v_0$, $m = \mu(u) - u_{xx}$, $n = \mu(v) - v_{xx}$. Simple calculation yields that w satisfies the following equation

$$\left[\partial_t + \sin(2\mu(u)u - u_x^2)u\partial_x\right] w = (\mu - \partial_x^2)^{-1} \left(\sum_{j=1}^{6} S_j(t,x)\right) \tag{4.5}$$

with

$$\begin{aligned}
S_1(t,x) &= -\partial_x^2\left[\sin(2\mu(u)u - u_x^2)u\right]\partial_x w,\\
S_2(t,x) &= -2\partial_x\left[\sin(2\mu(u)u - u_x^2)u\right]\partial_x^2 w,\\
S_3(t,x) &= 2\left[\sin(2\mu(v)v - v_x^2)v_x n - \sin(2\mu(u)u - u_x^2)u_x m\right],\\
S_4(t,x) &= 2\left[\cos(2\mu(v)v - v_x^2)v v_x n^2 - \cos(2\mu(u)u - u_x^2)u u_x m^2\right],\\
S_5(t,x) &= \left[\sin(2\mu(v)v - v_x^2)v - \sin(2\mu(u)u - u_x^2)u\right]\partial_x n,\\
S_6(t,x) &= \mu\left(\sin(2\mu(u)u - u_x^2)u\partial_x w\right).
\end{aligned}$$

Let $P(t) := e^{-CU(t)}\|w(t)\|_{B^{\frac{3}{2}}_{2,\infty}}$ with $U(t) = \int_0^t \left\|\partial_x[\sin(2\mu(u)u - u_x^2)]\right\|_{B^{\frac{1}{2}}_{2,1}}(\tau')\,\mathrm{d}\tau'$ and $Q(t) := \|u(t)\|_{B^{\frac{5}{2}}_{2,1}} + \|v(t)\|_{B^{\frac{5}{2}}_{2,1}} \le \tilde{Q} := 2\big(\|u_0\|_{B^{\frac{5}{2}}_{2,1}} + \|v_0\|_{\frac{5}{2}}\big)$. Then, we can refer to section 4 [35] to conclude for all $t \in [0,T]$ that

$$\frac{P(t)}{e\tilde{Q}} \le \left(\frac{P(0)}{e\tilde{Q}}\right)^{\exp(-C\tilde{Q}t)} \tag{4.6}$$

implying the uniqueness. This completes the proof of Theorem 2.2.

5 Blow-up criterion and quantity

In this section we would like to give the proofs of the blow-up criterion and precise blow-up quantity for the solution of the Cauchy problem (1.7).

Proof of Theorem 2.3. The proof contains three steps. The method is mainly induction with regard to the derivative index s.

Step 1. AS $s \in (1/2, 1)$, based on a priori estimate in the Sobolev space [2, 19], it follows from the transport form (3.12) of Eq. (1.7) that one has

$$\begin{aligned}
\|m\|_{H^s} \le \|m_0\|_{H^s} &+ C\int_0^t \|m\|_{H^s}\|\partial_x[u\sin(2\mu_0 u - u_x^2)]\|_{L^\infty}\mathrm{d}\tau + C\int_0^t \|\cos(2\mu_0 u - u_x^2)u u_x m^2\|_{H^s}\mathrm{d}\tau\\
&+C\int_0^t \|\sin(2\mu_0 u - u_x^2)u_x m\|_{H^s}\mathrm{d}\tau.
\end{aligned} \tag{5.1}$$

Employing $u = (\mu - \partial_x^2)^{-1}m = p * m$, one finds with the help of the Young inequality that

$$\begin{aligned}
&\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} \le C\|m\|_{L^\infty},\\
&\|u\|_{H^s} + \|u_x\|_{H^s} + \|u_{xx}\|_{H^s} \le C\|m\|_{H^s},
\end{aligned}$$

which further yield

$$\begin{aligned}\|\partial_x[u\sin(2\mu_0 u-u_x^2)]\|_{L^\infty} &\le C\|u_x\|_{L^\infty}+C\|u_x m u\|_{L^\infty}\le C(\|m\|_{L^\infty}^3+\|m\|_{L^\infty}),\\ \|\sin(2\mu_0 u-u_x^2)u_x m\|_{H^s} &\le \|u_x m\|_{H^s}\le C\|m\|_{L^\infty}\|m\|_{H^s},\\ \|\cos(2\mu_0 u-u_x^2)u_x u m^2\|_{H^s} &\le C\|u_x u m^2\|_{H^s}\\ &\le C\|m\|_{L^\infty}\|u_x u m\|_{H^s}+C\|m\|_{H^s}\|u_x u m\|_{L^\infty}\\ &\le C\|m\|_{L^\infty}^3\|m\|_{H^s}.\end{aligned}\tag{5.2}$$

It follows from Eqs. (5.1) and (5.2) that one has

$$\|m\|_{H^s}\le\|m_0\|_{H^s}\exp\left(C\int_0^t(\|m\|_{L^\infty}^3+1)\mathrm{d}\tau\right).\tag{5.3}$$

Therefore, if $\int_0^{T^*}\|m(\tau)\|_{L^\infty}^3\mathrm{d}\tau<\infty$ for the maximal existence time $T^*<\infty$, then the inequality (5.3) yields $\limsup\limits_{t\to T^*}\|m(t)\|_{H^s}<\infty$, which contradicts the assumption on T^*. Thus the proof of Theorem 2.3 is completed for $s\in(1/2,1)$.

Step 2. For $s\in[1,2)$, we differentiate (3.12) once with respect to x to find

$$\begin{aligned}m_{xt}+\sin(2\mu_0 u-u_x^2)u\partial_x m_x &= -2\cos(2\mu_0 u-u_x^2)u_x u m m_x-\sin(2\mu_0 u-u_x^2)u_x m_x\\ &\quad -2\partial_x[\cos(2\mu_0 u-u_x^2)u_x u m^2]-2\partial_x[\sin(2\mu_0 u-u_x^2)u_x m].\end{aligned}\tag{5.4}$$

We further have

$$\begin{aligned}\|m_x\|_{H^{s-1}}\le\|m_{0x}\|_{H^{s-1}}&+C\int_0^t\|m_x\|_{H^{s-1}}\|\partial_x[u\sin(2\mu_0 u-u_x^2)]\|_{L^\infty}\mathrm{d}\tau\\ &+C\int_0^t\|-2\cos(2\mu_0 u-u_x^2)u_x u m m_x-2\partial_x[\cos(2\mu_0 u-u_x^2)u_x u m^2]\|_{H^{s-1}}\mathrm{d}\tau\\ &+C\int_0^t\|-\sin(2\mu_0 u-u_x^2)u_x m_x-2\partial_x[\sin(2\mu_0 u-u_x^2)u_x m]\|_{H^{s-1}}\mathrm{d}\tau.\end{aligned}\tag{5.5}$$

Direct calculation generates

$$\begin{aligned}&\|-2\cos(2\mu_0 u-u_x^2)u_x u m m_x-2\partial_x[\cos(2\mu_0 u-u_x^2)u_x u m^2]\|_{H^{s-1}}\\ &\le C\|u_x u m m_x\|_{H^{s-1}}+C\|u_x u m^2\|_{H^s}\\ &\le C\|\partial_x(u_x u m^2)\|_{H^{s-1}}+C\|u_{xx}u m^2\|_{H^{s-1}}+C\|u_x^2 m^2\|_{H^{s-1}}+C\|m\|_{H^s}\|m\|_{L^\infty}^3\\ &\le C\|m\|_{H^s}\|m\|_{L^\infty}^3\end{aligned}\tag{5.6}$$

and

$$\begin{aligned}\|-\sin(2\mu_0 u-u_x^2)u_x m_x-2\partial_x[\sin(2\mu_0 u-u_x^2)u_x m]\|_{H^{s-1}} &\le C\|u_x m_x\|_{H^{s-1}}+C\|u_x m\|_{H^s}\\ &\le C\|u_x m\|_{H^s}+C\|u_{xx}m\|_{H^{s-1}}\\ &\le C\|m\|_{H^s}\|m\|_{L^\infty}.\end{aligned}\tag{5.7}$$

Combining (5.2) and (5.5)-(5.6), one deduces

$$\|m_x\|_{H^{s-1}}\le\|m_{0x}\|_{H^{s-1}}+C\int_0^t\|m\|_{H^s}(\|m\|_{L^\infty}^3+1)\mathrm{d}\tau.$$

By using the same argument as in **Step 1**, we know that this theorem also holds for the case $s\in[1,2)$.

Step 3: Let $2 \le l \in \mathbb{N}$ and (2.1) hold true for $l-1 \le s < l$. We will invoke induction to prove the validity of (2.1) for $l \le s < l+1$. Applying ∂_x^l to both sides of Eq. (3.12), one obtains

$$\begin{aligned}&\partial_t\partial_x^l m + \sin(2\mu_0 u - u_x^2)u\partial_x^{l+1}m\\&= -\sum_{i=0}^{l-1} C_l^i \partial_x^{l-i}[u\sin(2\mu_0 u - u_x^2)]\partial_x^{i+1}m - 2\partial_x^l[\cos(2\mu_0 u - u_x^2)u_x u m^2] - 2\partial_x^l[\sin(2\mu_0 u - u_x^2)u_x m] \coloneqq f_2.\end{aligned}$$

Moreover one has

$$\|\partial_x^l m\|_{H^{s-l}} \le \|m_0\|_{H^s} + C\int_0^t \|\partial_x^l m\|_{H^{s-l}}\|\partial_x[u\sin(2\mu_0 u - u_x^2)]\|_{L^\infty}\mathrm{d}\tau + C\int_0^t \|f_2\|_{H^{s-l}}\mathrm{d}\tau. \tag{5.8}$$

According to the Sobolev embedding inequality one has

$$\begin{aligned}&\|\sum_{i=0}^{l-1} C_l^i \partial_x^{l-i}[u\sin(2\mu_0 u - u_x^2)]\partial_x^{i+1}m\|_{H^{s-l}}\\&\le \sum_{i=0}^{l-1} C_l^i \|\partial_x^{l-i}[u\sin(2\mu_0 u - u_x^2)]\partial_x^{i+1}m\|_{H^{s-l}}\\&\le C\Big(\|\partial_x^{l-i}[u\sin(2\mu_0 u - u_x^2)]\|_{H^{s-l+1}}\|\partial_x^i m\|_{L^\infty} + \|\partial_x^{l-i}[u\sin(2\mu_0 u - u_x^2)]\|_{L^\infty}\|\partial_x^{i+1}m\|_{H^{s-l}}\Big)\\&\le C\|m\|_{H^{l+1/2+\epsilon}}\|u\sin(2\mu_0 u - u_x^2)\|_{H^{s-i+1}} + C\|m\|_{H^{s-l+i+1}}\|u\sin(2\mu_0 u - u_x^2)\|_{H^{l-i+1/2+\epsilon}}\\&\le C\|m\|_{H^{l+1/2+\epsilon}}\|u\|_{H^{s-i+1}} + C\|m\|_{H^{s-l+i+1}}\|u\|_{H^{l-i+1/2+\epsilon}}\\&\le C\|m\|_{H^{s-1/2+\epsilon}}\|m\|_{H^s},\end{aligned} \tag{5.9}$$

where $\epsilon \in (0, 1/8)$ such that one has $H^{\frac{1}{2}+\epsilon}(\mathbb{S}) \hookrightarrow L^\infty(\mathbb{S})$.

