Combined effects of double nonlocal terms in the nonlinear eigenvalue problems

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Abstract

In this paper, we study the following eigenvalue problem for Kirchhoff type equation with Hartree nonlinearity:

$$-M\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+\mu V(x)u=\left(I_\alpha * Q\,|u|^p\right)Q|u|^{p-2}u+\lambda f(x)u \quad \text{in} \quad \mathbb{R}^N, \tag{1}$$

where $N\geq 3, a, \mu>0$ parameters, $M(t)=at+1$, $V\in C(\mathbb{R}^N,\mathbb{R}^+)$, I_α is the Riesz potential, $Q(x)\in L^\infty(\mathbb{R}^N)$ with changes sign in $\overline{\Omega}:=\{V(x)=0\}$, and $0<p<2^*_\alpha:=\frac{N+\alpha}{N-2}$. By using mountain pass theory, new constraint manifold method and some approximation estimates, we mainly prove the existence and multiplicity of positive solutions when λ and p belongs to different intervals. Furthermore, we do not assume any sign condition on the integral $\int_{\mathbb{R}^N}\left(I_\alpha * Q\,|\phi_1|^p\right)Q|\phi_1|^pdx$, and the number of solutions in the neighborhood of the bifurcation point $\lambda_1(f_\Omega)$ is clearly presented, where $\lambda_1(f_\Omega)$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$ with weight function $f_\Omega:=f|_{\overline{\Omega}}$ and ϕ_1 is the corresponding principal eigenfunction.

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1 Introduction

We are concerned with the following nonlinear Kirchhoff equations:

$$\begin{cases} -\left(b+a\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+\mu V(x)u=g(x,u) \qquad \text{in} \quad \mathbb{R}^N, \\ u\in H^1(\mathbb{R}^N), \end{cases} \tag{2}$$

where $N\geq 3$, the parameters $a,b,\mu>0$, $g(x,u)=\left(I_\alpha * Q(x)\,|u|^p\right)Q(x)|u|^{p-2}u+\lambda f(x)u$, I_α is the Riesz potential of order $\alpha\in(0,N)$ defined by

$$I_\alpha=\frac{A\left(N,\alpha\right)}{|x|^{N-\alpha}} \quad \text{with} \quad A\left(N,\alpha\right)=\frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\pi^{N/2}2^\alpha\Gamma\left(\frac{\alpha}{2}\right)} \text{ for each } x\in\mathbb{R}^N\setminus\{0\},$$

and the weight functions satisfy the following conditions:

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(V_1) $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ and there exists $b > 0$ such that $|\{V < b\}|$ is the finite, where $|\cdot|$ is the Lebesgue measure;

(V_2) $\Omega = \text{int}\{x \in \mathbb{R}^N : V(x) = 0\}$ is nonempty and has smooth boundary with $\overline{\Omega} = \{x \in \mathbb{R}^N : V(x) = 0\}$;

(A_1) $Q \in L^\infty(\mathbb{R}^N)$ which $Q^+ := \max\{Q, 0\} \not\equiv 0$ in Ω;

(A_2) $f \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ which $f^+ := \max\{f, 0\} \not\equiv 0$ in Ω.

The hypotheses $(V_1) - (V_2)$, first suggested by Bartsch-Wang [4], imply that $\mu V(x)$ represents a potential well whose depth is controlled by μ. If μ is sufficiently large, then $\mu V(x)$ is known as the steep potential well. Consider the following nonlinear eigenvalue problem:

$$-\Delta u(x) = \lambda f_\Omega(x) u(x) \, \text{for} \, x \in \Omega; \; u(x) = 0 \, \text{for} \, x \in \partial\Omega, \tag{3}$$

where f_Ω is a restriction of f on $\overline{\Omega}$. By the assumption (A_2), we have $\{f > 0\} \cap \Omega$ has a positive Lebesgue measure, thus the problem (3) has a sequence of eigenvalues $0 < \lambda_1(f_\Omega) < \lambda_2(f_\Omega) \leq \cdots \leq \lambda_n(f_\Omega) \leq \cdots$, which are obtained by Krasnoselski genus techniques. It is well-known that $\lambda_1(f_\Omega)$ is the positive principal eigenvalue of problem (3) and $\lambda_1(f_\Omega)$ has a corresponding positive principal eigenfunction ϕ_1 with $\int_\Omega f_\Omega \phi_1^2 dx = 1$ and $\int_\Omega |\nabla \phi_1|^2 dx = \lambda_1(f_\Omega)$.

When $a = \lambda = 0$ and $V(x) = Q(x) = 1$, Eq. (2) becomes the well-known Choquard-Peark equation

$$-\Delta u + u = (I_\alpha * |u|^p) |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^N. \tag{4}$$

Such equations have an important physical background. When $N = 3$ and $p = \alpha = 2$, Eq. (4) was proposed by Pekar [26] to describe the quantum theory model of the polaron at rest, and was applied as an approximation to Hartree-Fock theory of one component plasma by Choquard [15]. After the pioneering work by Lieb [15], the existence of ground states, sign-changing solutions and their qualitative analysis for Eq. (4) has received much attention in recent years, see for example [2, 3, 17, 18, 23, 25]. In particular, Moroz-Van Schaftingen [23] studied the existence and qualitative properties of ground state solution for Eq. (4) in $\mathbb{R}^N (N \geq 3)$ within an optimal range on p by $2_\alpha < p < 2_\alpha^*$, where $2_\alpha = \frac{N+\alpha}{N}$ is termed as the lower critical exponent, $2_\alpha^* = \frac{N+\alpha}{N-2}$ is termed as the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality (see below Lemma 2.1). By using the Pohozaev identity, the nonexistence of nontrivial smooth H^1-solution of Eq. (4) when either $p \leq 2_\alpha$ or $p \geq 2_\alpha^*$ was proved.

On the other hand, the Kirchhoff type equation arises in an important physical context. In fact, if $\mu = 0$ and replace $\mathbb{R}^N$ by bounded domain $\Omega \subset \mathbb{R}^N$, then it reduces to the following Dirichlet problem:

$$\begin{cases} -\left(1 + a \int_\Omega |\nabla u|^2 dx\right) \Delta u = h(x, u) & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

which is related to the stationary analogue of time-dependent equation

$$u_{tt} - \left(1 + a \int_\Omega |\nabla u|^2 dx\right) \Delta u = g(x, u) \quad \text{in} \quad \Omega. \tag{5}$$

Equation (5) was first proposed by Kirchhoff [14] to extend the classical D'Alembert wave equation by taking into account the subsequent change in string length during the vibrations. The existence and qualitative properties of nontrivial solutions for the nonlinear Kirchhoff type equations have been extensively investigated in the literature. We refer the reader to [11, 12, 13, 20, 28, 31, 35] and the references therein. Let us briefly review some related work. For spatial dimension $N = 3$, Li and Ye [20] studied that $g(x, u) = |u|^{q-2}u$ with $q \in (3, 6)$, and they obtained the existence of positive ground states to Eq. (2). Later, for $q \in (2, 4)$, Sun and Wu [31] proved that the existence of two positive solutions under the suitable assumptions on potential $V(x)$. In [28], they proved that when $N \geq 4$ and $g(x, u)$ is superlinear and subcritical on u, two different positive solutions can be obtained by standard variational methods. Very recently, when $g(x, u) = \lambda f(x)u + h(x)|u|^{p-2}u$ with $2 < p < 2^*$, by using Nehari manifold method, Zhang et al. [36] studied the the existence and multiplicity of positive solutions when λ lies in the left and right neighborhood of $\lambda_1(f_\Omega)$. Later, the corresponding result was further improved by Sun et al. [32], the branch phenomenon was more clearly showed by using mountain pass theory.

Recently, the following Kirchhoff-Hartree type equations

$$-\left(b + a\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + u = \left(I_\alpha * |u|^p\right) |u|^{p-2}\, u \quad \text{in} \quad \mathbb{R}^N, \tag{6}$$

have begun to receive increasingly interest. But to our best knowledge, there are few results to Eq. (6), see for example [7, 8, 19, 21, 22, 27] and the references therein. In [19], by using Nehari manifold and the concentration compactness principle, they established the existence of ground states for $N = 3$, $\alpha \in (0, N)$ and $2 < p < 3 + \alpha$. Later, Chen and Liu [7] obtained existence of ground states for the full range $(3 + \alpha)/3 < p < 3 + \alpha$. In [21], when $N \geq 3$, $\max\{0, N - 4\} < \alpha < N$ and $2 < p < 2^*_\alpha$, they proved that there admits a positive ground state solution by using global compactness lemma and monotonicity tricks. We notice that there seems to be a rare concern on the eigenvalue problem for Kirchhoff-Hartree type equations in the existing literature.

Inspired by the fact mentioned above, the purpose of present paper is to study this case. The problem we consider is thus

$$-\left(1 + a\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + \mu V(x)u = \left(I_\alpha * Q\,|u|^p\right) Q\,|u|^{p-2}\, u + \lambda f(x)u \quad \text{in} \quad \mathbb{R}^N, \tag{$K^\mu_{a,\lambda}$}$$

where $N \geq 3, a, \lambda, \mu > 0$. We need to separate the problem in five cases as follows:
Case (a): $\alpha \in (\max\{0, N - 4\}, N)$, $2 < p < 2^*_\alpha$ for $N \geq 3$;
Case (b): $\alpha \in (\max\{0, N - 4\}, N)$, $p = 2$ for $N \geq 3$;
Case (c): $(c-i) : \alpha \in (0, N)$, $2_\alpha < p < 2$ for $N = 3, 4$; $(c-ii) : \alpha \in (0, N-4)$, $2_\alpha < p \leq 1 + \frac{\alpha}{N-4}$ for $N \geq 5$; $(c - iii) : \alpha \in [N - 4, N)$, $2_\alpha < p < 2$ for $N \geq 5$;
Case (d): $\alpha \in (0, N - 4)$, $1 + \frac{\alpha}{N-4} < p < 2^*_\alpha$ for $N \geq 5$;
Case (e): $\alpha \in (0, N)$, $0 < p < 1$ for $N \geq 3$, and we assume that the weight function $Q(x)$ satisfies the following condition:

(A_3) $Q \in L^{\frac{2N}{N+\alpha-Np}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ which $Q^+ := \max\{Q, 0\} \not\equiv 0$ in Ω.

We now summarize our main results as follows.