Moreover we have

$$\begin{aligned}&\|2\partial_x^l[\cos(2\mu_0 u - u_x^2)u_x u m^2]\|_{H^{s-l}} \le C\|u_x u m^2\|_{H^s} \le C\|m\|_{H^s}\|m\|^3_{H^{l-1/2+\epsilon}},\\&\|2\partial_x^l[\sin(2\mu_0 u - u_x^2)u_x m]\|_{H^{s-l}} \le C\|u_x m\|_{H^s} \le C\|m\|_{H^s}\|m\|_{H^{l-1/2+\epsilon}}.\end{aligned} \tag{5.10}$$

Eq. (5.2) and Eqs. (5.8)-(5.10) leads to

$$\|\partial_x^l m\|_{H^{s-l}} \le \|m_0\|_{H^s} + C\int_0^t \|m\|_{H^s}(\|m\|^3_{H^{l-1/2+\epsilon}} + 1)\mathrm{d}\tau,$$

that is,

$$\|m(t)\|_{H^s} \le \|m_0\|_{H^s}\exp\left\{C\int_0^t (\|m(\tau)\|^3_{H^{l-\frac{1}{2}+\epsilon}} + 1)\mathrm{d}\tau\right\}. \tag{5.11}$$

Thus, if $\int_0^{T^\star} \|m(\tau)\|^3_{L^\infty}\mathrm{d}\tau < \infty$ for the maximal existence time $T^\star < \infty$, then the uniqueness of the solution given by Theorem 2.1 makes sure the uniform boundedness of $\|m(t)\|_{H^{l-\frac{1}{2}+\epsilon}}$ in $t \in (0, T^*)$, which and (5.11) give rise to the contradiction $\limsup_{t\to T^*}\|m(t)\|_{H^s} < \infty$. Therefore, the proof of Theorem 2.3 is completed. $\square$

Before we deduce the precise blow-up quantity for strong solutions of (1.7), we first give the following proposition. Let $q(t,x)$ solve the following trajectory equation:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}q(t,x) = [u\sin(2\mu_0 u - u_x^2)](t, q(t,x)), \quad x \in \mathbb{S}, \ \ t \in [0,T),\\ q(0,x) = x, \quad x \in \mathbb{S}.\end{cases} \tag{5.12}$$

Then one has

Proposition 5.1. *Suppose $u_0 \in H^s(\mathbb{S})$ with $s > \frac{5}{2}$. Let $T^* > 0$ be the maximal existence time of the solution u to Eq. (1.7). Then there exists a unique solution $q(t,x) \in C^1([0,T^*) \times \mathbb{S}; \mathbb{S})$ to Eq. (5.12) satisfying*

$$q_x(t,x) = \exp\left\{ \int_0^t \left[\sin(2\mu_0 u - u_x^2)u_x + 2\cos(2\mu_0 u - u_x^2)uu_x m\right](s, q(x,s))ds \right\} > 0. \tag{5.13}$$

Moreover, the momentum density $m(t, q(t,x))$ satisfies

$$m(t,q(t,x))q_x(t,x) = m_0(x)\exp\left\{ -\int_0^t [\sin(2\mu_0 u - u_x^2)u_x](s, q(s,x))ds \right\}, \tag{5.14}$$

which implies that the sign and zeros of $m(x,t)$ and $m_0(x)$ are same.

Proof. It follows from Eq. (5.12) that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} q_x(t,x) = [\sin(2\mu_0 u - u_x^2)u_x + 2\cos(2\mu_0 u - u_x^2)uu_x m](t, q(t,x))q_x(t,x), \\ q_x(0,x) = 1. \end{cases} \tag{5.15}$$

Solving (5.15) produces the solution $q_x(t,x)$ given by (5.13). Eqs. (1.7) and (5.12) give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[m(t,q(t,x))q_x(t,x)] &= [m_t(t,q(t,x)) + m_x(t,q(t,x))q_t(t,x)]q_x(t,x) + m(t,q(t,x))q_{xt}(t,x) \\ &= q_x[m_t + m_x \sin(2\mu_0 u - u_x^2)u + m\partial_x(\sin(2\mu_0 u - u_x^2)u)](t, q(t,x)) \end{aligned}$$

implying

$$\frac{\mathrm{d}}{\mathrm{d}t}(mq_x) + \sin(2\mu_0 u - u_x^2)u_x m q_x = 0,$$

which further yields Eq. (5.14). Therefore, the proof of Proposition 5.1 is completed. □

We next deduce the precise blow-up quantity of the Cauchy problem (1.7), that is, the proof of Theorem 2.4.

Proof of Theorem 2.4. It suffices to consider the case $s = 3$. We will prove this Theorem by contradiction. Suppose there exists a positive constant K_1 such that

$$\inf_{x\in\mathbb{S}}[\cos(2\mu_0 u - u_x^2)u_x u m(t,x) + \sin(2\mu_0 u - u_x^2)u_x(t,x)] \geq -K_1, \quad 0 \leq t \leq T^*. \tag{5.16}$$

From Eq. (1.7), one obtains

$$\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} m^{2k}\mathrm{d}x = -\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2)u_x m^{2k}\mathrm{d}x - \frac{2k-1}{2k}\int_{\mathbb{S}} \partial_x[\sin(2\mu_0 u - u_x^2)u]m^{2k}\mathrm{d}x, \tag{5.17}$$

which with $k = 1$ generates

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} m^2\mathrm{d}x = -\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2)u_x m^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{S}}[\sin(2\mu_0 u - u_x^2)u]_x m^2\mathrm{d}x. \tag{5.18}$$

It follows from Eq. (5.18) along with the following two inequalities

$$\|u_x\|_{L^\infty}^2 \leq \|u\|_{H^1}^2 + \|m\|_{L^2}^2, \quad \|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|u\|_{H^1} = \frac{1}{\sqrt{2}}\|u_0\|_{H^1}$$

that one has

$$\|m\|_{L^2} \leq C(\|u_0\|_{H^1}, \|m_0\|_{L^2}), \quad \|u_x\|_{L^\infty} \leq C(\|u_0\|_{H^1}, \|m_0\|_{L^2}). \tag{5.19}$$

Combining (5.17), (5.19) and the Gronwall inequality yields

$$\|m\|_{L^{2k}} \leq C(\|u_0\|_{H^1}, \|m_0\|_{L^{2k}}). \tag{5.20}$$

Then we applying ∂_x to Eq. (1.7) and dotting the result equation with $\partial_x m$ to obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} m_x^2 \mathrm{d}x = \quad & -2\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_x^2 m^2 m_x \mathrm{d}x - 2\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2) u_{xx} m m_x \mathrm{d}x \\
& -\frac{5}{2}\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2) u_x m_x^2 \mathrm{d}x + 4\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2) u_x^2 u m^3 m_x \mathrm{d}x \\
& -2\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_{xx} u m^2 m_x \mathrm{d}x - 5\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_x u m m_x^2 \mathrm{d}x \\
& -4\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_x^2 m^2 m_x \mathrm{d}x.
\end{aligned} \tag{5.21}$$

Combining Eqs. (5.18) and (5.21) yields

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} (m^2 + m_x^2) \mathrm{d}x = \quad & -\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2) u_x m^2 \mathrm{d}x \underbrace{-\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_x u m^3 \mathrm{d}x}_{E_1} \\
& -\frac{1}{2}\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2) u_x m^2 \mathrm{d}x - 2\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_x^2 m^2 m_x \mathrm{d}x \\
& -2\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2) u_{xx} m m_x \mathrm{d}x - \frac{5}{2}\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2) u_x m_x^2 \mathrm{d}x \\
& +4\int_{\mathbb{S}} \sin(2\mu_0 u - u_x^2) u_x^2 u m^3 m_x \mathrm{d}x - 2\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_{xx} u m^2 m_x \mathrm{d}x \\
& \underbrace{-5\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_x u m m_x^2 \mathrm{d}x}_{E_2} - 4\int_{\mathbb{S}} \cos(2\mu_0 u - u_x^2) u_x^2 m^2 m_x \mathrm{d}x.
\end{aligned} \tag{5.22}$$

The integrals E_1 and E_2 in (5.22) can be controlled by using (5.19) and the assumption (5.16), since it indicates

$$-\cos(2\mu_0 u - u_x^2) u_x u m(t,x) \leq \sin(2\mu_0 u - u_x^2) u_x(t,x) + K_1. \tag{5.23}$$

The remaining integrals in (5.22) can be estimated by just employing (5.19). This argument finally enables us to arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} \left(m^2 + m_x^2\right) \mathrm{d}x \leq \left(6K_1 + C_1\left(u_0, m_0\right)\right)\int_{\mathbb{S}} \left(m^2 + m_x^2\right) \mathrm{d}x + C_2\left(t, u_0, m_0\right) \|m_x\|_{L^2}$$

which combined with the Gronwall inequality yields

$$\|m\|_{H^1} \leq e^{[6K_1 + C_1(u_0,m_0)]t}\|m_0\|_{H^1} + \int_0^t e^{[6K_1 + C_1(u_0,m_0)](t-\tau)} C_2(\tau, u_0, m_0) \mathrm{d}\tau$$

for $t \in [0, T^*)$ and consequently $m(t,x)$ will not blow up in finite time recalling Theorem 2.3. However, if Eq. (2.2) holds true, then the Sobolev embedding ensure that $m(t,x)$ or u_x will blow up in finite time.

We next consider the proof of Eq. (2.3). Let $M(t,x) = \cos(2\mu_0 u - u_x^2)uu_x m + \sin(2\mu_0 u - u_x^2)u_x$. It follows from Corollary 2.1 that $M \in \mathcal{C}([0,T^*);H^s)\cap\mathcal{C}^1([0,T^*);H^{s-1})$. For the given $t\in[0,T^*)$, there exists some point $x_0(t)\in\mathbb{R}$ such that

$$M\left(t,x_0(t)\right)=\sup_{x\in\mathbb{R}} M(t,x), \quad \text{i.e.,} \quad M_x\left(t,x_0(t)\right)=0, \text{ a.e. on } (0,T^*). \tag{5.24}$$

It follows from Theorem 2.1 in [7], Corollary 2.1 and $H^s(\mathbb{S})\hookrightarrow\mathcal{C}_0(\mathbb{S})$ with $s>1/2$ that

$$M\left(t,x_0(t)\right)\geq 0 \quad \text{for all} \quad t\in[0,T^*). \tag{5.25}$$

Eq. (5.13) implies that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{S}$ such that $q\left(t,\xi_0(t)\right)=x_0(t)$ for some $\xi_0(t)\in\mathbb{S}$. For the point $(t,x_0(t))=(t,q\left(t,\xi_0(t)\right))$, it follows from Eq. (1.7) that

$$m(t,x_0(t))=m_0(x_0(0))\exp\left(-2\int_0^t M(\tau,x_0(\tau))\mathrm{d}\tau\right)\leq m_0(x_0(0))\leq\sup_{x\in\mathbb{S}} m_0(x). \tag{5.26}$$

On the other hand, the non-sign-changing condition indicates $|u_x|\leq u$, which combined with (5.26) and (**??**) leads to

$$\begin{aligned}
M(t,x_0(t)) &= \cos(2\mu_0 u-u_x^2)uu_x m+\sin(2\mu_0 u-u_x^2)u_x\\
&\leq \|u\|_{L^\infty}\|u_x\|_{L^\infty}\sup_{x\in\mathbb{S}} m_0(x)+\|u_x\|_{L^\infty}\\
&\leq \|u\|^2_{L^\infty}\sup_{x\in\mathbb{S}} m_0(x)+\|u\|_{L^\infty}\\
&\leq C\|u_0\|^2_{H^1}\sup_{x\in\mathbb{S}} m_0(x)+C\|u_0\|_{H^1},
\end{aligned}$$

which gives rise to (2.3). We thus complete the proof of Theorem 2.4. □

6 The wave-breaking phenomenon

We now give the proof Theorem 2.5.

Proof of Theorem 2.5. Let the precise blow-up quantity M be

$$M=\cos(2\mu_0 u-u_x^2)uu_x m+\sin(2\mu_0 u-u_x^2)u_x. \tag{6.1}$$

In what follows, we would like to estimate the dynamics of M along the characteristic. Firstly, one has

$$\begin{aligned}
&(\mu-\partial_x^2)[u_t+\sin(2\mu_0 u-u_x^2)uu_x]\\
&=m_t+(\mu-\partial_x^2)[\sin(2\mu_0 u-u_x^2)uu_x]\\
&=\mu\left(\sin(2\mu_0 u-u_x^2)uu_x\right)+\sin(2\mu_0 u-u_x^2)mu_x-3\mu_0\sin(2\mu_0 u-u_x^2)u_x\\
&\quad -2\mu_0\cos(2\mu_0 u-u_x^2)muu_x-2\partial_x[\cos(2\mu_0 u-u_x^2)muu_x^2]-2\cos(2\mu_0 u-u_x^2)mu_x^3,
\end{aligned} \tag{6.2}$$

from which we have

$$\begin{aligned}
u' &\equiv u_t+\sin(2\mu_0 u-u_x^2)uu_x\\
&=(\mu-\partial_x^2)^{-1}[\sin(2\mu_0 u-u_x^2)mu_x-3\mu_0\sin(2\mu_0 u-u_x^2)u_x-2\mu_0\cos(2\mu_0 u-u_x^2)muu_x\\
&\quad -2\cos(2\mu_0 u-u_x^2)mu_x^3]-2(\mu-\partial_x^2)^{-1}\partial_x[\cos(2\mu_0 u-u_x^2)muu_x^2]+\mu\left(\sin(2\mu_0 u-u_x^2)uu_x\right).
\end{aligned} \tag{6.3}$$

It follows from Eq. (6.3) that one has

$$
\begin{aligned}
u_x' &\equiv u_{xt} + \sin(2\mu_0 u - u_x^2) u u_{xx} \\
&= (\mu - \partial_x^2)^{-1}\partial_x[\sin(2\mu_0 u - u_x^2) m u_x - 3\mu_0 \sin(2\mu_0 u - u_x^2) u_x - 2\mu_0 \cos(2\mu_0 u - u_x^2) m u u_x \\
&\quad -2\cos(2\mu_0 u - u_x^2) m u_x^3] - \sin(2\mu_0 u - u_x^2) u_x^2 - 2\mu \left(\cos(2\mu_0 u - u_x^2) m u u_x^2\right).
\end{aligned} \tag{6.4}
$$

Combining (6.3)-(6.4) and using $m_t + \sin(2\mu_0 u - u_x^2) u m_x = -2mM$, one obtains

$$
\begin{aligned}
& M_t + \sin(2\mu_0 u - u_x^2) u M_x \\
&= -\sin(2\mu_0 u - u_x^2) m u_x u (2\mu_0 u' - 2u_x u_x') + \cos(2\mu_0 u - u_x^2) u_x u(-2mM) + \cos(2\mu_0 u - u_x^2) m u u_x' \\
&\quad + \cos(2\mu_0 u - u_x^2) u_x m u' + \cos(2\mu_0 u - u_x^2) u_x (2\mu_0 u' - 2u_x u_x') + \sin(2\mu_0 u - u_x^2) u_x'.
\end{aligned} \tag{6.5}
$$

From the inequality $|u_x| \le u$ and the assumption $|2\mu \left(\cos(2\mu_0 u - u_x^2) m u u_x^2\right)| \le C_*$, one concludes

$$
\begin{aligned}
(6.5) &\le -2M^2 + 2\sin(2\mu_0 u - u_x^2) u_x M + (m\|u\|_{L^\infty}^2 + \|u\|_{L^\infty})\big[2|\mu_0|\big(\|u\|_{L^\infty}^2 + 3|\mu_0|\|u\|_{L^\infty} + 2|\mu_0|\|u\|_{L^\infty}^3 \\
&\quad + 4\|u\|_{L^\infty}^4 + C_*\big) + 2\|u\|_{L^\infty}\big(2\|u\|_{L^\infty}^2 + 3|\mu_0|\|u\|_{L^\infty} + 2|\mu_0|\|u\|_{L^\infty}^3 + 2\|u\|_{L^\infty}^4 + C_*\big)\big] \\
&\quad + (m\|u\|_{L^\infty} + 1)\big(2\|u\|_{L^\infty}^2 + 3|\mu_0|\|u\|_{L^\infty} + 2|\mu_0|\|u\|_{L^\infty}^3 + 2\|u\|_{L^\infty}^4 + C_*\big) \\
&\quad + m\|u\|_{L^\infty}\big(\|u\|_{L^\infty}^2 + 3|\mu_0|\|u\|_{L^\infty} + 2|\mu_0|\|u\|_{L^\infty}^3 + 4\|u\|_{L^\infty}^4 + C_*\big) \\
&\le -2M^2 + m\big\{(2|\mu_0| + 1)C_*\|u\|_{L^\infty} + 3|\mu_0|\|u\|_{L^\infty}^2 + (6\mu_0^2 + 2C_* + 4)\|u\|_{L^\infty}^3 + 10|\mu_0|\|u\|_{L^\infty}^4 \\
&\quad + (4|\mu_0| + 6)\|u\|_{L^\infty}^5 + 12|\mu_0|\|u\|_{L^\infty}^6 + 4\|u\|_{L^\infty}^7\big\} + \big\{(2|\mu_0| + 1)C_* + 3|\mu_0|\|u\|_{L^\infty} \\
&\quad + (6\mu_0^2 + 2C_* + 4)\|u\|_{L^\infty}^2 + 10|\mu_0|\|u\|_{L^\infty}^3 + (4|\mu_0| + 6)\|u\|_{L^\infty}^4 + 12|\mu_0|\|u\|_{L^\infty}^5 + 4\|u\|_{L^\infty}^6\big\} \\
&\le -2M^2 + (m+1)\big\{(2|\mu_0| + 1)C_* + (5|\mu_0| + 1)C_*\|u\|_{L^\infty} + (6\mu_0^2 + 3|\mu_0| + 2C_* + 4)\|u\|_{L^\infty}^2 \\
&\quad + (6\mu_0^2 + 10|\mu_0| + 2C_* + 4)\|u\|_{L^\infty}^3 + (14|\mu_0| + 6)\|u\|_{L^\infty}^4 + (16|\mu_0| + 6)\|u\|_{L^\infty}^5 \\
&\quad + (12|\mu_0| + 4)\|u\|_{L^\infty}^6 + 4\|u\|_{L^\infty}^7\big\} \\
&\le -2M^2 + C_1(m+1),
\end{aligned} \tag{6.6}
$$

where we use $\|u(t,\cdot)\|_{L^\infty} \leqslant |\mu_0| + \sqrt{3}/6\mu_1$ [18] to control $\|u\|_{L^\infty}$ and the constant C_1 is defined as

$$
\begin{aligned}
C_1 &= 4\mu_0^2 + 7|\mu_0|^3 + 22\mu_0^4 + 26|\mu_0|^5 + 20\mu_0^6 + 16|\mu_0|^7 + \frac{\mu_1}{\sqrt{3}}(4|\mu_0| + 9\mu_0^2 + 33|\mu_0|^3 \\
&\quad + 52\mu_0^4 + 52|\mu_0|^5 + 50\mu_0^6) + \mu_1^2(\frac{1}{3} + \frac{5}{4}|\mu_0| + 6\mu_0^2 + \frac{27}{2}|\mu_0|^3 + \frac{55}{3}\mu_0^4 + 22|\mu_0|^5) \\
&\quad + \frac{\mu_1^3}{12\sqrt{3}}(2 + 17|\mu_0| + 61\mu_0^2 + 120|\mu_0|^3 + 190\mu_0^4) \\
&\quad + \mu_1^4(\frac{1}{24} + \frac{11}{36}|\mu_0| + \frac{35}{36}\mu_0^2 + \frac{20}{9}|\mu_0|^3) + \mu_1^5(\frac{1}{48\sqrt{3}} + \frac{5}{36\sqrt{3}}|\mu_0| + \frac{13}{24\sqrt{3}}\mu_0^2) \\
&\quad + \mu_1^6(\frac{1}{432} + \frac{5}{216}|\mu_0|) + \frac{\mu_1^7}{864\sqrt{3}} + C_*(1 + 3|\mu_0| + 7\mu_0^2 + 2|\mu_0|^3 + \frac{1}{2\sqrt{3}}\mu_1 \\
&\quad + \frac{3\sqrt{3}}{2}|\mu_0|\mu_1 + \sqrt{3}\mu_0^2\mu_1 + \frac{1}{6}\mu_1^2 + \frac{1}{2}|\mu_0|\mu_1^2 + \frac{1}{12\sqrt{3}}\mu_1^3).
\end{aligned} \tag{6.7}
$$