Theorem 1.1 *Suppose that the **Case (a)**, conditions* $(V_1)-(V_2)$ *and* $(A_1)-(A_2)$ *hold. Then for each* $a>0$ *and* $0<\lambda\leq\lambda_1(f_\Omega)$*, Eq.* $(K_{a,\lambda}^\mu)$ *admits at least a positive solution* $\overline{u}\in H^1(\mathbb{R}^N)$ *with positive energy* $I_{a,\lambda}^\mu(\overline{u})>0$ *for* $\mu>0$ *sufficiently large.*

Theorem 1.2 *Suppose that the **Case (a)**, conditions* $(V_1)-(V_2)$ *and* $(A_1)-(A_2)$ *hold. Then there exists* $\delta_{a,\mu}$ *such that for each* $a>0$ *and* $\lambda_1(f_\Omega)<\lambda<\lambda_1(f_\Omega)+\delta_{a,\mu}$*, Eq.* $(K_{a,\lambda}^\mu)$ *admits at least two positive solutions* $u^{(1)}$ *and* $u^{(2)}$ *satisfying* $I_{a,\lambda}^\mu(u^{(2)})<0<I_{a,\lambda}^\mu(u^{(1)})$ *for* $\mu>0$ *sufficiently large.*

To consider the **Case (b)**, we need the following maximum problem:

$$\Gamma^*:=\sup_{u\in E}\frac{\int_{\mathbb{R}^N}(I_\alpha*Q|u|^2)Q|u|^2}{\|\nabla u\|_{L^2}^4}>0.$$

Theorem 1.3 *Suppose that the **Case (b)**, conditions* $(V_1)-(V_2)$ *and* $(A_1)-(A_2)$ *hold. Then we have the following results:*
(i) For each $0<a<\Gamma^*$ *and* $0<\lambda<\lambda_1(f_\Omega)$*, Eq.* $(K_{a,\lambda}^\mu)$ *admits a positive solution* $\overline{u}$ *satisfying* $I_{a,\lambda}^\mu(\overline{u})>0$ *for* $\mu>0$ *sufficiently large;*
(ii) If $\Gamma^*<\infty$*, then for each* $a\geq\Gamma_*$ *and* $0<\lambda<\lambda_1(f_\Omega)$*, Eq.* $(K_{a,\lambda}^\mu)$ *does not admit nontrivial solution for* $\mu>0$ *sufficiently large;*
(iii) If $\Gamma^*<\infty$*, then for each* $a>\Gamma^*$ *and* $\lambda\geq\lambda_1(f_\Omega)$*, Eq.* $(K_{a,\lambda}^\mu)$ *admits a positive solution* $\widetilde{u}$ *satisfying* $I_{a,\lambda}^\mu(\widetilde{u})<0$ *for* $\mu>0$ *sufficiently large;*
(iv) If $\Gamma^*<\infty$ *and* Γ^* *is not attained, then for* $a=\Gamma^*$ *and* $\lambda\geq\lambda_1(f_\Omega)$*, Eq.* $(K_{a,\lambda}^\mu)$ *admits a positive solution* $\widehat{u}$ *satisfying* $I_{a,\lambda}^\mu(\widehat{u})<0$ *for* $\mu>0$ *sufficiently large.*

Theorem 1.4 *Suppose that the **Case (b)**, conditions* $(V_1)-(V_2)$ *and* $(A_1)-(A_2)$ *hold. Then for each* $\lambda_1^{-2}(f_\Omega)\int_\Omega(I_\alpha*Q|\phi_1|^2)Q|\phi_1|^2dx<a<\Gamma^*$*, there exists* $\widehat{\delta}>0$ *such that for each* $\lambda_1(f_\Omega)\leq\lambda<\lambda_1(f_\Omega)+\widehat{\delta}$*, Eq.* $(K_{a,\lambda}^\mu)$ *admits at least two positive solutions* $u^{(1)}$ *and* $u^{(2)}$ *satisfying* $I_{a,\lambda}^\mu(u^{(2)})<0<I_{a,\lambda}^\mu(u^{(1)})$ *for* $\mu>0$ *sufficiently large.*

Set

$$A_p^*:=\sup\left\{\frac{\int_\Omega(I_\alpha*Q|u|^p)Q|u|^pdx}{(\int_\Omega|\nabla u|^2dx)^p}|u\in H_0^1(\Omega)\setminus\{0\},\int_\Omega f_\Omega(x)u^2dx\geq0\right\}$$

and

$$a_*(p)=\left(\frac{2(p-1)A_p^*}{p}\right)^{\frac{1}{p-1}}\left(\frac{2-p}{2(p-1)}\right)^{\frac{2-p}{p-1}}.$$

By conditions (A_1) and (A_2), following the idea in [6], we can choose a function $\varphi\in H_0^1(\Omega)$ such that $\int_\Omega(I_\alpha*Q|\varphi|^p)Q|\varphi|^pdx>0$ and $\int_\Omega f_\Omega(x)\varphi^2dx>0$. Then, it is easy to deduce that $0<A_p^*<\infty$ by Hardy-Littlewood-Sobolev inequality (see below Lemma 2.1).

Theorem 1.5 *Suppose that the **Case (c)** and conditions* $(V_1)-(V_2)$ *and* $(A_1)-(A_2)$ *hold. In addition, we assume that* $0<a<a_*(p)$*,* $0<\lambda<\lambda_1(f_\Omega)$ *and the following condition hold:*

(V_3) *There exist two positive constants $C_*, R_* > 0$ such that*

$$|x|^{\frac{N+\alpha-4-(N-4)p}{2}} Q(x) \leq C_* V(x)^{\frac{2-p}{2}} \quad \text{for all} |x| > R_*.$$

Then Eq. $(K^{\mu}_{a,\lambda})$ admits at least two positive solutions $u^{(1)}$ and $u^{(2)}$ satisfying $I^{\mu}_{a,\lambda}(u^{(2)}) < 0 < I^{\mu}_{a,\lambda}(u^{(1)})$ for $\mu > 0$ sufficiently large.

Theorem 1.6 *Suppose that the **Case (c)** and conditions $(V_1) - (V_2)$ and $(A_1) - (A_2)$ hold. Then for each $a > 0$ and $\lambda \geq \lambda_1(f_\Omega)$, Eq. $(K^{\mu}_{a,\lambda})$ admits at least a positive solution $\overline{u}$ satisfying $I^{\mu}_{a,\lambda}(\overline{u}) < 0$ for $\mu > 0$ sufficiently large.*

Denote

$$\lambda_{1,\mu}(f) = \inf \left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \mu V(x) u^2 \right) dx | u \in E, \int_{\mathbb{R}^N} f(x) u^2 dx = 1 \right\}.$$

By using condition (A_2) and Hölder inequality, we get

$$\frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + \mu V(x) u^2) dx}{\int_{\mathbb{R}^N} f(x) u^2 dx} \geq \frac{\|\nabla u\|^2_{L^2}}{\|f\|_{L^{N/2}} \mathcal{S}^{-2} \|\nabla u\|^2_{L^2}} > 0,$$

this implies that $\lambda_{1,\mu}(f) \geq \|f\|^{-1}_{L^{N/2}} \mathcal{S}^2 > 0$. Moreover, by condition (V_2), one has

$$\inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + \mu V(x) u^2) dx}{\int_{\mathbb{R}^N} f(x) u^2 dx} \leq \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega f_\Omega(x) u^2 dx},$$

which indicates that $\lambda_{1,\mu}(f) \leq \lambda_1(f_\Omega)$ for all $\mu > 0$. By Lemma 2.6, for each $0 < \lambda < \lambda_1(f_\Omega)$, there exists $\overline{\mu_*}(\lambda)$ with $\overline{\mu_*}(\lambda) \to \infty$ as $\lambda \to \lambda_1(f_\Omega)$ such that for every $\mu \geq \overline{\mu_*}(\lambda)$, there holds $0 < \lambda < \lambda_{1,\mu}(f) < \lambda_1(f_\Omega)$. Thus, for $0 < \lambda < \lambda_1(f_\Omega)$, we can set

$$\overline{a}_{**}(p) := \frac{p-1}{2(2-p)} \left(\frac{2p\beta(2-p)}{p(p-1)} \right)^{\frac{1}{p-1}} > 0,$$

and

$$\widehat{a}_{**}(p, \lambda) := \frac{(p-1)^2 (\lambda_{1,\mu}(f) - \lambda)^2 (2-p)^{(2-p)/(p-1)}}{4\beta \lambda^2_{1,\mu}(f) p^{p/(p-1)}} > 0,$$

where $\beta > 0$ is the energy level of ground state solution for the following Hartree type equation:

$$\begin{cases} -\Delta u = \lambda f_\Omega u + (I_\alpha * Q_\Omega |u|^p) Q_\Omega |u|^{p-2} u & \text{in } \Omega, \\ u \in H^1_0(\Omega). \end{cases}$$

Theorem 1.7 *Suppose that the **Case (d)** and conditions $(V_1) - (V_2)$ and $(A_1) - (A_2)$ hold. Then the following statements are true:*
(i) For each $0 < a < a_{**}(p) := \min \{\overline{a}_{**}(p), \widehat{a}_{**}(p, \lambda)\}$ and $0 < \lambda < \lambda_1(f_\Omega)$, Eq. $(K^{\mu}_{a,\lambda})$ admits at least two positive solutions $u^{(1)}$ and $u^{(2)}$ satisfying $I^{\mu}_{a,\lambda}(u^{(2)}) < 0 < I^{\mu}_{a,\lambda}(u^{(1)}) < \frac{1}{2} \left(\frac{1}{2-p} \right)^{\frac{1}{p-1}} \beta$ for $\mu > 0$ sufficiently large;
(ii) For each $a > 0$ and $\lambda \geq \lambda_1(f_\Omega)$, Eq. $(K^{\mu}_{a,\lambda})$ admits at least a positive solution $\overline{u}$ satisfying $I^{\mu}_{a,\lambda}(\overline{u}) < 0$ for $\mu > 0$ sufficiently large.

Theorem 1.8 *Suppose that the **Case (e)** and conditions $(V_1)-(V_2)$ and $(A_2)-(A_3)$ hold. Then for each $a > 0$ and $0 < \lambda < \lambda_1(f_\Omega)$, Eq. $(K^{\mu}_{a,\lambda})$ admits at least a positive solution $\overline{u}$ satisfying $I^{\mu}_{a,\lambda}(\overline{u}) < 0$ for $\mu > 0$ sufficiently large.*

The following table sums up the main results of present paper:

Cases \ λ	$0 < \lambda < \lambda_1(f_\Omega)$	$\lambda = \lambda_1(f_\Omega)$	$\lambda_1(f_\Omega) < \lambda < \lambda_1(f_\Omega) + \delta$
Case (a)	one solution	one solution	two solutions
Case (b)	one solution $(0 < a < \Gamma_*)$	one solution $(a > \Gamma_*)$	one solution $(a > \Gamma_*)$ two solutions $(\lambda_1^{-2}(f_\Omega) \int_\Omega (I_\alpha * Q\lvert\phi_1\rvert^2) Q\lvert\phi_1\rvert^2 dx < a < \Gamma^*)$
Case (c)	two solutions $(0 < a < a_*(p))$	one solution	one solution
Case (d)	two solutions $(0 < a < a_{**}(p))$	one solution	one solution
Case (e)	one solution	-	-

In the above table, we assume that "one solution" (respectively "two solutions") means that there exists at least one positive solution (respectively two positive solutions) of Eq. $(K^{\mu}_{a,\lambda})$.