Let $\widehat{M}(t) = M(t, q(t, x_0))$ and $\widehat{m}(t) = m(t, q(t, x_0))$ with $q(t,x)$ being defined in (5.12), then Eq. (6.6) can generate the relation about $\bar{M}(t)$ as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\widehat{M}(t) = (M_t + \sin(2\mu_0 u - u_x^2) u M_x)(t, q(t, x_0)) \le -2\widehat{M}^2(t) + C_1(\widehat{m} + 1). \tag{6.8}
$$

Moreover, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{m}(t) = -2\widehat{m}\widehat{M}. \tag{6.9}$$

It follows from (6.8)-(6.9) that we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widehat{M}(t)}{\widehat{m}(t)}\right) &= \frac{1}{\widehat{m}^2(t)}\left[\widehat{M}'(t)\widehat{m}(t) - \widehat{M}(t)\widehat{m}'(t)\right] \\ &\leq \frac{1}{\widehat{m}^2(t)}\left[\widehat{m}(t)(-2\widehat{M}^2(t) + C_1\widehat{m} + C_1) - \widehat{M}(-2\widehat{m}\bar{M})\right] \\ &= C_1(1 + \frac{1}{\widehat{m}(t)}) \leq C_1(1 + \frac{1}{\varepsilon}) \leq C_2. \end{aligned} \tag{6.10}$$

Integrating (6.10) from 0 to t can give rise to

$$\widehat{M}(t) \leq \left(\frac{\widehat{M}(0)}{\widehat{m}(0)} + C_2 t\right)\widehat{m}(t). \tag{6.11}$$

Therefore, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\widehat{m}(t)}\right) = 2\frac{\widehat{M}(t)}{\widehat{m}(t)} \leq 2\left(\frac{\widehat{M}(0)}{\widehat{m}(0)} + C_2 t\right).$$

Integrating once again, we obtain

$$0 < \frac{1}{\widehat{m}(t)} \leq C_2 t^2 + \frac{2\widehat{M}(0)}{\widehat{m}(0)}t + \frac{1}{\widehat{m}(0)} := h(t).$$

Since $\widehat{M}(0) < 0$, thus $h'(0) < 0$. According to $\lim_{t\to\infty} h'(t) = +\infty$ and the continuity of $h'(t)$, there exists a $\xi > 0$ such that $h'(\xi) = 0$. Under the assumption (2.4), we have $h(\xi) < 0$. Note that $h(0) = \frac{1}{\widehat{m}(0)} > 0$ and $h(t) \in C[0, +\infty)$, one can find some $T^* \in (0, \xi)$ such that

$$0 < \frac{1}{\widehat{m}(t)} \leq h(t) \to 0, \quad \text{as} \quad t \to T^*, \tag{6.12}$$

which indicates that $\lim_{t\to T^*} \widehat{m}(t) = +\infty$. Since $\frac{\widehat{M}(0)}{\widehat{m}(0)} + C_2 T^* < \frac{\widehat{M}(0)}{\widehat{m}(0)} + C_2\xi = 0$, then by using Eqs. (6.11) and (6.12), one can derive

$$\inf_{x\in\mathbb{R}}\left[\cos(2\mu_0 u - u_x^2)uu_x m + \sin(2\mu_0 u - u_x^2)u_x\right](t,x) \to -\infty \quad \text{as} \quad t \to T^*.$$

Using Theorem 2.4, one can know that the solution m blows up at the time $T^* \in (0, \xi]$.

To prove the blow-up rate (2.5), one solves the algebraic equation $h(t) = 0$ to find

$$t_{\pm} := -\frac{\widehat{M}(0)}{C_2\widehat{m}(0)} \pm \frac{1}{2}\sqrt{\left(\frac{2\widehat{M}(0)}{C_2\widehat{m}(0)}\right)^2 - \frac{2}{C_2\widehat{m}(0)}} \tag{6.13}$$

implying

$$0 \leq \frac{1}{\widehat{m}(t)} \leq C_2\left(t - t_-\right)\left(t - t_+\right), \tag{6.14}$$

which combined with Eq. (6.11) generates

$$
\begin{aligned}
&2\,(T^*-t)\inf_{x\in\mathbb{R}} M(t,x) \le 2\,(T^*-t)\,M(t) \le 2\,(T^*-t)\,\widehat{m}(t)\left(\frac{\widehat{M}(0)}{\bar{m}(0)}+C_2 t\right)\\
&\le \frac{2\,(T^*-t)}{(t-t_-)\,(t-t_+)}\left(t+\frac{\widehat{M}(0)}{C_2\widehat{m}(0)}\right),
\end{aligned}
$$

and consequently there holds (2.5) when $T^*=t_-$. This completes the proof of Theorem 2.5. $\square$

7 Hölder continuity

In this section, we would like to prove Theorem 2.6 in the spirit of [22] for the mCH equation. we firstly consider the Lipschitz continuity, namely, $\beta=1$ in the region D_1. Eq. (1.7) can be rewritten as

$$
\begin{aligned}
&u_t+\sin(2\mu(u)u-u_x^2)uu_x\\
&=(\mu-\partial_x^2)^{-1}\partial_x\bigg\{-2\mu_0\sin(2\mu_0u-u_x^2)u-\cos(2\mu_0u-u_x^2)\left[2\mu_0uu_x^2-\frac{2}{3}\partial_x(uu_x^3)+\frac{2}{3}u_x^4\right]\bigg\}\\
&\quad+(\mu-\partial_x^2)^{-1}\bigg\{-\frac{1}{2}\sin(2\mu_0u-u_x^2)\partial_x(u_x^2)+\mu_0\cos(2\mu_0u-u_x^2)[\mu_0\partial_x(u^2)-\partial_x(uu_x^2)-u_x^3]\\
&\qquad+\frac{1}{2}\cos(2\mu_0u-u_x^2)\partial_x(u_x^4)\bigg\}+\mu\left(\sin(2\mu(u)u-u_x^2)uu_x\right).
\end{aligned}
\tag{7.1}
$$

Differentiating Eq. (7.1) once with respect to x and employing the relation $(\mu-\partial_x^2)^{-1}\partial_x^2 f=\mu(f)-f$ lead to

$$
\begin{aligned}
&u_{xt}+\sin(2\mu(u)u-u_x^2)u_x^2+\sin(2\mu(u)u-u_x^2)uu_{xx}-2\mu_0\sin(2\mu(u)u-u_x^2)u\\
&=(\mu-\partial_x^2)^{-1}\partial_x\bigg\{-\frac{1}{2}\sin(2\mu_0u-u_x^2)\partial_x(u_x^2)+\mu_0\cos(2\mu_0u-u_x^2)[\mu_0\partial_x(u^2)-\partial_x(uu_x^2)-u_x^3]\\
&\qquad+\frac{1}{2}\cos(2\mu_0u-u_x^2)\partial_x(u_x^4)\bigg\}\\
&\quad+\mu\bigg(-2\mu_0\sin(2\mu_0u-u_x^2)u-\cos(2\mu_0u-u_x^2)\left[2\mu_0uu_x^2-\frac{2}{3}\partial_x(uu_x^3)+\frac{2}{3}u_x^4\right]\bigg).
\end{aligned}
\tag{7.2}
$$

Set $w=u_x$. Then Eqs. (7.1) and (7.2) yield

$$
\begin{cases}
u_t=-\sin(2\mu_0u-w^2)uw-F(u,w),\\
w_t=-\sin(2\mu_0u-w^2)uw_x-\sin(2\mu_0u-w^2)w^2+2\mu_0\sin(2\mu_0u-w^2)u-G(u,w),\\
u(0,x)=u_0(x),\quad w(0,x)=\partial_xu_0(x)=w_0(x),
\end{cases}
\tag{7.3}
$$

where the nonlocal terms F and G are defined as

$$
\begin{aligned}
F(u,w)&=-p_x*\bigg\{-2\mu_0\sin(2\mu_0u-w^2)u-\cos(2\mu_0u-w^2)\left[2\mu_0uw^2-\frac{2}{3}\partial_x(uw^3)+\frac{2}{3}w^4\right]\bigg\}\\
&\quad-p*\bigg\{-\frac{1}{2}\sin(2\mu_0u-w^2)\partial_x(w^2)+\mu_0\cos(2\mu_0u-w^2)[\mu_0\partial_x(u^2)-\partial_x(uw^2)-w^3]\\
&\qquad+\frac{1}{2}\cos(2\mu_0u-w^2)\partial_x(w^4)\bigg\}-\mu\left(\sin(2\mu(u)u-w^2)uw\right),
\end{aligned}
\tag{7.4}
$$

$$
\begin{aligned}
G(u,w) = -p_x * \Big\{ &-\frac{1}{2}\sin(2\mu_0 u - w^2)\partial_x(w^2) + \mu_0\cos(2\mu_0 u - w^2)[\mu_0\partial_x(u^2) - \partial_x(uw^2) - w^3] \\
&+\frac{1}{2}\cos(2\mu_0 u - w^2)\partial_x(w^4)\Big\} \\
-\mu\Big(&- 2\mu_0\sin(2\mu_0 u - w^2)u - \cos(2\mu_0 u - w^2)\left[2\mu_0 uw^2 - \frac{2}{3}\partial_x(uw^3) + \frac{2}{3}w^4\right]\Big).
\end{aligned} \tag{7.5}
$$

Using similar method as that in [21], we can prove that for $u_0(x) \in H^s$ $(s > 5/2)$ the solution of system (7.3) corresponding to initial data $(u_0(x), w_0(x))$ satisfies $(u,w) \in C([0,T]; H^{s-1})$ and the following size estimates

$$
\|(u,w)\|_{H^{s-1}} \le C\|u_0\|_{H^s}, \quad (s > 5/2) \tag{7.6}
$$

in the lifespan of the solution with C a generic constant.