It is worth pointing that the Eq. $(K^{\mu}_{a,\lambda})$ has two nonlocal terms, this brings some mathematical difficulties. Now, we give some brief strategy for the proof of the above Theorems. The compactness condition is a difficult issue here, since the equation is considered in the whole space $\mathbb{R}^N$ and the Sobolev embedding is not compact any more. To overcome this difficulty we apply a potential well method and concentration compactness principle. But this leads to another difficulty, the first eigenvalue of problem $-\Delta u + \mu V(x)u = \lambda f(x)u$ is less than $\lambda_1(f_\Omega)$, thus it is difficult to judge the linear part is coercive, even if in the case of $0 < \lambda < \lambda_1(f_\Omega)$. Here, following the ideas in [36], we prove our main results by using an approximation estimate of first eigenvalue.

For the **Cases (d)**, we need to face more challenges. Because the standard method of getting bounded (PS) sequence is not applicable. The standard Nehari manifold method does not work as well, since the energy functional is not bounded below on the Nehari manifold. In addition, we are more interested in multiplicity results and branch phenomena. Hence, some new ideas and estimates are proposed. To overcome this obstacle, we shall construct a new constraint manifold proposed in [29, 30]. That is, we introduce the filtration of Nehari manifold $\mathbf{N}_{\mu,\lambda}$, a sub-level set on $\mathbf{N}_{\mu,\lambda}$:

$$\mathbf{N}_{\mu,\lambda}(c) = \left\{ u \in \mathbf{N}_{\mu,\lambda} : I^{\mu}_{a,\lambda}(u) < c \right\} \text{ for some } c > 0.$$

Under some suitable assumptions, it can be shown that

$$\mathbf{N}_{\mu,\lambda}(c) = \mathbf{N}^{(1)}_{\mu,\lambda}(c) \cup \mathbf{N}^{(2)}_{\mu,\lambda}(c),$$

where

$$\mathbf{N}^{(1)}_{\mu,\lambda}(c) = \left\{ u \in \mathbf{N}_{\mu,\lambda}(c) \ \middle|\ \|u\|_{\mu} < A_1 \right\} \text{ and } \mathbf{N}^{(2)}_{\mu,\lambda}(c) = \left\{ u \in \mathbf{N}_{\mu,\lambda}(c) \ \middle|\ \|u\|_{\mu} > A_2 \right\}$$

for $0 < A_1 < A_2$. The key of the filtration of Nehari manifold is to find a suitable energy level c, then it can be decomposed into the above two submanifolds $\mathbf{N}_{\mu,\lambda}^{(1)}$ and $\mathbf{N}_{\mu,\lambda}^{(2)}$. For different problem, the selected energy level is different. By some detailed estimates and analysis, for Eq. $(K_{a,\lambda}^{\mu})$, the suitable energy level c is given in present paper. In addition, we can illustrate that each local minimizer of the functional $I_{a,\lambda}^{\mu}(u)$ restricted on $\mathbf{N}_{\mu,\lambda}^{(1)}(c)$ and $\mathbf{N}_{\mu,\lambda}^{(2)}(c)$ is a critical point of $I_{a,\lambda}^{\mu}(u)$ in $H^1(\mathbb{R}^N)$. Hence, we can find two critical points of the functional $I_{a,\lambda}^{\mu}(u)$.

The remainder of this paper is organized as follows. After presenting some preliminary results in Section 2, we prove Theorem 1.1 and Theorem 1.2 in Section 3. We give the proof of Theorems 1.3-1.4 in Section 4, and Theorems 1.5-1.8 in Section 5, respectively.

2 Preliminaries

Let

$$E = \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \right\}$$

associated the inner product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x) uv) dx, \quad \|u\| = \langle u, u \rangle^{1/2}.$$

For $\mu > 0$, we also need the following inner product and norm

$$\langle u, v \rangle_{\mu} = \int_{\mathbb{R}^N} (\nabla u \nabla v + \mu V(x) uv) dx, \quad \|u\|_{\mu} = \langle u, u \rangle_{\lambda}^{1/2}.$$

It is clear that $\|u\| \leq \|u\|_{\mu}$ for $\lambda \geq 1$. Now we set $E_{\mu} = (E, \|u\|_{\mu})$. By conditions $(V_1) - (V_2)$, the Hölder and Sobolev inequalities, we have

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 \right) dx \leq \max\left\{ 1 + |\{V < b\}|^{\frac{2}{N}} \mathcal{S}^{-2}, \frac{1}{b} \right\} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx,$$

where $\mathcal{S}$ is the best constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$. This implies that the imbedding $E \hookrightarrow H^1(\mathbb{R}^N)$ is continuous. Moreover, using the conditions $(V_1) - (V_2)$, the Hölder and Sobolev inequalities again, we have for any $r \in (2, 2^*)$,

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^r dx &\leq \left(\int_{\{V<b\}} u^2 dx + \int_{\{V \geq b\}} u^2 dx \right)^{\frac{2^*-r}{2^*-2}} \left(\mathcal{S}^{-2^*} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{2^*}{2}} \right)^{\frac{r-2}{2^*-2}} \\
&\leq \left(\frac{1}{\mu b} \int_{\mathbb{R}^N} \mu V(x) u^2 + |\{V < b\}|^{\frac{2}{N}} \mathcal{S}^{-2} \int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{2^*-r}{2^*-2}} \left(\mathcal{S}^{-2^*} \|u\|_{\mu}^{2^*} \right)^{\frac{r-2}{2^*-2}} \\
&\leq |\{V < b\}|^{\frac{2^*-r}{2^*}} \mathcal{S}^{-r} \|u\|_{\mu}^{r} \quad \text{for} \quad \mu \geq \mu_* := b^{-1} \mathcal{S}^2 |\{V < b\}|^{-2/N}. \qquad (7)
\end{aligned}$$

We give the classical Hardy-Littlewood-Sobolev inequality will be frequently used.

Lemma 2.1 *[16] (Hardy-Littlewood-Sobolev inequality) Let $s,r>1$ and $0<\alpha<N$ with $\frac{1}{s}+\frac{N-\alpha}{N}+\frac{1}{r}=2$. For $u\in L^s(\mathbb{R}^N)$ and $v\in L^r(\mathbb{R}^N)$,there exists a sharp constant $C\left(N,\alpha,s\right)>0$, independent of u,v, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u\left(x\right)v\left(y\right)}{\left|x-y\right|^{N-\alpha}}dydx\leq C\left(N,\alpha,s\right)\left(\int_{\mathbb{R}^N}\left|u\right|^s dx\right)^{1/s}\left(\int_{\mathbb{R}^N}\left|v\right|^r dx\right)^{1/r}.$$

Remark 2.2 *By the above inequality and (7), for $p\in(2_\alpha,2_\alpha^*)$ and $Q\in L^\infty(\mathbb{R}^N)$, there exists a best constant $\mathcal{C}_{HLS}:=C(N,\alpha,\frac{2N}{N+\alpha})>0$ such that*

$$\begin{aligned}\int_{\mathbb{R}^N}\left(I_\alpha*Q\left|u\right|^p\right)Q\left|u\right|^p dx &\leq \mathcal{C}_{HLS}\left\|Q\right\|_{L^\infty}^2\left(\int_{\mathbb{R}^N}\left|u\right|^{\frac{2Np}{N+\alpha}}\right)^{\frac{N+\alpha}{N}}\\ &\leq \mathcal{C}_{HLS}\left\|Q\right\|_{L^\infty}^2\left|\{V<b\}\right|^{\frac{N+\alpha-p(N-2)}{N}}\mathcal{S}^{-2p}\left\|u\right\|_\mu^{2p}\ \textit{for all}\quad \mu\geq\mu_*.\end{aligned}\tag{8}$$

It is well-known that Eq. $(K_{a,\lambda}^\mu)$ is variational and its solutions are the critical points of the energy functional $I_{a,\lambda}^\mu:E\to\mathbb{R}$ given by

$$I_{a,\lambda}^\mu\left(u\right)=\frac{1}{2}\|u\|_\mu^2+\frac{a}{4}\|u\|_{L^2}^2-\frac{1}{2p}\int_{\mathbb{R}^N}\left(I_\alpha*Q\left|u\right|^p\right)Q\left|u\right|^p dx-\int_{\mathbb{R}^N}f(x)u^2dx.$$

For $p\in(2_\alpha,2_\alpha^*)$ and $q\in(2,2^*)$, the Sobolev inequality and the Hardy-Littlewood-Sobolev inequality imply that the energy functional $I_{a,\lambda}^\mu\in C^1\left(E,\mathbb{R}\right)$ whose Fréchet derivative is

$$\begin{aligned}\left\langle (I_{a,\lambda}^\mu)'\left(u\right),\varphi\right\rangle &= \left(1+a\int_{\mathbb{R}^N}\left|\nabla u\right|^2\right)\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx+\int_{\mathbb{R}^N}\mu V(x)u\varphi dx\\ &\quad -\int_{\mathbb{R}^N}\left(I_\alpha*Q\left|u\right|^p\right)Q\left|u\right|^{p-2}u\varphi dx-\int_{\mathbb{R}^N}f(x)u\varphi dx\end{aligned}$$

for any $\varphi\in H^1(\mathbb{R}^N)$.

Set

$$B\left(u\right)=\int_{\mathbb{R}^N}\left(I_\alpha*Q|u|^p\right)Q\left|u\right|^p dx=A\left(N,\alpha\right)\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{Q(x)\left|u\left(x\right)\right|^pQ(y)\left|u\left(y\right)\right|^p}{\left|x-y\right|^{N-\alpha}}dxdy.$$

Next, we show a splitting property for the nonlocal term B, which is similar to the Brezis-Lieb type Lemma [1, 2].

Lemma 2.3 *Assume that $Q\in L^\infty(\mathbb{R}^N)$. Let u_n be a bounded sequence in $H^1(\mathbb{R}^N)$. If $u_n\to u$ a.e. in $\mathbb{R}^N$, then*

$$B\left(u_n-u\right)=B\left(u_n\right)-B\left(u\right)+o\left(1\right).$$

Lemma 2.4 *Suppose that $Q\in L^\infty(\mathbb{R}^N)$, $2_\alpha<p<2_\alpha^*$ and $2<q<2^*$. Let $\{u_n\}\subset H^1(\mathbb{R}^N)$ be a sequence satisfying $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then for any $\varphi\in H^1(\mathbb{R}^N)$, there holds*

$$\int_{\mathbb{R}^N}\left(I_\alpha*Q\left|u_n\right|^p\right)Q\left|u_n\right|^{p-2}u_n\varphi dx\to\int_{\mathbb{R}^N}\left(I_\alpha*Q\left|u\right|^p\right)Q\left|u\right|^{p-2}u\varphi dx$$

as $n\to\infty$.

Proof. The proof can be found in [24], we omit it here. ■

Now, let us consider the following eigenvalue problem:

$$-\Delta u + \mu V(x)u = \lambda f(x)u, \quad \text{in } E. \tag{9}$$

In order to find the positive principal eigenvalue of Eq. (9), we need to solve the following minimization problem:

$$\lambda_{1,\mu}(f) = \inf\left\{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \mu V(x)u^2\right) dx | u \in E, \int_{\mathbb{R}^N} f(x)u^2 dx = 1\right\}.$$

Then we have the following results.