Let $(v,z) \in C([0,T]; H^{s-1})$ with initial data $(v_0(x), z_0(x))$ be another solution to system (7.3). Let $\varphi = u - v$, $\psi = w - z$. Then we find

$$
\begin{aligned}
\partial_t\varphi = -2&\left[\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2}\sin\frac{2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi}{2}\right]uw \\
&-\sin(2\mu_0' v - z^2)w\varphi - \sin(2\mu_0' v - z^2)v\psi - F(u,w) + F(v,z)
\end{aligned} \tag{7.7}
$$

and

$$
\begin{aligned}
\partial_t\psi = -2&\left[\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2}\sin\frac{2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi}{2}\right]uw_x \\
&-\sin(2\mu_0' v - z^2)w_x\varphi - \sin(2\mu_0' v - z^2)v\psi_x \\
&-2\left[\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2}\sin\frac{2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi}{2}\right]w^2 \\
&-\sin(2\mu_0' v - z^2)\psi(w+z) \\
&+2\mu_0 u\left[\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2}\sin\frac{2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi}{2}\right] \\
&+2\sin(2\mu_0' v - z^2)\mu(\varphi)u + 2\sin(2\mu_0' v - z^2)\mu_0'\varphi - G(u,w) + G(v,z),
\end{aligned} \tag{7.8}
$$

where μ_0' corresponds to v_0.

In view of Eqs. (7.7)-(7.8), one can find from the energy method that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{H^r}^2 = \int_{\mathbb{R}} D^r\left[-2\left[\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2}\sin\frac{2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi}{2}\right]uw\right]\cdot D^r\varphi\mathrm{d}x \tag{7.9a}
$$

$$
-\int_{\mathbb{R}} D^r[\sin(2\mu_0' v - z^2)w\varphi]\cdot D^r\varphi\mathrm{d}x \tag{7.9b}
$$

$$
-\int_{\mathbb{R}} D^r[\sin(2\mu_0' v - z^2)v\psi]\cdot D^r\varphi\mathrm{d}x \tag{7.9c}
$$

$$
-\int_{\mathbb{R}} D^r[F(u,w) - F(v,z)]\cdot D^r\varphi\mathrm{d}x \tag{7.9d}
$$

and

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{H^r}^2 = \int_{\mathbb{R}} D^r\left[-2\left[\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2}\sin\frac{2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi}{2}\right]uw_x\right]\cdot D^r\psi\mathrm{d}x \tag{7.10a}
$$

$$- \int_{\mathbb{R}} D^r\{\sin(2\mu_0' v - z^2) w_x \varphi\} \cdot D^r \psi \mathrm{d}x \tag{7.10b}$$

$$- \int_{\mathbb{R}} D^r\{\sin(2\mu_0' v - z^2) v \psi_x\} \cdot D^r \psi \mathrm{d}x \tag{7.10c}$$

$$+ \int_{\mathbb{R}} D^r \left[-2\left[\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi}{2} \right] w^2 \right] \cdot D^r \psi \mathrm{d}x \tag{7.10d}$$

$$- \int_{\mathbb{R}} D^r\{\sin(2\mu_0' v - z^2)(w+z)\psi\} \cdot D^r \psi \mathrm{d}x \tag{7.10e}$$

$$+ \int_{\mathbb{R}} D^r \left[-2\mu_0 u \left[\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi}{2} \right] \right] \cdot D^r \psi \mathrm{d}x \tag{7.10f}$$

$$+ 2 \int_{\mathbb{R}} D^r\{\sin(2\mu_0' v - z^2) u \mu(\varphi)\} \cdot D^r \psi \mathrm{d}x \tag{7.10g}$$

$$+ 2\mu_0' \int_{\mathbb{R}} D^r\{\sin(2\mu_0' v - z^2)\varphi\} \cdot D^r \psi \mathrm{d}x \tag{7.10h}$$

$$- \int_{\mathbb{R}} D^r[G(u,w) - G(v,z)] \cdot D^r \psi \mathrm{d}x. \tag{7.10i}$$

We first deal with the terms in (7.9). When $1/2 < r \le s-1$, we use the algebra property of H^r and (7.6) to derive

$$\begin{aligned}
|(7.9a)| &\le C \left\| [2\mu(\varphi)u + 2\mu(v)\varphi - (w+z)\psi] uw \right\|_{H^r} \|\varphi\|_{H^r} \\
&\le C[\|w\|_{H^r}\|u\|_{H^r}(\|u\|_{H^r} + \|v\|_{H^r}) + (\|w\|_{H^r} + \|z\|_{H^r})](\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C(\|w\|_{H^{s-1}}^3 + \|z\|_{H^{s-1}}^3 + \|u\|_{H^{s-1}}^3 + \|v\|_{H^{s-1}}^3)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C(\|u_0\|_{H^{s-1}}^3 + \|v_0\|_{H^{s-1}}^3)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C\rho^3(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r})
\end{aligned} \tag{7.11}$$

remembering $\|u_0\|_{H^s} \le \rho$ and $\|v_0\|_{H^s} \le \rho$ in the assumption of Theorem 2.6.

As $-1/2 < r \le 1/2$ and $r \le s-2$, Lemma **??** and (7.6) give rise to

$$\begin{aligned}
|(7.9a)| &\le C[\|w\|_{H^{r+1}}\|u\|_{H^{r+1}}(\|u\|_{H^{r+1}} + \|v\|_{H^{r+1}} + \|w\|_{H^{r+1}} + \|z\|_{H^{r+1}})](\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C(\|w\|_{H^{s-1}}^3 + \|z\|_{H^{s-1}}^3 + \|u\|_{H^{s-1}}^3 + \|v\|_{H^{s-1}}^3)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C(\|u_0\|_{H^{s-1}}^3 + \|v_0\|_{H^{s-1}}^3)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C\rho^3(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}).
\end{aligned} \tag{7.12}$$

When $-1 \le r \le -1/2$, invoking the inequality

$$\|fg\|_{H^r} \le C\|f\|_{H^{s-1}}\|g\|_{H^r} \quad (-1 \le r \le 0, s > 3/2) \tag{7.13}$$

given by Lemma 2 in [22], we find

$$\begin{aligned}
|(7.9a)| &\le C[\|w\|_{H^{s-1}}\|u\|_{H^{s-1}}(\|u\|_{H^{s-1}} + \|v\|_{H^{s-1}} + \|w\|_{H^{s-1}} + \|z\|_{H^{s-1}})](\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C(\|w\|_{H^{s-1}}^3 + \|z\|_{H^{s-1}}^3 + \|u\|_{H^{s-1}}^3 + \|v\|_{H^{s-1}}^3)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C(\|u_0\|_{H^{s-1}}^3 + \|v_0\|_{H^{s-1}}^3)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
&\le C\rho^3(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}).
\end{aligned} \tag{7.14}$$

Using similar procedure as above, we can estimate (7.9b)-(7.9c) as

$$|(7.9b)| \leq C\rho\|\varphi\|_{H^r}^2, \quad |(7.9c)| \leq C\rho\|\varphi\|_{H^r}\|\psi\|_{H^r} \tag{7.15}$$

for $r \in \{-1 \leq r \leq -1/2\} \cup \{-1/2 < r \leq 1/2,\, r \leq s-2\} \cup \{1/2 < r \leq s-1\}$.

For the nonlocal terms (7.9d), a simple computation yields

$$\begin{aligned}
|(7.9d)| \leq & \left| \int_{\mathbb{S}} D^{r-2}\partial_x \left[4\mu_0 u \cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + 2\left| \int_{\mathbb{S}} D^{r-2}\partial_x \left[(\mu(\varphi)u + \mu_0'\varphi)\sin(2\mu_0' v - z^2) \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + 4\left| \int_{\mathbb{S}} D^{r-2}\partial_x \left[\mu_0 u w^2 \sin\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + 2\left| \int_{\mathbb{S}} D^{r-2}\partial_x \left[\cos(2\mu_0' v - z^2)[\mu(\varphi)uw^2 + \mu_0'\varphi w^2 + \mu_0' v(w+z)\psi] \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + \frac{4}{3}\left| \int_{\mathbb{S}} D^{r-2}\partial_x \left[\partial_x(uw^3)\sin\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + \frac{2}{3}\left| \int_{\mathbb{S}} D^{r-2}\partial_x \left[\cos(2\mu_0' v - z^2)\partial_x[\varphi w^3 + v\psi(w^2 + wz + z^2)] \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + \frac{4}{3}\left| \int_{\mathbb{S}} D^{r-2}\partial_x \left[w^4 \sin\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + \frac{2}{3}\left| \int_{\mathbb{S}} D^{r-2}\partial_x \left[\cos(2\mu_0' v - z^2)[\psi(w+z)(w^2+z^2)] \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + 2\left| \int_{\mathbb{S}} D^{r-2} \left[\partial_x(w^2)\cos\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + \left| \int_{\mathbb{S}} D^{r-2} \left[\partial_x[\psi(w+z)]\sin(2\mu_0' v - z^2) \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + 2\left| \int_{\mathbb{S}} D^{r-2} \left[\mu_0^2 \partial_x(u^2)\sin\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + 2\left| \int_{\mathbb{S}} D^{r-2} \left[\mu_0 \partial_x(uw^2)\sin\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + 2\left| \int_{\mathbb{S}} D^{r-2} \left[\mu_0 w^3 \sin\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + \left| \int_{\mathbb{S}} D^{r-2} \left[[\mu(\varphi)\mu(u+v)\partial_x(u^2) + (\mu_0')^2 \partial_x(\varphi(u+v))]\cos(2\mu_0' v - z^2) \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + \left| \int_{\mathbb{S}} D^{r-2} \left[[\mu(\varphi)\partial_x(uw^2) + \mu_0'\partial_x(\varphi w^2 + v\psi(w+z))]\cos(2\mu_0' v - z^2) \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + \left| \int_{\mathbb{S}} D^{r-2} \left[[\mu(\varphi)w^3 + \mu_0'\psi(w^2 + wz + z^2)]\cos(2\mu_0' v - z^2) \right] \cdot D^r\varphi \mathrm{d}x \right| \\
& + 2\left| \int_{\mathbb{S}} D^{r-2} \left[\partial_x(w^4)\sin\frac{2\mu_0 u - w^2 + 2\mu_0' v - z^2}{2} \sin\frac{2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi}{2} \right] \cdot D^r\varphi \mathrm{d}x \right| \quad (7.16a) \\
& + \left| \int_{\mathbb{S}} D^{r-2} \left[\partial_x[\psi(w+z)(w^2+z^2)]\cos(2\mu_0' v - z^2) \right] \cdot D^r\varphi \mathrm{d}x \right| \quad (7.16b) \\
& + \left| \int_{\mathbb{S}} D^{r} \left[\mu\left(\sin(2\mu_0 u - w^2)uw - \sin(2\mu_0' v - z^2)vz \right) \right] \cdot D^r\varphi \mathrm{d}x \right|.
\end{aligned}$$