Lemma 2.5 *[36, Lemma 3.1] Let $\mu_n \to \infty$ as $n \to \infty$ and $\{v_n\} \subset E$ with $\|v_n\|_{\mu_n} \le M_0$ for some $M_0 > 0$. Then there exist a subsequence $\{v_n\}$ and $v_0 \in H_0^1(\Omega)$ such that $v_n \rightharpoonup v_0$ in E and $v_n \to v_0$ in $L^r(\mathbb{R}^N)$ for all $r \in [2, 2^*)$.*

Lemma 2.6 *[36, Lemma 3.2] For each $\mu \ge \mu_*$ there exists a positive function $\phi_{1,\mu} \in E$ with $\int_{\mathbb{R}^N} f\phi_{1,\mu}^2 dx = 1$ such that*

$$\lambda_{1,\mu}(f) = \int_{\mathbb{R}^N} |\nabla \phi_{1,\mu}|^2 + \mu V \phi_{1,\mu}^2 dx < \lambda_1(f_\Omega).$$

Moreover, $\lambda_{1,\mu}(f) \to \lambda_1^-(f_\Omega)$ and $\phi_{1,\mu} \to \phi_1$ as $\mu \to \infty$, where ϕ_1 is positive eigenvalue of problem (3).

Note that we can find the other positive eigenvalues of Eq. (9) by solving the following problem:

$$\lambda_{2,\mu}(f) = \inf\left\{\int_{\mathbb{R}^N} (|\nabla u|^2 + \mu V(x)u^2) dx | u \in E, \int_{\mathbb{R}^N} f(x)u^2 dx = 1 \, \text{and} \, \langle u, \phi_{1,\mu}\rangle_\mu = 0\right\}. \tag{10}$$

In order to solve problem (10), we need the following lemmas.

Lemma 2.7 *[34, Lemma 2.13] If $N \ge 3$ and $f(x) \in L^{N/2}(\mathbb{R}^N)$, the functional $u \mapsto \int_{\mathbb{R}^N} f(x)u^2 dx$ is weakly continuous on $H^1(\mathbb{R}^N)$.*

By Lemma 2.5 and Lemma 2.7, we can get the following result.

Lemma 2.8 *[32, Lemma 2.4] For each $\mu > 0$, there exists a function $\phi_{2,\mu} \in E$ with $\int_{\mathbb{R}^N} f(x)\phi_{2,\mu}^2 dx = 1$ and $\langle \phi_{2,\mu}, \phi_{1,\mu}\rangle_\lambda = 0$ such that*

$$\lambda_{2,\mu}(f) = \int_{\mathbb{R}^N} (|\nabla \phi_{2,\mu}|^2 + \mu V(x)\phi_{2,\mu}^2) dx.$$

Moreover, it holds that

$$\frac{\lambda_1(f_\Omega) + \lambda_2(f_\Omega)}{2} < \lambda_{2,\mu}(f) \quad \text{for } \mu \text{ sufficiently large.} \tag{11}$$

Let us recalling the well-known the mountain pass theorem as follows:

Theorem 2.9 (Mountain Pass Theorem) *Let X be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $\rho > 0$ be such that $\|e\| > \rho$, and*

$$b := \inf_{\|u\|=\rho} J(u) > J(0) \geq J(e).$$

If J satisfies the Palais-Smale condition at level $c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$ with

$$\Gamma := \{\gamma \in C([0,1], X) | \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of J and $c \geq b$.

In the end in this section, we give the following compactness proposition.

Proposition 2.10 *Suppose that the conditions $(V_1)-(V_2)$ and $(A_1)-(A_2)$ hold. Let $c \in \mathbb{R}$ and $\{u_n\}$ be a $(PS)_c$ sequence for energy functional $I^\mu_{a,\lambda}$. If there exists $\widehat{M_0} > 0$ such that $\|u_n\|_\mu < \widehat{M_0}$, then $I^\mu_{a,\lambda}$ satisfies $(PS)_c$–condition for μ sufficiently large, that is, $\{u_n\}$ strongly converges in E_μ up to subsequence for μ sufficiently large.*

Proof. Let $\{u_n\}$ be a $(PS)_c$–sequence for $I^\mu_{a,\lambda}$ and $\{u_n\}$ is bounded in E_μ. Then there exist a subsequence $\{u_n\}$ and u_0 in E_μ such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \text{ weakly in } E_\mu; \\ u_n &\to u_0 \text{ strongly in } L^r_{loc}(\mathbb{R}^N) \text{ for } 2 \leq r < 2^*; \\ u_n(x) &\to u_0(x) \text{ a.e. on } \mathbb{R}^N. \end{aligned}$$

Then by condition (A_2),

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f u_n^2 dx = \int_{\mathbb{R}^N} f u_0^2 dx. \tag{12}$$

Moreover, we obtain that

$$\|u_0\|_\mu \leq \liminf_{n\to\infty} \|u_n\|_\mu < \widehat{M_0}.$$

Now we prove that $u_n \to u_0$ strongly in E_μ. Let $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in E_μ and

$$\|v_n\|_\mu \leq 2\widehat{M_0} + o(1). \tag{13}$$

From Lemma 2.3 it follows that

$$\int_{\mathbb{R}^N} (I_\alpha * Q|v_n|^p) Q|v_n|^p dx = \int_{\mathbb{R}^N} (I_\alpha * Q|u_n|^p) Q|u_n|^p dx - \int_{\mathbb{R}^N} (I_\alpha * Q|u_0|^p) Q|u_0|^p dx + o(1).$$

Moreover, it follows from condition (V_2) that

$$\begin{aligned} \int_{\mathbb{R}^N} v_n^2 dx &= \int_{\{V\geq b\}} v_n^2 dx + \int_{\{V<b\}} v_n^2 dx \\ &\leq \frac{1}{\mu b} \int_{\mathbb{R}^N} \mu V(x) v_n^2 dx + \int_{\{V<b\}} v_n^2 dx \leq \frac{1}{\mu b} \|v_n\|_\mu^2 + o(1). \end{aligned}$$

Using this, together with Sobolev inequalities, for any $r \in (2, 2^*)$ we have

$$\int_{\mathbb{R}^N} |v_n|^r \, dx \leq \left(\frac{1}{\mu b}\right)^{\frac{(2^*-r)(N-2)}{4}} \mathcal{S}^{-\frac{N(r-2)}{2}} \|v_n\|_\mu^r + o(1). \tag{14}$$

By (8) and (14) one has

$$\int_{\mathbb{R}^N} (I_\alpha * Q |v_n|^p) Q |v_n|^p \, dx \leq \mathcal{C}_{HLS} \|Q\|_{L^\infty}^2 \mathcal{S}^{N+\alpha-Np} \left(\frac{1}{\mu b}\right)^{\frac{N+\alpha-(N-2)p}{2}} \|v_n\|_\lambda^{2p} + o(1). \tag{15}$$

Since the sequence $\{u_n\}$ is bounded in E_μ, there exists a constant $A > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to A \quad \text{as}\, n \to \infty.$$

Hence, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} o(1) = \left\langle (I_{a,\lambda}^\mu)'(u_n), \varphi \right\rangle \quad \to \quad & \int_{\mathbb{R}^N} \nabla u_0 \nabla \varphi + \int_{\mathbb{R}^N} \mu V(x) u_0 \varphi + aA \int_{\mathbb{R}^N} \nabla u_0 \nabla \varphi \\ & - \int_{\mathbb{R}^N} (I_\alpha * Q |u_0|^p) Q |u_0|^{p-2} u_0 \varphi dx - \int_{\mathbb{R}^N} f u_0 \varphi dx \quad \text{as}\, n \to \infty, \end{aligned}$$

which implies that

$$\|u_0\|_\mu^2 + aA \int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \int_{\mathbb{R}^N} (I_\alpha * Q |u_0|^p) Q |u_0|^p \, dx - \int_{\mathbb{R}^N} f u_0^2 dx = 0. \tag{16}$$

Thus, it follows from (12) $-$ (16) that

$$\begin{aligned} o(1) &= \|u_n\|_\mu^2 + a \left(\int_{\mathbb{R}^N} |\nabla u_n|^2\right)^2 - \int_{\mathbb{R}^N} (I_\alpha * Q |u_n|^p) Q |u_n|^p \, dx - \int_{\mathbb{R}^N} f u_n^2 dx \\ & \quad - \|u_0\|_\mu^2 - aA \int_{\mathbb{R}^N} |\nabla u_0|^2 + \int_{\mathbb{R}^N} (I_\alpha * Q |u_0|^p) Q |u_0|^p \, dx + \int_{\mathbb{R}^N} f u_0^2 dx \\ &= \|v_n\|_\mu^2 + a \int_{\mathbb{R}^N} |\nabla u_n|^2 \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 - \int_{\mathbb{R}^N} |\nabla u_0|^2\right) - \int_{\mathbb{R}^N} (I_\alpha * Q |v_n|^p) Q |v_n|^p \, dx \\ &= \|v_n\|_\mu^2 + a \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \int_{\mathbb{R}^N} (I_\alpha * Q |v_n|^p) Q |v_n|^p \, dx \\ &\geq \|v_n\|_\mu^2 - \int_{\mathbb{R}^N} (I_\alpha * Q |v_n|^p) Q |v_n|^p \, dx \\ &\geq \|v_n\|_\mu^2 \left[1 - \mathcal{C}_{HLS} \|Q\|_{L^\infty}^2 \mathcal{S}^{N+\alpha-Np} \left(\frac{1}{\mu b}\right)^{\frac{N+\alpha-(N-2)p}{2}} \left(2\widehat{M_0}\right)^{2p-2}\right] + o(1), \end{aligned}$$

which implies that $v_n \to 0$ strongly in E_μ for $\mu > 0$ sufficiently large. This completes the proof.
■

3 The proof of Theorem 1.1 and 1.2

Lemma 3.1 *Suppose that **Case (a)** and conditions* $(V_1)-(V_2)$, $(A_1)-(A_2)$ *hold. Then, for each* $a>0$, *there exists a positive number* $\delta_a>0$ *such that* $0<\lambda<\lambda_1(f_\Omega)+\delta_a$, *there exist* $\rho_{a,\lambda}>0$ *and* $e_0\in H_0^1(\Omega)$ *such that*

$$\|e_0\|_\mu>\rho_{a,\lambda}\ \textit{and}\ \inf_{\|u\|_\mu=\rho_{a,\lambda}} I^\mu_{a,\lambda}(u)>0>I^\mu_{a,\lambda}(e_0)$$

for μ *sufficiently large.*

Proof. Now, we need to separate the proof in two cases as follows.

Case (i): $0<\lambda<\lambda_1(f_\Omega)$. Since $\lambda_{1,\mu}(f)\to\lambda_1^-(f_\Omega)$ for μ sufficiently large, then we have

$$\lambda_{1,\mu}(f)\geq\frac{\lambda_1(f_\Omega)+\lambda}{2}\quad\text{for}\,\mu\,\text{sufficiently large},$$

and

$$\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{1,\mu}(f)}\right)\geq\frac{1}{2}\left(\frac{\lambda_1(f_\Omega)-\lambda}{\lambda_1(f_\Omega)+\lambda}\right)\quad\text{for}\,\mu\,\text{sufficiently large}.\tag{17}$$

By (8) and (17), one has

$$I^\mu_{a,\lambda}(u)\geq\frac{1}{2}\left(\frac{\lambda_1(f_\Omega)-\lambda}{\lambda_1(f_\Omega)+\lambda}\right)\|u\|_\mu^2+\frac{a}{4}\|\nabla u\|_{L^2}^4-\frac{\mathcal{C}_{HLS}\,\|Q\|^2_{L^\infty}\,|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}{2p\mathcal{S}^{2p}}\|u\|_\mu^{2p}\tag{18}$$

for μ sufficiently large. Let

$$\rho_\lambda=\left[\frac{1}{4}\left(\frac{\lambda_1(f_\Omega)-\lambda}{\lambda_1(f_\Omega)+\lambda}\right)\frac{2p\mathcal{S}^{2p}}{\mathcal{C}_{HLS}\,\|Q\|^2_{L^\infty}\,|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}\right]^{1/(2p-2)}>0.$$

Then for all $u\in E$ with $\|u\|_\mu=\rho_\lambda$, we have

$$I^\mu_{a,\lambda}(u)\geq\frac{1}{4}\left(\frac{\lambda_1(f_\Omega)-\lambda}{\lambda_1(f_\Omega)+\lambda}\right)\rho_\lambda^2>0,$$

which implies that $\inf_{\|u\|_\mu=\rho_\lambda}I^\mu_{a,\lambda}(u)>0$.

Case (ii): $\lambda\geq\lambda_1(f_\Omega)$. For each $u\in E$, by the orthogonal decomposition theorem, there exist $t\in\mathbb{R}$ and $\omega\in E$ with $\langle\omega,\phi_{1,\mu}\rangle_\mu=0$ such that $u=t\phi_{1,\mu}+\omega$. Clearly,

$$\|u\|_\mu^2=\lambda_{1,\mu}(f)t^2+\|\omega\|_\mu^2.\tag{19}$$

Moreover, we get

$$\lambda_{2,\mu}(f)\int_{\mathbb{R}^N}f(x)\omega^2dx\leq\|\omega\|_\mu^2\tag{20}$$

and

$$\lambda_{1,\mu}(f)\int_{\mathbb{R}^N}f(x)\phi_{1,\mu}\omega dx=\int_{\mathbb{R}^N}(\nabla\phi_{1,\mu}\nabla\omega+\mu V(x)\phi_{1,\mu}\omega)dx=0.\tag{21}$$

It follows from (19)-(21) that

$$\begin{aligned}
I_{a,\lambda}^{\mu}(u) &= \frac{1}{2}\left(\lambda_{1,\mu}(f)t^2+\|\omega\|_\mu^2\right)+\frac{a}{4}\|\nabla u\|_{L^2}^4-\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p dx\\
&\quad -\frac{\lambda}{2}\int_{\mathbb{R}^N}(t^2f(x)\phi_{1,\mu}^2+2tf(x)\phi_{1,\mu}\omega+f(x)\omega^2)dx\\
&\geq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{1,\mu}(f)}\right)\lambda_{1,\mu}(f)t^2+\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{2,\mu}(f)}\right)\|\omega\|_\mu^2+\frac{a}{4}\|\nabla u\|_{L^2}^4\\
&\quad -\frac{\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}{2p\mathcal{S}^{2p}}\|u\|_\mu^{2p}\\
&\geq \frac{a}{4}\|\nabla u\|_{L^2}^4-\frac{1}{2}\left(\frac{\lambda}{\lambda_{1,\mu}(f)}-1\right)\|u\|_\mu^2+\frac{\lambda}{2}\left(\frac{\lambda_{2,\mu}(f)-\lambda_{1,\mu}(f)}{\lambda_{1,\mu}(f)\lambda_{2,\mu}(f)}\right)\|\omega\|_\mu^2\\
&\quad -\frac{\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}{2p\mathcal{S}^{2p}}\|u\|_\mu^{2p}. \qquad (22)
\end{aligned}$$

Since $\lambda_{1,\mu}(f)<\lambda_1(f_\Omega)$, by (11), we obtain

$$\frac{\lambda}{2}\left(\frac{\lambda_{2,\mu}(f)-\lambda_{1,\mu}(f)}{\lambda_{1,\mu}(f)\lambda_{2,\mu}(f)}\right)\geq\frac{1}{2}\left(\frac{\lambda_{2,\mu}(f)-\lambda_{1,\mu}(f)}{\lambda_{2,\mu}(f)}\right)\geq\frac{\lambda_2(f_\Omega)-\lambda_1(f_\Omega)}{2(\lambda_1(f_\Omega)+\lambda_2(f_\Omega))}=:C_0 \qquad (23)$$

for μ sufficiently large. Moreover, since $\phi_{1,\mu}\to\phi_1$ in E as $\mu\to\infty$, we conclude that $\phi_{1,\mu}\to\phi_1$ in $D^{1,2}(\mathbb{R}^N)$ as $\mu\to\infty$, which implies that

$$\|\nabla\phi_{1,\mu}\|_{L^2}^4\geq\frac{1}{2}\lambda_1^2(f_\Omega)$$

for μ sufficiently large. By the similar arguments in [32], we have

$$\|\nabla u\|_{L^2}^4\geq\frac{\|\nabla\phi_{1,\mu}\|_{L^2}^4}{4\lambda_{1,\mu}^2(f)}\left(\|u\|_\mu^2-\|\omega\|_\mu^2\right)^2-17\|\nabla\omega\|_{L^2}^4\geq\frac{1}{16}\|u\|_\mu^4-\frac{137}{8}\|\omega\|_\mu^4, \qquad (24)$$

and

$$\begin{aligned}
I_{a,\lambda}^{\mu}(u) &\geq \frac{a}{64}\|u\|_\mu^4-\frac{137a}{32}\|\omega\|_\mu^4-\frac{1}{2}\left(\frac{\lambda}{\lambda_{1,\mu}(f)}-1\right)\|u\|_\mu^2+C_0\|\omega\|_\mu^2\\
&\quad -\frac{\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}{2p\mathcal{S}^{2p}}\|u\|_\mu^{2p}\\
&= -\frac{1}{2}\left(\frac{\lambda}{\lambda_{1,\mu}(f)}-1\right)\|u\|_\mu^2+\left(C_0-\frac{137a}{32}\right)\|\omega\|_\mu^4\\
&\quad +\|u\|_\mu^4\left(\frac{a}{64}-\frac{\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}{2p\mathcal{S}^{2p}}\|u\|_\mu^{2p-4}\right).
\end{aligned}$$

This implies that there exists a number

$$\rho_a=\min\left\{\left(\frac{ap\mathcal{S}^{2p}}{64\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}\right)^{1/(2p-4)},\left(\frac{32C_0}{137a}\right)^{1/2}\right\} \qquad (25)$$

such that for all $u \in E$ with $\|u\|_\mu = \rho_a$,

$$I^\mu_{a,\lambda}(u) \geq -\frac{1}{2}\left(\frac{\lambda}{\lambda_{1,\mu}(f)} - 1\right)\|u\|^2_\mu + \frac{a}{128}\rho_a^4.$$

Thus, we deduce that

$$I^\mu_{a,\lambda}(u) \geq \frac{a}{256}\rho_a^4 > 0$$

for each $\lambda_{1,\mu}(f) \leq \lambda < \lambda_{1,\mu}(f) + \delta_{a,\mu}$, where

$$\delta_{a,\mu} := \frac{\lambda_{1,\mu}(f)}{128} a\rho_a^2.$$

Hence,for each $a > 0$ and $0 < \lambda < \lambda_1(f_\Omega) + \delta_{a,\mu}$, we have

$$\inf_{\|u\|_\mu = \rho_{a,\lambda}} I^\mu_{a,\lambda}(u) > 0$$

for μ sufficiently large, where

$$\rho_{a,\lambda} = \begin{cases} \rho_\lambda & \text{for} \quad 0 < \lambda < \lambda_1(f_\Omega), \\ \rho_a & \text{for} \quad \lambda_1(f_\Omega) \leq \lambda < \lambda_1(f_\Omega) + \delta_{a,\mu}. \end{cases}$$

Now, we show that there exists $e_0 \in H^1_0(\Omega)$ such that $\|e_0\|_\mu > \rho_{a,\lambda}$ and $I^\mu_{a,\lambda}(e_0) < 0$. Owing to condition (A_2), we can take $\varphi \in H^1_0(\Omega)$ such that $\int_{\mathbb{R}^N}(I_\alpha * Q|\varphi|^p)Q|\varphi|^p dx > 0$. Then for any $t > 0$, we have

$$I^\mu_{a,\lambda}(t\varphi) = \frac{t^2}{2}(\|\varphi\|^2_\mu - \int_{\mathbb{R}^N} f\varphi^2 dx) + \frac{at^4}{4}\|\nabla\varphi\|^4_{L^2} - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha * Q|\varphi|^p)Q|\varphi|^p dx.$$

This implies that there exists $t_0 > 0$ such that $\|e_0\|_\mu := \|t_0\varphi\|_\mu > \rho_{a,\lambda}$ and $I^\mu_{a,\lambda}(t_0\varphi) < 0$. Consequently, we complete the proof. ■

Now, we give the proof Theorem 1.1: By Lemma 3.1, for each $a > 0$ and μ sufficiently large, the functional $I^\mu_{a,\lambda}$ has the mountain pass geometry. Let

$$\beta_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I^\mu_{a,\lambda}(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], E) | \gamma(0) = 0, \gamma(1) = e_0\}.$$

Let $\{u_n\}$ be a $(PS)_{\beta_\mu}$ sequence, that is $I^\mu_{a,\lambda}(u_n) \to \beta_\mu$ and $(I^\mu_{a,\lambda})'(u_n) \to 0$ as $n \to \infty$. In fact, since

$$\begin{aligned} 2p\beta_\mu + 1 &\geq 2pI^\mu_{a,\lambda}(u_n) - \left\langle (I^\mu_{a,\lambda})'(u_n), u_n \right\rangle \\ &= (p-1)\|u_n\|^2_\mu + \frac{a(p-2)}{2}\|\nabla u_n\|^4_{L^2} - \lambda(p-1)\int_{\mathbb{R}^N} f(x)u_n^2 dx. \end{aligned}$$

Using condition (A_2) and Young's inequality gives

$$\begin{aligned} \lambda(p-1)\int_{\mathbb{R}^N} f(x)u_n^2 dx &\leq \lambda(p-1)\mathcal{S}^{-2}\|f\|_{L^{N/2}}\|\nabla u_n\|^2_{L^2} \\ &\leq \frac{a(p-2)}{2}\|\nabla u_n\|^4_{L^2} + \frac{\lambda^2(p-1)^2\|f\|^2_{L^{N/2}}}{2(p-2)a\mathcal{S}^4}. \end{aligned}$$

It follows that

$$2p\beta_\mu + 1 \geq (p-1)\|u_n\|_\mu^2 - \frac{\lambda^2(p-1)^2\|f\|_{L^{N/2}}^2}{2(p-2)a\mathcal{S}^4},$$

which indicates that there exists C such that $\|u_n\|_\mu \leq C$ for μ sufficiently large. Thus, by Proposition 2.10, the functional $I_{a,\lambda}^\mu$ satisfies the $(PS)_{\beta_\mu}$-condition. Hence, there exists $0 \leq \overline{u} \in E$ such that $I_{a,\lambda}^\mu(\overline{u}) = \beta_\mu$ and $(I_{a,\lambda}^\mu)'(\overline{u}) = 0$ for μ sufficiently large, this implies that $\overline{u}$ is a nontrivial nonnegative solution of Eq. $(K_{a,\lambda}^\mu)$. The strong maximum principle implies that $\overline{u} > 0$. Therefore, the proof of Theorem 1.1 is completed.