Not all terms in the above will be estimated. Instead, we will just evaluate two terms containing the derivatives of ψ or w. The remaining terms can be estimated in a similar way. We first handle the term (7.16a). When $1/2 < r \le s-2$, this term can be controlled as

$$
\begin{aligned}
& C\|w^3 w_x[2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi]\|_{H^r}\|\varphi\|_{H^r} \\
& \le C\|w_x\|_{H^{r-1}}\|w^3[2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi]\|_{H^r}\|\varphi\|_{H^r} \\
& \le C[\|w\|_{H^{s-1}}^8 + \|w\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2 + \|z\|_{H^{s-1}}^2](\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
& \le C(\rho^8 + \rho^2)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}).
\end{aligned}
$$

When $-1/2 < r \le 1/2$ and $r \le s-3$, we can control it as

$$
\begin{aligned}
& C\|w_x w^3[2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi]\|_{H^r}\|\varphi\|_{H^r} \\
& \le C\|w_x\|_{H^{r+1}}\|w^3[2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi]\|_{H^r}\|\varphi\|_{H^r} \\
& \le C[\|w\|_{H^{s-1}}^8 + \|w\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2 + \|z\|_{H^{s-1}}^2](\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
& \le C(\rho^8 + \rho^2)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}).
\end{aligned}
$$

When $-1 \le r \le -1/2$ and $r+s \ge 2$, the following inequality [22]

$$
\|fg\|_{H^r} \le c_{r,s}\|f\|_{H^{s-2}}\|g\|_{H^r} \quad (-1 \le r \le 0,\ r+s \ge 2,\ s > 5/2) \tag{7.17}
$$

will be employed to handle it as

$$
\begin{aligned}
& C\|w_x w^3[2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi]\|_{H^r}\|\varphi\|_{H^r} \\
& \le C\|w_x\|_{H^{s-2}}\|w^3[2\mu(\varphi)u + 2\mu_0'\varphi - (w+z)\psi]\|_{H^r}\|\varphi\|_{H^r} \\
& \le C[\|w\|_{H^{s-1}}^8 + \|w\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^4 + \|z\|_{H^{s-1}}^2](\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}) \\
& \le C(\rho^8 + \rho^2)(\|\varphi\|_{H^r}^2 + \|\varphi\|_{H^r}\|\psi\|_{H^r}).
\end{aligned}
$$

The evaluation of this term is finished. We next estimate the term (7.16b). When $1/2 < r \le s-2$, one finds this term can be controlled by

$$
C\|(w+z)(w^2+z^2)\psi\|_{H^r}\|\varphi\|_{H^r} \le C(\|w\|_{H^r}^3 + \|v\|_{H^r}^3 + \|z\|_{H^r}^3)\|\psi\|_{H^r}\|\varphi\|_{H^r} \le C\rho^3\|\psi\|_{H^r}\|\varphi\|_{H^r}.
$$

As $-1/2 < r \le 1/2$ and $r \le s-3$, we evaluate it as

$$
\begin{aligned}
C\|(w+z)(w^2+z^2)\psi\|_{H^r}\|\varphi\|_{H^r} & \le C(\|w^3\|_{H^{r+1}} + \|z^3\|_{H^{r+1}})\|\psi\|_{H^r}\|\varphi\|_{H^r} \\
& \le C(\|w^3\|_{H^{s-1}} + \|z^3\|_{H^{s-1}})\|\psi\|_{H^r}\|\varphi\|_{H^r} \\
& \le C\rho^3\|\varphi\|_{H^r}\|\psi\|_{H^r}.
\end{aligned}
$$

When $-1 \le r \le -1/2$ and $r+s \ge 2$, this term can be estimated as

$$
\begin{aligned}
C\|(w+z)(w^2+z^2)\psi\|_{H^r}\|\varphi\|_{H^r} & \le C(\|w^3\|_{H^{s-2}} + \|z^3\|_{H^{s-2}})\|\psi\|_{H^r}\|\varphi\|_{H^r} \\
& \le C(\|w^3\|_{H^{s-1}} + \|z^3\|_{H^{s-1}})\|\psi\|_{H^r}\|\varphi\|_{H^r} \\
& \le C\rho^3\|\varphi\|_{H^r}\|\psi\|_{H^r}.
\end{aligned}
$$

Combining all the above estimations, we find

$$
|(7.9d)| \le C(\rho^8 + \rho)[\|\varphi\|_{H^r}\|\psi\|_{H^r} + \|\varphi\|_{H^r}^2].
$$

Consequently, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{H^r} \le C(\rho^8+\rho)(\|\varphi\|_{H^r}+\|\psi\|_{H^r}) \tag{7.18}$$

for $r\in\{-1\le r\le -1/2, r+s\ge 2\}\cup\{-1/2<r\le 1/2,\ r\le s-3\}\cup\{1/2<r\le s-2\}$.

We next estimate $\|\psi\|_{H^r}$. We first estimate the term (7.10c), To do this, we will employ he following Calderon-Coifman-Meyer commutator estimate [22, 33]

$$\|[D^r\partial_x, f]\,g\|_{L^2} \le C\|f\|_{H^{s-1}}\|g\|_{H^r}, (0\le r+1\le s-1; s-1>3/2), \tag{7.19}$$

where $[A,B]=AB-BA$ represents the commutator. We then recast (7.10c) as

$$\begin{aligned}(7.10c) &= \int_{\mathbb{R}}\left[D^r\partial_x, v\sin(2\mu_0'v-z^2)\right]\psi\cdot D^r\psi\mathrm{d}x && (7.20\text{a})\\ &\quad+\int_{\mathbb{R}} v\sin(2\mu_0'v-z^2)D^r\partial_x\psi\cdot D^r\psi\mathrm{d}x && (7.20\text{b})\\ &\quad-\int_{\mathbb{R}} D^r\left[\psi\partial_x\left(v\sin(2\mu_0'v-z^2)\right)\right]\cdot D^r\psi\mathrm{d}x. && (7.20\text{c})\end{aligned}$$

From (7.19), there holds

$$\begin{aligned}|(7.20a)| &\le C\|[D^r\partial_x, v\sin(2\mu_0'v-z^2)]\psi\|_{L^2}\|\psi\|_{H^r} \le C\|v\sin(2\mu_0'v-z^2)\|_{H^{s-1}}\|\psi\|_{H^r}^2\\ &\le C\rho\|\psi\|_{H^r}^2\end{aligned} \tag{7.21}$$

for $r\in\{-1\le r\le -1/2, r+s\ge 2\}\cup\{-1/2<r\le 1/2, r\le s-3\}\cup\{r>1/2, r\le s-2\}$.

Invoking integration by parts and Sobolev embedding inequality, one obtains

$$|(7.20b)| \le C\|\partial_x[v\sin(2\mu_0'v-z^2)]\|_{L^\infty}\|\psi\|_{H^r}^2 \le C\|v\sin(2\mu_0'v-z^2)\|_{H^{s-1}}\|\psi\|_{H^r}^2 \le C\rho\|\psi\|_{H^r}^2. \tag{7.22}$$

(7.20c) can be estimated as

$$\begin{aligned}|(7.20c)| &\le C\|\psi\partial_x[v\sin(2\mu_0'v-z^2)]\|_{H^r}\|\psi\|_{H^r}\\ &\le \begin{cases} C\|\partial_x[v\sin(2\mu_0'v-z^2)]\|_{H^r}\|\psi\|_{H^r}^2, (1/2<r\le s-2),\\ C\|\partial_x[v\sin(2\mu_0'v-z^2)]\|_{H^{r+1}}\|\psi\|_{H^r}^2\\ \quad\le C\|v\sin(2\mu_0'v-z^2)\|_{H^{r+2}}\|\psi\|_{H^r}^2, (-1/2<r\le 1/2; r\le s-3),\\ C\|\partial_x[v\sin(2\mu_0'v-z^2)]\|_{H^{s-2}}\|\psi\|_{H^r}^2, (-1\le r\le -1/2; r+s\ge 2)\end{cases}\\ &\le C\|v\sin(2\mu_0'v-z^2)\|_{H^{s-1}}\|\psi\|_{H^r}^2 \le C\rho\|\psi\|_{H^r}^2,\end{aligned} \tag{7.23}$$

recalling the inequality [22]

$$\|fg\|_{H^r} \le c_{r,s}\|f\|_{H^{s-2}}\|g\|_{H^r} \quad (-1\le r\le 0,\ r+s\ge 2,\ s>5/2) \tag{7.24}$$

used to handle the third case in the brace.