Now, we give the proof Theorem 1.2: By Lemma 3.1, for each $a > 0$ and μ sufficiently large, there exists $\delta_a > 0$ such that the functional $I_{a,\lambda}^\mu$ has the mountain pass geometry for $\lambda_1(f_\Omega) < \lambda < \lambda_1(f_\Omega) + \delta_a$. By similar arguments of Theorem 1.1, there exists a positive solution $u^{(1)}$ for Eq. $(K_{a,\lambda}^\mu)$. Next, we consider the infimum of $I_{a,\lambda}^\mu$ on the ball $B_{\rho_{a,\lambda}} := \{u \in E | \|u\|_\mu \leq \rho_{a,\lambda}\}$ with $\rho_{a,\lambda}$ being as in Lemma 3.1. Set

$$\overline{\beta_\mu} = \inf_{\|u\|_\mu \leq \rho_{a,\lambda}} I_{a,\lambda}^\mu(u).$$

Let

$$I_{a,\lambda}^\mu(t\phi_1) = -\frac{\lambda - \lambda_1(f_\Omega)}{2}t^2 + \frac{a\lambda_1^2(f_\Omega)}{4}t^4 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha * Q|\phi_1|^p)Q|\phi_1|^p dx$$

for $t > 0$. Then for each $\lambda > \lambda_1(f_\Omega)$, there exists $t_0 > 0$ such that $\|t_0\phi_1\|_\mu \leq \rho_{a,\lambda}$ and $I_{a,\lambda}^\mu(t_0\phi_1) < 0$. Moreover, we have

$$\begin{aligned} I_{a,\lambda}^\mu(u) &\geq -\frac{\lambda\|f\|_{L^{N/2}}}{2\mathcal{S}^2}\|u\|_\mu^2 - \frac{\mathcal{C}_{HLS}\,\|Q\|_{L^\infty}^2\,|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}{2p\mathcal{S}^{2p}}\|u\|_\mu^{2p} \\ &\geq -\frac{\lambda\|f\|_{L^{N/2}}}{2\mathcal{S}^2}\rho_{a,\lambda}^2 - \frac{\mathcal{C}_{HLS}\,\|Q\|_{L^\infty}^2\,|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}{2p\mathcal{S}^{2p}}\rho_{a,\lambda}^{2p}, \end{aligned}$$

which implies that $-\infty < \overline{\beta_\mu} < 0$. By the Ekeland variational principle [9] and $I_{a,\lambda}^\mu(u_n) = I_{a,\lambda}^\mu(|u_n|)$, there exists a $(PS)_{\overline{\beta_\mu}}$ sequence $\{u_n\} \subset B_{\rho_{a,\lambda}}$. Thus, by Proposition 2.10, the functional $I_{a,\lambda}^\mu$ satisfies the $(PS)_{\beta_\mu}$-condition. Hence, there existss $0 \leq u^{(2)} \in E$ such that $I_{a,\lambda}^\mu(u^{(2)}) = \beta_\mu$ and $(I_{a,\lambda}^\mu)'(u^{(2)}) = 0$ for μ sufficiently large, this implies that $u^{(2)}$ is a nontrivial nonnegative solution of Eq. $(K_{a,\lambda}^\mu)$. The strong maximum principle implies that $u^{(2)} > 0$. Therefore, the proof of Theorem 1.2 is completed.

4 The proof of Theorem 1.3 and 1.4

First of all, we define the Nehari manifold

$$\mathbf{N}_{\mu,\lambda} = \left\{u \in E_\mu \backslash \{0\} \;\middle|\; \left\langle (I_{a,\lambda}^\mu)'(u), u\right\rangle = 0\right\}.$$

Then, $u \in \mathbf{N}_{\mu,\lambda}$ if and only if

$$\|u\|_\mu^2 + a\|\nabla u\|_{L^2}^4 - \int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)\,Q\,|u|^p\,dx - \lambda\int_{\mathbb{R}^N} fu^2 dx = 0.$$

Note that the Nehari manifold $\mathbf{N}_{\mu,\lambda}$ is closed linked to the behavior of the function of the form $h_u : t \to I^{\mu}_{a,\lambda}(tu)$ given by

$$h_u(t) = I^{\mu}_{a,\lambda}(tu) = \frac{1}{2}\|tu\|^2_{\mu} + \frac{a}{4}\|\nabla(tu)\|^4_{L^2} - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * Q|tu|^p)Q|tu|^p\,dx - \lambda\int_{\mathbb{R}^N} f(tu)^2dx$$

for $t > 0$. Then, we have

$$\begin{aligned} h'_u(t) &= t\left(\|u\|^2_{\mu} - \lambda\int_{\mathbb{R}^N} fu^2dx\right) + at^3\|\nabla u\|^4_{L^2} - t^{2p-1}\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p\,dx; \\ h''_u(t) &= \|u\|^2_{\mu} - \lambda\int_{\mathbb{R}^N} fu^2dx + 3at^2\|\nabla u\|^4_{L^2} - (2p-1)t^{2p-2}\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p\,dx. \end{aligned}$$

It is easy to see that

$$th'_u(t) = \|tu\|^2_{\mu} + a\|\nabla(tu)\|^4_{L^2} - \int_{\mathbb{R}^N}(I_\alpha * Q|tu|^p)Q|tu|^p\,dx - \lambda\int_{\mathbb{R}^N} f(tu)^2dx,$$

which implies that for $u \in E\backslash\{0\}$ and $t > 0$, $h'_u(t) = 0$ if and only if $tu \in \mathbf{N}_{\mu,\lambda}$. In particular, $h'_u(1) = 0$ if and only if $u \in \mathbf{N}_{\mu,\lambda}$. Thus, it is natural split $\mathbf{N}_{\mu,\lambda}$ into three parts corresponding to local minima, local maxima and points of inflection. Following [33], we define

$$\mathbf{N}^+_{\mu,\lambda} = \{u \in \mathbf{N}_{\mu,\lambda} \mid h''_u(1) > 0\};\mathbf{N}^0_{\mu,\lambda} = \{u \in \mathbf{N}_{\mu,\lambda} \mid h''_u(1) = 0\};\mathbf{N}^-_{\mu,\lambda} = \{u \in \mathbf{N}_{\mu,\lambda} \mid h''_u(1) < 0\}.$$

For each $u \in \mathbf{N}_{\mu,\lambda}$, there holds

$$\begin{aligned} h''_u(1) &= \|u\|^2_{\mu} + 3a\|\nabla u\|^4_{L^2} - (2p-1)\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p\,dx - \lambda\int_{\mathbb{R}^N} fu^2dx \\ &= -2\left(\|u\|^2_{\mu} - \lambda\int_{\mathbb{R}^N} fu^2dx\right) + 2(2-p)\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p\,dx \qquad (26) \\ &= -2(p-1)\left(\|u\|^2_{\mu} - \lambda\int_{\mathbb{R}^N} fu^2dx\right) + 2a(2-p)\|\nabla u\|^4_{L^2}. \qquad (27) \end{aligned}$$

Define

$$\psi_{\mu,\lambda}(u) = \left\langle (I^{\mu}_{a,\lambda})'(u), u\right\rangle = \|u\|^2_{\mu} + a\|\nabla u\|^4_{L^2} - \int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p\,dx - \lambda\int_{\mathbb{R}^N} fu^2dx.$$

Then for $u \in \mathbf{N}_{\mu,\lambda}$ we have

$$\left\langle \psi'_{\mu,\lambda}(u), u\right\rangle = \|u\|^2_{\mu} + 3a\|\nabla u\|^4_{L^2} - (2p-1)\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p\,dx - \lambda\int_{\mathbb{R}^N} fu^2dx = h''_u(1).$$

Next, we define

$$\begin{aligned} \Sigma^+_{\mu,\lambda} &= \left\{u \in E \mid \|u\|_{\mu} = 1, \|u\|^2_{\mu} - \lambda\int_{\mathbb{R}^N} fu^2dx > 0\right\}; \\ \Sigma^0_{\mu,\lambda} &= \left\{u \in E \mid \|u\|_{\mu} = 1, \|u\|^2_{\mu} - \lambda\int_{\mathbb{R}^N} fu^2dx = 0\right\}; \\ \Sigma^-_{\mu,\lambda} &= \left\{u \in E \mid \|u\|_{\mu} = 1, \|u\|^2_{\mu} - \lambda\int_{\mathbb{R}^N} fu^2dx < 0\right\}, \end{aligned}$$

and

$$\begin{aligned}\Psi_{\mu,\lambda}^{+} &= \{u \in E \mid \|u\|_\mu = 1, \Phi_p(u) > 0\}\,; \\ \Psi_{\mu,\lambda}^{0} &= \{u \in E \mid \|u\|_\mu = 1, \Phi_p(u) = 0\}\,; \\ \Psi_{\mu,\lambda}^{-} &= \{u \in E \mid \|u\|_\mu = 1, \Phi_p(u) < 0\}\,,\end{aligned}$$

where

$$\Phi_p(u) = \int_{\mathbb{R}^N} (I_\alpha * Q|u|^p)Q|u|^p - a\|\nabla u\|_{L^2}^4. \tag{28}$$

Lemma 4.1 *Suppose that the **Case (b)** holds. If $u \in E_\mu \backslash \{0\}$, then*
(i) a multiple of u lies in $\mathbf{N}_{\mu,\lambda}^{-}$ if and only if $\frac{u}{\|u\|_\mu}$ lies in $\Sigma_{\mu,\lambda}^{+} \cap \Psi_{\mu,\lambda}^{+}$;
(ii) a multiple of u lies in $\mathbf{N}_{\mu,\lambda}^{+}$ if and only if $\frac{u}{\|u\|_\mu}$ lies in $\Sigma_{\mu,\lambda}^{-} \cap \Psi_{\mu,\lambda}^{-}$;
(iii) when $\overline{\Sigma_{\mu,\lambda}^{+}} \cap \overline{\Psi_{\mu,\lambda}^{-}}$ or $\overline{\Sigma_{\mu,\lambda}^{-}} \cap \overline{\Psi_{\mu,\lambda}^{+}}$, no multiple of u lies in $\mathbf{N}_{\mu,\lambda}$.

Furthermore, similar to the argument in Brown-Zhang [5, Theorem 2.3], we can conclude the following result.