Combining Eqs. (7.21)-(7.23) produces $|(7.10c)|\le C\rho\|\psi\|_{H^r}^2$ for $r\in\{-1\le r\le -1/2, r+s\ge 2\}\cup\{-1/2<r\le 1/2,\ r\le s-3\}\cup\{r>1/2, r\le s-2\}$.

The remaining terms in (7.10) can handled similarly as those in (7.9). We finally deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{H^r} \le C(\rho^8+\rho)(\|\varphi\|_{H^r}+\|\psi\|_{H^r}) \tag{7.25}$$

for $r\in\{-1\le r\le -1/2,\ r+s\ge 2\}\cup\{-1/2<r\le 1/2,\ r\le s-3\}\cup\{1/2<r\le s-2\}$.

It follows from Eqs. (7.18) and (7.25) that we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi\|_{H^r}+\|\psi\|_{H^r}) \le C(\rho^8+\rho)(\|\varphi\|_{H^r}+\|\psi\|_{H^r}) \tag{7.26}$$

or

$$\|\varphi\|_{H^r}+\|\psi\|_{H^r} \le C(\|\varphi(0)\|_{H^r}+\|\psi(0)\|_{H^r})e^{(\rho^8+\rho)t}, \quad (t\in[0,T]) \tag{7.27}$$

for $r\in\{-1\le r\le -1/2, r+s\ge 2\}\cup\{-1/2<r\le 1/2, r\le s-3\}\cup\{r>1/2, r\le s-2\}$.

Accordingly, there holds

$$\|u-v\|_{H^{r+1}} \le C\|u_0-v_0\|_{H^{r+1}}e^{(\rho^8+\rho)T}. \tag{7.28}$$

Replacing $r+1$ with r in (7.28) leads to

$$\|u-v\|_{H^r} \le C\|u_0-v_0\|_{H^r}e^{(\rho^8+\rho)T} \tag{7.29}$$

for $r\in\{0\le r\le 1/2,\ r+s\ge 3\}\cup\{1/2<r\le 3/2,\ r\le s-2\}\cup\{3/2<r\le s-1\}$. We therefore have established the Lipschitz continuity in the region D_1.

Next, we show the Hölder continuity in the region $D_2\cup D_3\cup D_4$. The method is interpolation based on the Lipschitz continuity proved previously. The inequality [22]

$$\|f\|_{H^\sigma} \le \|f\|_{H^{\sigma_1}}^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}}\|f\|_{H^{\sigma_2}}^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}, \quad (\sigma_1<\sigma<\sigma_2) \tag{7.30}$$

will be used frequently in the remaining parts of this section.

When $(s,r)\in D_2$, we have

$$\begin{aligned}
\|u-v\|_{H^r} &\le \|u-v\|_{H^{3-s}}\\
&\le C\|u_0-v_0\|_{H^{3-s}}e^{(\rho^8+\rho)T} \quad \text{(by (7.29))}\\
&\le C\|u_0-v_0\|_{H^r}^{\frac{2s-3}{s-r}}\|u_0-v_0\|_{H^s}^{\frac{3-s-r}{s-r}}e^{(\rho^8+\rho)T} \quad \text{(by (7.30))}\\
&\le C\rho^{\frac{3-s-r}{s-r}}\|u_0-v_0\|_{H^r}^{\frac{2s-3}{s-r}}e^{(\rho^8+\rho)T}.
\end{aligned}$$

When $(s,r)\in D_3$, one finds $s-2\le r<s$, consequently

$$\begin{aligned}
\|u-v\|_{H^r} &\le \|u-v\|_{H^{s-2}}^{\frac{s-r}{2}}\|u-v\|_{H^s}^{\frac{r-s+2}{2}} \quad \text{(by (7.30))}\\
&\le C\rho^{\frac{r-s+2}{2}}\|u_0-v_0\|_{H^{s-2}}^{\frac{s-r}{2}}e^{(\rho^8+\rho)T} \quad \text{(by (7.29))}\\
&\le C\rho^{\frac{r-s+2}{2}}\|u_0-v_0\|_{H^r}^{\frac{s-r}{2}}e^{(\rho^8+\rho)T}.
\end{aligned}$$

When $(s,r)\in D_4$, there holds

$$\begin{aligned}
\|u-v\|_{H^r} &\le \|u-v\|_{H^{s-1}}^{s-r}\|u-v\|_{H^s}^{r-s+1} \quad \text{(by (7.30))}\\
&\le C\rho^{r-s+1}\|u_0-v_0\|_{H^{s-1}}^{s-r}e^{(\rho^8+\rho)T} \quad \text{(by (7.29))}\\
&\le C\rho^{r-s+1}\|u_0-v_0\|_{H^r}^{s-r}e^{(\rho^8+\rho)T}.
\end{aligned}$$

We thus complete the proof of Theorem 2.6.

8 Peakon solutions

We here give the proof of Theorem 2.7) about the peakon solutions of Eq. (1.7):

Proof. Let $u_c(t,x)$ be defined as in (2.8) with a to be determined. Substituting (2.8) into definition 2.1 yields

$$\int_0^T \int_{\mathbb{S}} \Big[u_{c,t}\varphi + \sin(2\mu(u_c)u_c - u_{c,x}^2)u_c u_{c,x}\varphi \\ -p_x * [-2\mu(u_c)\sin(2\mu(u_c)u_c - u_{c,x}^2)u_c - \cos(2\mu(u_c)u_c - u_{c,x}^2)(2\mu(u_c)u_c u_{c,x}^2 - 2/3\partial_x(u_c u_{c,x}^3) + 2/3u_{c,x}^4)]\varphi \\ -p * [-1/2\sin(2\mu(u_c)u_c - u_{c,x}^2)\partial_x(u_{c,x}^2) + \mu(u_c)\cos(2\mu(u_c)u_c - u_{c,x}^2)(\mu(u_c)\partial_x(u_c^2) - \partial_x(u_c u_{c,x}^2) - u_{c,x}^3) \\ +1/2\cos(2\mu(u_c)u_c - u_{c,x}^2)\partial_x(u_{c,x}^4)]\varphi - \mu\left(\sin(2\mu(u_c)u_c - u_{c,x}^2)u_c u_{c,x}\right)\varphi \Big] \mathrm{d}x\mathrm{d}t = 0, \tag{8.1}$$

where $p(x) = \frac{1}{2}(x - [x] - \frac{1}{2})^2 + \frac{23}{24}$. For $x \in \mathbb{S}$, we have

$$\mu(u_c) = a\int_0^{ct} \left[\frac{1}{2}\left(x - ct + \frac{1}{2}\right)^2 + \frac{23}{24}\right] dx + a\int_{ct}^1 \left[\frac{1}{2}\left(x - ct - \frac{1}{2}\right)^2 + \frac{23}{24}\right] dx = a.$$

Simple computation leads to

$$u_{c,x} = \begin{cases} a(x - ct - \frac{1}{2}), & (x > ct) \\ a(x - ct + \frac{1}{2}), & (x \le ct) \end{cases}$$

implying

$$2\mu(u_c)u_c - u_{c,x}^2 = \frac{23}{12}a^2$$

whenever $x > ct$ or $x \le ct$. Gathering the above, one can simplify (8.1) as

$$\int_0^T \int_{\mathbb{S}} u_{c,t}\varphi \mathrm{d}x\mathrm{d}t + \sin(\frac{23}{12}a^2)\int_0^T \int_{\mathbb{S}} u_c u_{c,x}\varphi \mathrm{d}x\mathrm{d}t + 3a\sin(\frac{23}{12}a^2)\int_0^T \int_{\mathbb{S}} p_x * u_c\varphi \mathrm{d}x\mathrm{d}t = 0 \tag{8.2}$$

by noticing $u_{c,xx} = a$.

On the other hand, for $x > ct$, one has

$$\begin{aligned} p_x * u_c &= a\int_0^1 \left(x - y - [x-y] - \frac{1}{2}\right)\left[\frac{1}{2}\left(y - ct - [y - ct] - \frac{1}{2}\right)^2 + \frac{23}{24}\right]\mathrm{d}y \\ &= a\int_0^{ct} \left(x - y - \frac{1}{2}\right)\left[\frac{1}{2}\left(y - ct + \frac{1}{2}\right)^2 + \frac{23}{24}\right]\mathrm{d}y \\ &\quad +a\int_{ct}^{x} \left(x - y - \frac{1}{2}\right)\left[\frac{1}{2}\left(y - ct - \frac{1}{2}\right)^2 + \frac{23}{24}\right]\mathrm{d}y \\ &\quad +a\int_{x}^{1} \left(x - y + \frac{1}{2}\right)\left[\frac{1}{2}\left(y - ct - \frac{1}{2}\right)^2 + \frac{23}{24}\right]\mathrm{d}y \\ &= \frac{1}{12}a(x - ct)[1 - 2(x - ct)][(x - ct) - 1]. \end{aligned} \tag{8.3}$$

For $x \le ct$, one has

$$p_x * u_c = a\int_0^1 \left(x - y - [x-y] - \frac{1}{2}\right)\left[\frac{1}{2}\left(y - ct - [y - ct] - \frac{1}{2}\right)^2 + \frac{23}{24}\right]\mathrm{d}y$$

$$
\begin{aligned}
&= a\int_0^x \left(x-y-\frac{1}{2}\right)\left[\frac{1}{2}\left(y-ct+\frac{1}{2}\right)^2+\frac{23}{24}\right]\mathrm{d}y \\
&\quad +a\int_x^{ct} \left(x-y+\frac{1}{2}\right)\left[\frac{1}{2}\left(y-ct+\frac{1}{2}\right)^2+\frac{23}{24}\right]\mathrm{d}y \\
&\quad +a\int_{ct}^{1} \left(x-y+\frac{1}{2}\right)\left[\frac{1}{2}\left(y-ct-\frac{1}{2}\right)^2+\frac{23}{24}\right]\mathrm{d}y \\
&= -\frac{1}{12}a(x-ct)[1+2(x-ct)][(x-ct)+1]. \qquad (8.4)
\end{aligned}
$$

Plugging $u_{c,t} = -ac(x-ct-[x-ct]-1/2)$ and $u_{c,x} = a(x-ct-[x-ct]-1/2)$ into (8.2) yields $c=\sin(\frac{23}{12}a^2)$. $\square$

Gathering the above together, we find (8.2) is equivalent to

$$\int_0^T\int_{\mathbb{S}}\Big\{-ac(\xi-\frac{1}{2})+a^2\sin(\frac{23}{12}a^2)(\xi-\frac{1}{2})\Big[\frac{1}{2}(\xi-\frac{1}{2})^2+\frac{23}{24}\Big]-\frac{1}{2}a^2\sin(\frac{23}{12}a^2)(\xi-\frac{1}{2})\xi(\xi+1)\Big\}\varphi\mathrm{d}x\mathrm{d}t=0$$

for $x>ct$ and

$$\int_0^T\int_{\mathbb{S}}\Big\{-ac(\xi+\frac{1}{2})+a^2\sin(\frac{23}{12}a^2)(\xi+\frac{1}{2})\Big[\frac{1}{2}(\xi+\frac{1}{2})^2+\frac{23}{24}\Big]-\frac{1}{2}a^2\sin(\frac{23}{12}a^2)(\xi+\frac{1}{2})\xi(\xi+1)\Big\}\varphi\mathrm{d}x\mathrm{d}t=0$$

for $x\le ct$, which indicates

$$c=\frac{13}{12}a\sin(\frac{23}{12}a^2).$$

We thus complete the proof of Theorem 2.7.

Acknowledgement

This work was partially supported by the National Natural Science Foundation of China (Nos. 11925108 and 11731014).

Appendix A.

Let $\mathcal{C}\equiv\{\xi\in\mathbb{S}^d|4/3\le|\xi|\le 8/3\}$ and $\tilde{\mathcal{C}}\equiv B(0,2/3)+\mathcal{C}$, where $B(x_0,r)$ stands for the open ball with center x_0 and radius r. Then there exist two functions $\chi\in\mathcal{D}(B(0,4/3))$ and $\varphi\in\mathcal{D}(\mathcal{C})$ which are both radial satisfying

$$
\begin{cases}
\chi(\xi)+\sum_{q\ge 0}\varphi(2^{-q}\xi)=1, \quad 1/3\le\chi^2(\xi)+\sum_{q\ge 0}\varphi^2(2^{-q}\xi)\le 1 \quad (\forall\xi\in\mathbb{S}^d),\\
|q-q'|\ge 2\Rightarrow \operatorname{Supp}\varphi(2^{-q}\cdot)\cap\operatorname{Supp}\varphi(2^{-q'}\cdot)=\varnothing,\\
q\ge 1\Rightarrow \operatorname{Supp}\chi(\cdot)\cap\operatorname{Supp}\varphi(2^{-q'}\cdot)=\varnothing, \quad |q-q'|\ge 5\Rightarrow 2^{q'}\tilde{\mathcal{C}}\cap 2^q\mathcal{C}=\varnothing.
\end{cases}
$$

In the periodic setting, the functions on the torus $\mathbb{S}^d$ are decomposed in Fourier series:

$$u(x)=\sum_{\alpha\in\mathbb{Z}^d}u_\alpha\exp(i2\pi\alpha\cdot x)\quad\text{where } u_\alpha=\int_{\mathbb{S}^d}u(x)\exp(-i2\pi\alpha\cdot x)\mathrm{d}x. \qquad (A.1)$$

Let

$$h_q(x)=\sum_{\alpha\in\mathbb{Z}^d}\varphi\left(2^{-q}\alpha\right)\exp(i2\pi\alpha\cdot x)\quad\text{and}\quad\tilde{h}(x)=\sum_{\alpha\in\mathbb{Z}^d}\chi(\alpha)\exp(i2\pi\alpha\cdot x). \qquad (A.2)$$

Then two dyadic operators Δ_q and S_q acting on functions $u(t,x)$ belonging to $S'(\mathbb{S}^d)$ are defined as

$$\Delta_q u = \begin{cases} 0, & q \leq -2, \\ \chi(D)u = \int_{\mathbb{S}^d} \tilde{h}(y)u(x-y)dy, & q = -1, \\ \varphi\left(2^{-q}D\right)u = 2^{qd}\int_{\mathbb{S}^d} h\left(2^q y\right)u(x-y)dy, & q \geq 0, \end{cases}$$
$$S_q u = \sum_{q' \leq q-1} \Delta_{q'} u.$$

The Besov space is defined as $B^s_{p,r}(\mathbb{S}^d) = \left\{ u \in S'(\mathbb{S}^d) \,\middle|\, \|u\|_{B^s_{p,r}(\mathbb{S}^d)} = \left(\sum_{j\geq -1} 2^{rjs}\|\Delta_j u\|^r_{L^p(\mathbb{S}^d)}\right)^{1/r} < \infty \right\}$.
Let $E^s_{p,r}(T)$ with $T > 0$, $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$ be defined as

$$E^s_{p,r}(T) \triangleq \begin{cases} C\left([0,T]; B^s_{p,r}\right) \cap C^1\left([0,T]; B^{s-1}_{p,r}\right), & \text{as } r < \infty, \\ C_w\left([0,T]; B^s_{p,\infty}\right) \cap C^{0,1}\left([0,T]; B^{s-1}_{p,\infty}\right), & \text{as } r = \infty, \end{cases}$$

and $E^s_{p,r} = \bigcap_{T>0} E^s_{p,r}(T)$.

References

[1] S. C. Anco, E. Recio, A general family of multi-peakon equations and their properties, *J. Phys. A: Math. Theor.* 52 (2019) 125203.

[2] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer, Berlin-Heidelberg-New York, 2011.

[3] R. Beals, D. Sattinger, J. Szmigielski, Multipeakons and the classical moment problem, *Adv. Math.* 154 (2000) 229-57.

[4] R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 71 (1993) 1661-1664.

[5] J. Chemin, Localization in Fourier space and Navier-Stokes system, in: Phase Space Analysis of Partial Differential Equations. Proceedings, in: CRM Series, Pisa, 2004, pp. 53-136.

[6] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.* 166 (2006) 23-535.

[7] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.* 181 (1998) 229-243.

[8] A. Constantin, J. Escher, Particle trajectories in solitary water waves, *Bull. Am. Math. Soc.* 44 (2007) 423-431.

[9] A. Constantin, J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. Math.* 173 (2011) 559-568.

[10] A. Constantin, V. Gerdjikov, I. Ivanov, Inverse scattering transform for the Camassa-Holm equation, *Inverse Probl.* 22 (2006) 2197-2207.

[11] A. Constantin, W. Strauss, Stability of peakons, *Commun. Pure Appl. Math.* 53 (2000) 603-610.

[12] A. Constantin, W. Strauss, Stability of a class of solitary waves in compressible elastic rods, *Phys. Lett. A* 270 (2000) 140-148.

[13] H. Dai, Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod, *Acta Mechanica* 127 (1998) 193-207.

[14] R. Danchin, A few remarks on the Camassa-Holm equation, *Differ. Integral Equ.* 14 (2001) 953-988.

[15] R. Danchin, Fourier Analysis Methods for PDEs, Lecture Notes, 14 November 2003.

[16] R. Dachin, A note on well-posedness for Camassa-Holm equation, *J. Differential Equations* 192 (2003) 429-444.

[17] A. Fokas, B. Fuchssteiner, Symplectic structures, their Backlund transformation and hereditary symmetries, *Physica D* 4 (1981) 47-66.

[18] Y. Fu, Y. Liu, C. Qu, On the blow-up structure for the generalized periodic Camassa-Holm and Degasperis-Procesi equations, *J. Funct. Anal.* 262 (2012) 3125-3158.

[19] G. Gui, Y. Liu, On the global existence and wave-breaking criteria for the two-component Camassa-Holm system, *J. Funct. Anal.* 258 (2010) 4251-4278.

[20] G. Gui, Y. Liu, M. Zhu, On the wave-breaking phenomena and global existence for the generalized periodic Camassa-Holm equation, *Int. Math. Res. Not.* 21 (2012) 4858-4903.

[21] A. Himonas, D. Mantzavinos, The Cauchy problem for the Fokas-Olver-Rosenau-Qiao equation, *Nonlinear Anal.* 95 (2014) 499-529.

[22] A. Himonas, D. Mantzavinos, Hölder continuity for the Fokas-Olver-Rosenau-Qiao equation, *J. Nonlinear Sci.* 24 (2014) 1105-1124.

[23] J.K. Hunter, R. Saxton, Dynamics of director fields, *SIAM J. Appl. Math.* 51 (1991) 1498-1521.

[24] B. Khesin, J. Lenells, G. Misiołek, Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms, *Math. Ann.* 342 (2008) 617-656.

[25] J. Lenells, Stability of periodic peakons, *Int. Math. Res. Not.* 10 (2004) 485-99.

[26] J. Lenells, Traveling wave solutions of the Camassa-Holm equation, *J. Differential Equations* 217 (2005) 393-430.

[27] J. Lenells, G. Misiołek, F. Tiğlay, Integrable evolution equations on spaces of tensor densities and their peakon solutions, *Comm. Math. Phys.* 299 (2010) 129-161.

[28] Y. Liu, C. Qu, Y. Zhang, Stability of periodic peakons for the modified μ-Camassa-Holm equation, *Physica D* 250 (2013) 66-74.

[29] P. Olver, P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E* 53 (1996) 1900-1906.

[30] Y. Fu, C. Qu, Y.Liu, Well-posedness, wave breaking and peakons for a modified μ-Camassa-Holm equation, *J. Funct. Anal.* 266 (2014) 433-477.

[31] G. Qin, Z. Yan, B. Guo, The Cauchy problem and wave-breaking phenomenon for a generalized sine-type FORQ/mCH equation, *Monatsh. Math.* (to appear, 2021).

[32] G. Qin, Z. Yan, B. Guo, A sine-type Camassa-Holm equation: Local well-posedness, Hölder continuity, and wave-breaking analysis (to be submitted, 2021).

[33] M. Taylor, Commutator estimates, *Proc. Am. Math. Soc.* 131 (2003) 1501-1507.

[34] H. Wang, Y. Fu, Non-uniform dependence on initial data for the modified μ-Camassa-Holm equation, *J. Differential Equations*, 261 (2016) 6099-6124.

[35] M. Yang, Y. Li, Y. Zhao, On the Cauchy problem of generalized Fokas-Olver-Resenau-Qiao equation, *Appl. Anal.* 97 (2018) 2246-2268.