Lemma 4.2 *Suppose that u_0 is a local minimizer for $I_{a,\lambda}^{\mu}$ on $\mathbf{N}_{\mu,\lambda}$ and that $u_0 \notin \mathbf{N}_{\mu,\lambda}^{0}$. Then $(I_{a,\lambda}^{\mu})'(u_0) = 0$ in E_μ^{-1}.*

By Lemma 2.6, for each $0 < \lambda < \lambda_1(f_\Omega)$, there exists $\overline{\mu_*}(\lambda) \geq \mu_*$ with $\overline{\mu_*}(\lambda) \to \infty$ as $\lambda \to \lambda_1(f_\Omega)$ such that for every $\mu \geq \overline{\mu_*}(\lambda)$, there holds $0 < \lambda < \lambda_{1,\mu}(f) < \lambda_1(f_\Omega)$, which indicates that

$$\|u\|_\mu^2 - \lambda \int_{\mathbb{R}^N} fu^2 dx \geq \frac{\lambda_{1,\mu}(f) - \lambda}{\lambda_{1,\mu}(f)} \|u\|_\mu^2 > 0 \tag{29}$$

for all $u \in E_\mu$. Moreover, it is easy to show that

$$h_u''(1) = -2\,(p-1)\left(\|u\|_\mu^2 - \lambda \int_{\mathbb{R}^N} fu^2 dx\right) + 2a\,(2-p)\,\|\nabla u\|_{L^2}^4 < 0 \tag{30}$$

for $u \in \mathbf{N}_{\mu,\lambda}$. Furthermore, we have the following results.

Lemma 4.3 *Suppose that the **Case (b)** holds and $\Gamma_* = \infty$. Then for each $a > 0$ and $0 < \lambda < \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^{-}$ and*

$$\mathbf{N}_{\mu,\lambda}^{-} = \left\{t_{max}(u)u : u \in \Psi_{\mu,\lambda}^{+}\right\}$$

for $\mu > 0$ sufficiently large.

Proof. By (29), we have $\Sigma_{\mu,\lambda}^{+} \neq \emptyset$ and $\Sigma_{\mu,\lambda}^{+} \cup \Sigma_{\mu,\lambda}^{0} = \emptyset$. This implies that the submanifolds $\mathbf{N}_{\mu,\lambda}^{+}$ and $\mathbf{N}_{\mu,\lambda}^{0}$ are empty and

$$\mathbf{N}_{\mu,\lambda}^{-} = \left\{t_{max}(u)u : u \in \Psi_{\mu,\lambda}^{+}\right\}$$

for $\mu > 0$ sufficiently large. ■

Lemma 4.4 *Suppose that the **Case (b)** holds and $\Gamma_* < \infty$. Then we have the following results.*
(i) For each $0 < a < \Gamma_*$ and $0 < \lambda < \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^-$ and

$$\mathbf{N}_{\mu,\lambda}^- = \left\{ t_{max}(u)u : u \in \Psi_{\mu,\lambda}^+ \right\}$$

for $\mu > 0$ sufficiently large;
(ii) For each $a \geq \Gamma_*$ and $0 < \lambda < \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \emptyset$ for $\mu > 0$ sufficiently large;
(iii) For each $a > \Gamma_*$ and $\lambda \geq \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^+$ and

$$\mathbf{N}_{\mu,\lambda}^+ = \left\{ t_{min}(u)u : u \in \Psi_{\mu,\lambda}^+ \right\}$$

for $\mu > 0$ sufficiently large;
(iv) If Γ_* is not attained and $a = \Gamma_*$, then for each $\lambda \geq \lambda_1(f_\Omega)$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^+$ and

$$\mathbf{N}_{\mu,\lambda}^+ = \left\{ t_{min}(u)u : u \in \Psi_{\mu,\lambda}^+ \right\}$$

for $\mu > 0$ sufficiently large.

Lemma 4.5 *Suppose that the **Case (b)** holds. Then for each $0 < \lambda < \lambda_1(f_\Omega)$ there exists $\overline{\mu_*}(\lambda) \geq \mu_*$ with $\overline{\mu_*}(\lambda) \to \infty$ as $\lambda \to \lambda_1(f_\Omega)$ such that for every $\mu > \overline{\mu_*}(\lambda)$, then energy functional $I_{a,\lambda}^\mu$ is bounded below and coercive on $\mathbf{N}_{\mu,\lambda}^-$.*

Proof. By (29) and (8), for each $\mu > \overline{\mu_*}(\lambda)$ and $u \in \mathbf{N}_{\mu,\lambda}^-$, we get

$$\begin{aligned} \frac{\lambda_{1,\mu}(f) - \lambda}{\lambda_{1,\mu}(f)} \|u\|_\mu^2 &\leq \|u\|_\mu^2 - \lambda \int_{\mathbb{R}^N} f u^2 dx + a \|\nabla u\|_{L^2}^4 \\ &= \int_{\mathbb{R}^N} (I_\alpha * Q|u|^p) Q|u|^p dx \leq \mathcal{C}_{HLS} \|Q\|_{L^\infty}^2 |\{V < b\}|^{\frac{N+\alpha-p(N-2)}{N}} \mathcal{S}^{-2p} \|u\|_\mu^{2p}, \end{aligned}$$

which implies that

$$\|u\|_\mu \geq \left(\frac{(\lambda_{1,\mu}(f) - \lambda)\mathcal{S}^{2p}}{\lambda_{1,\mu}(f)\mathcal{C}_{HLS} \|Q\|_{L^\infty}^2 |\{V < b\}|^{\frac{N+\alpha-p(N-2)}{N}}} \right)^{\frac{1}{2(p-1)}}.$$

Thus, we have

$$\begin{aligned} I_{a,\lambda}^\mu(u) &\geq \frac{1}{4} \left(\|u\|_\mu^2 - \lambda \int_{\mathbb{R}^N} f u^2 dx \right) \geq \frac{\lambda_{1,\mu}(f) - \lambda}{4\lambda_{1,\mu}(f)} \|u\|_\mu^2 \\ &\geq \frac{\lambda_{1,\mu}(f) - \lambda}{4\lambda_{1,\mu}(f)} \left(\frac{(\lambda_{1,\mu}(f) - \lambda)\mathcal{S}^{2p}}{\lambda_{1,\mu}(f)\mathcal{C}_{HLS} \|Q\|_{L^\infty}^2 |\{V < b\}|^{\frac{N+\alpha-p(N-2)}{N}}} \right)^{\frac{1}{p-1}} > 0, \end{aligned}$$

this implies that the energy functional $I_{a,\lambda}^\mu$ is bounded below and coercive on $\mathbf{N}_{\mu,\lambda}^-$. ■

Now, we are ready to prove Theorem 1.3: (i) For $0 < \lambda < \lambda_1(f_\Omega)$, by Lemma 4.5 and Ekeland variational principle [9], for each $\mu > \overline{\mu_*}(\lambda)$ there exists a bounded minimizing sequence $\{u_n\} \subset \mathbf{N}_{\mu,\lambda}^-$ such that

$$\lim_{n\to\infty} I_{a,\lambda}^\mu(u_n) = \inf_{u\in\mathbf{N}_{\mu,\lambda}^-} I_{a,\lambda}^\mu(u) \text{ and } (I_{a,\lambda}^\mu)'(u_n) = o(1).$$

By Proposition 2.10, there exist a subsequence $\{u_n\}$ and $\overline{u}$ such that $(I^{\mu}_{a,\lambda})'(\overline{u}) = 0$ and $u_n \to \overline{u}$ strongly in E_μ for $\mu > 0$ sufficiently large, which implies that $I^{\mu}_{a,\lambda}$ has minimizer $\overline{u}$ in $\mathbf{N}_{\mu,\lambda} = \mathbf{N}^-_{\mu,\lambda}$ for μ sufficiently large. Since $I^{\mu}_{a,\lambda}(\overline{u}) = I^{\mu}_{a,\lambda}(|\overline{u}|)$, by Lemma 4.2, we may assume that $\overline{u}$ is a positive solution of Eq. $(K^{\mu}_{a,\lambda})$ such that $I^{\mu}_{a,\lambda}(\overline{u}) > 0$.

(ii) Since $\Gamma_* < \infty$, by Lemma 4.4 (ii) for each $a \geq \Gamma_*$ and $0 < \lambda < \lambda_1(f_\Omega)$, we have $\mathbf{N}_{\mu,\lambda} = \emptyset$ for μ sufficiently large, this implies that for each $a \geq \Gamma_*$ and $0 < \lambda < \lambda_1(f_\Omega)$, Eq. $(K^{\mu}_{a,\lambda})$ does not admit nontrivial solution.

(iii) Since $\Gamma_* < \infty$, by Lemma 4.4 (iii), for each $a > 0$ and $\lambda \geq \lambda_1(f_\Omega)$, we have $\mathbf{N}_{\mu,\lambda} = \mathbf{N}^+_{\mu,\lambda}$ and $\mathbf{N}^+_{\mu,\lambda} = \left\{t_{\min}(u)u : u \in \Psi^-_{\mu,\lambda}\right\}$ for $\mu > 0$ sufficiently large. Now, we will prove that $\mathbf{N}^+_{\mu,\lambda}$ is uniform bounded for $\mu > 0$ sufficiently large. Suppose on the contrary. Then there exist sequences $\{\mu_n\} \subset \mathbb{R}$ and $u_n \in \mathbf{N}^+_{\mu_n,\lambda}$ such that $\mu_n \to \infty$ and $\|u_n\|_{\mu_n} \to \infty$ as $n \to \infty$. Clearly,

$$\|u_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^N} f u_n^2 dx = \int_{\mathbb{R}^N} (I_\alpha * Q|u_n|^p)Q|u_n|^p dx - a\|\nabla u_n\|^4_{L^2} < 0. \tag{31}$$

Let $v_n := \frac{u_n}{\|u_n\|_{\mu_n}}$. By Lemma 2.5, we may assume that for every $\mu > 0$ there exists $v_0 \in H^1_0(\Omega)$ such that $v_n \rightharpoonup v_0$ in E_μ and $v_n \to v_0$ in $L^r(\mathbb{R}^N)$ for all $r \in [2, 2^*)$. Thus

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f v_n^2 dx = \int_{\mathbb{R}^N} f v_0^2 dx$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * Q|v_n|^p)Q|v_n|^p dx = \int_{\mathbb{R}^N} (I_\alpha * Q|v_0|^p)Q|v_0|^p dx.$$

Moreover, by Fatou's Lemma,

$$\int_{\mathbb{R}^N} |\nabla v_0|^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx.$$

Dividing (39) by $\|u_n\|_{\mu_n}$ gives

$$\|v_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^N} f v_n^2 dx = \|u_n\|^2_{\mu_n} \left(\int_{\mathbb{R}^N} (I_\alpha * Q|v_n|^p)Q|v_n|^p dx - a\|\nabla v_n\|^4_{L^2} \right) < 0.$$

Since

$$\lim_{n\to\infty} \left(\|v_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^N} f v_n^2 dx \right) = 1 - \lambda \lim_{n\to\infty} \int_{\mathbb{R}^N} f v_n^2 dx = 1 - \lambda \int_{\mathbb{R}^N} f v_0^2 dx$$

and $\|u_n\|_{\mu_n} \to \infty$, then we get

$$\int_{\mathbb{R}^N} (I_\alpha * Q|v_0|^p)Q|v_0|^p dx - a\|\nabla v_0\|^4_{L^2} \geq 0$$

and

$$\int_{\mathbb{R}^N} f v_0^2 dx > 0.$$

Moreover, for every $\mu > 0$,

$$\begin{aligned} \|v_0\|^2_\mu - \lambda \int_{\mathbb{R}^N} f v_0^2 dx &= \int_{\mathbb{R}^N} |\nabla v_0|^2 dx - \lambda \int_{\mathbb{R}^N} f v_0^2 dx \\ &\leq \liminf_{n\to\infty} \left(\|v_n\|^2_{\mu_n} - \lambda \int_{\mathbb{R}^N} f v_n^2 dx \right) \leq 0. \end{aligned}$$

We now show that $v_n \to v_0$ in E_μ. Suppose on the contrary. Then

$$\begin{aligned}\|v_0\|_\mu^2 - \lambda \int_{\mathbb{R}^N} f v_0^2 dx &= \int_{\mathbb{R}^N} |\nabla v_0|^2 dx - \lambda \int_{\mathbb{R}^N} f v_0^2 dx \\ &< \liminf_{n\to\infty} \left(\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^N} f v_n^2 dx \right) \le 0.\end{aligned}$$

since $\int_{\mathbb{R}^N} V(x) v_0^2 dx = 0$. Hence $\frac{v_0}{\|v_0\|_\mu} \in \overline{\Sigma_{\mu,\lambda}^-} \cap \overline{\Psi_{\mu,\lambda}^+}$ which is impossible. Since $v_n \to v_0$ in E_μ, then $\|v_0\|_\mu = 1$. Hence $v_0 \in \Psi_{\mu,\lambda}^0$ and so $v_0 \in \overline{\Psi_{\mu,\lambda}^+}$. Moreover,

$$\|v_0\|_\mu^2 - \lambda \int_{\mathbb{R}^N} f v_0^2 dx = \liminf_{n\to\infty} \left(\|v_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^N} f v_n^2 dx \right) \le 0,$$

and so $v_0 \in \overline{\Sigma_{\mu,\lambda}^-}$. Thus, $v_0 \in \overline{\Sigma_{\mu,\lambda}^-} \cap \overline{\Psi_{\mu,\lambda}^+}$ which is impossible. Therefore, we can deduce that $\mathbf{N}_{\mu,\lambda}^+$ is uniform bounded for $\mu > 0$ sufficiently large. Then there exists $C > 0$ such that $\|u\|_\mu \le C$ for all $u \in \mathbf{N}_{\mu,\lambda}^+$. By using (8), for $u \in \mathbf{N}_{\mu,\lambda}^+$, we have

$$I_{a,\lambda}^\mu(u) \ge -\frac{a}{4} \|\nabla u\|_{L^2}^4 + \frac{p-1}{2p} \int_{\mathbb{R}^N} (I_\alpha * Q|u_n|^p) Q|u_n|^p dx \ge -\frac{a}{4} C^4,$$

which implies that $I_{a,\lambda}^\mu(u)$ is bounded from below on $\mathbf{N}_{\mu,\lambda}^+$. Then similar to the argument of proof in Theorem 1.3 (i), $I_{a,\lambda}^\mu$ has minimizer $\widetilde{u}$ in $\mathbf{N}_{\mu,\lambda}^+$ for $\mu > 0$ sufficiently large such that $I_{a,\lambda}^\mu(\widetilde{u}) < 0$.

(iv) The proof is essentially same as that in part (iii), so we omit it here.

Lemma 4.6 *Suppose that the **Case (b)** holds and $\Phi_p(\phi_1) < 0$. Then for each $a > 0$ there exists $\widehat{\delta} > 0$ such that for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \widehat{\delta}$, there holds $\overline{\Sigma_{\mu,\lambda}^-} \cap \overline{\Psi_{\mu,\lambda}^+} = \emptyset$ for $\mu > 0$ sufficiently large.*

Lemma 4.7 *Suppose that the **Case (b)** holds and $\Phi_p(\phi_1) < 0$. Then for each $a > 0$ there exists $\widehat{\delta} > 0$ such that for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \widehat{\delta}$, there holds $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^+ \cup \mathbf{N}_{\mu,\lambda}^-$ for $\mu > 0$ sufficiently large. Moreover, $\mathbf{N}_{\mu,\lambda}^\pm$ are nonempty for $\mu > 0$ sufficiently large.*

Lemma 4.8 *Suppose that the **Case (b)** holds and $\Phi_p(\phi_1) < 0$. Then for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \widehat{\delta}$, we have the following results.*
(i) There exists $C_1 > 0$ such that $\|u\|_\mu \le C_1$ for all $u \in \mathbf{N}_{\mu,\lambda}^-$ and for $\mu > 0$ sufficiently large;
(ii) We have

$$\inf_{u \in \mathbf{N}_{\mu,\lambda}^-} I_{a,\lambda}^\mu(u) > 0$$

for $\mu > 0$ sufficiently large.

Now, we are ready to prove Theorem 1.4: Since $\lambda_1^{-2}(f_\Omega) \int_\Omega (I_\alpha * Q|\phi_1|^2) Q|\phi_1|^2 dx < a < \Gamma^*$, then

$$\Phi_p(\phi_1) = \int_\Omega (I_\alpha * Q|\phi_1|^2) Q|\phi_1|^2 dx - a \left(\int_\Omega |\nabla \phi_1|^2 dx \right)^2 < 0.$$

By Lemma 4.7, there exists $\widehat{\delta}$ such that for every $\lambda_1(f_\Omega) \le \lambda < \lambda_1(f_\Omega) + \widehat{\delta}$, $\mathbf{N}_{\mu,\lambda}^\pm$ are nonempty sets and $\mathbf{N}_{\mu,\lambda} = \mathbf{N}_{\mu,\lambda}^+ \cup \mathbf{N}_{\mu,\lambda}^-$ for $\mu > 0$ sufficiently large. Then similar to the argument of proof in Theorem 1.3 (iii) , Eq. $(K_{a,\lambda}^\mu)$ admits at least two positive solutions $u^{(1)}$ and $u^{(2)}$ satisfying $I_{a,\lambda}^\mu(u^{(2)}) < 0 < I_{a,\lambda}^\mu(u^{(1)})$ for $\mu > 0$ sufficiently large.

5 The proof of Theorem 1.5-1.8

Lemma 5.1 *Suppose that the* ***Case (c)*** *and conditions* $(V_1)-(V_2)$ *and* $(A_1)-(A_2)$ *hold. Then for each* $0<a<a_*(p)$ *and* $0<\lambda<\lambda_1(f_\Omega)$*, there exist a number* $\overline{\rho}_{a,\lambda}>0$ *and* $e_0\in H_0^1(\Omega)$ *such that*

$$\|e_0\|_\mu>\overline{\rho}_{a,\lambda} \text{ and } \inf_{\|e_0\|_\mu=\overline{\rho}_{a,\lambda}} I^\mu_{a,\lambda}(u)>0>I^\mu_{a,\lambda}(e_0)$$

for μ *sufficiently large.*

Proof. By (18), we have

$$I^\mu_{a,\lambda}(u)\geq\frac{1}{2}\left(\frac{\lambda_1(f_\Omega)-\lambda}{\lambda_1(f_\Omega)+\lambda}\right)\|u\|_\mu^2+\frac{a}{4}\|\nabla u\|^4_{L^2}-\frac{\mathcal{C}_{HLS}\|Q\|^2_{L^\infty}|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}{2p\mathcal{S}^{2p}}\|u\|_\mu^{2p} \tag{32}$$

Let $\overline{\rho}_{a,\lambda}=\min\{\overline{\rho}_\lambda,\overline{\rho}_a\}>0$, where

$$\overline{\rho}_\lambda:=\left[\frac{1}{4}\left(\frac{\lambda_1(f_\Omega)-\lambda}{\lambda_1(f_\Omega)+\lambda}\right)\frac{2p\mathcal{S}^{2p}}{\mathcal{C}_{HLS}\|Q\|^2_{L^\infty}|\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}\right]^{\frac{1}{2(p-1)}},\text{ and } \overline{\rho}_a:=\left(\frac{(p-1)A_p^*}{ap}\right)^{\frac{1}{2(2-p)}}.$$

Then for all $u\in E$ with $\|u\|_\mu=\overline{\rho}_{a,\lambda}$, one has

$$I^\mu_{a,\lambda}(u)\geq\frac{1}{4}\left(\frac{\lambda_1(f_\Omega)-\lambda}{\lambda_1(f_\Omega)+\lambda}\right)\overline{\rho}^2_{a,\lambda}>0.$$

Since

$$0<a<a_*(p)=\left(\frac{2(p-1)A_p^*}{p}\right)^{\frac{1}{p-1}}\left(\frac{2-p}{2(p-1)}\right)^{\frac{2-p}{p-1}},$$

there exists $\varphi_a\in H_0^1(\Omega)$ with $\int_\Omega(I_\alpha*Q|\varphi_a|^p)Q|\varphi_a|^pdx>0$ and $\int_\Omega f_\Omega(x)\varphi_a^2dx>0$ such that

$$0<a<\left(\frac{2(p-1)}{p}\right)^{\frac{1}{p-1}}\left(\frac{2-p}{2(p-1)}\right)^{\frac{2-p}{p-1}}\left(\frac{\int_\Omega(I_\alpha*Q|\varphi_a|^p)Q|\varphi_a|^pdx}{2p(\int_\Omega|\nabla\varphi_a|^2dx)^p}\right)^{1/(p-1)}\leq a_*(p).$$

Let

$$t_a=\left(\frac{2(p-1)\int_\Omega(I_\alpha*Q|\varphi_a|^p)Q|\varphi_a|^pdx}{ap(\int_\Omega|\nabla\varphi_a|^2dx)^p}\right)^{\frac{1}{2(2-p)}}\left(\int_\Omega|\nabla\varphi_a|^2dx\right)^{-\frac{1}{2}}.$$

Then we have $\|t_a\varphi_a\|_\mu>\overline{\rho}_a\geq\overline{\rho}_{a,\lambda}$ and

$$\begin{aligned}
I^\mu_{a,\lambda}(t_a\varphi_a) &= \frac{t_a^2}{2}\left[\int_\Omega|\nabla\varphi_a|^2dx-\lambda\int_\Omega f_\Omega\varphi_a^2dx+\frac{a}{2}\left(\int_\Omega|\nabla\varphi_a|^2dx\right)^2t_a^2\right.\\
&\quad\left.-\frac{\int_\Omega(I_\alpha*Q|\varphi_a|^p)Q|\varphi_a|^pdx}{p}t_a^{2(p-1)}\right]\\
&= \frac{t_a^2}{2}\left[\int_\Omega|\nabla\varphi_a|^2dx\left(1-\frac{a(2-p)}{2(p-1)}\left(\frac{2(p-1)\int_\Omega(I_\alpha*Q|\varphi_a|^p)Q|\varphi_a|^pdx}{ap\left(\int_\Omega|\nabla\varphi_a|^2dx\right)^p}\right)^{\frac{1}{2-p}}\right)\right.\\
&\quad\left.-\lambda\int_\Omega f_\Omega\varphi_a^2dx\right]<0.
\end{aligned}$$

■

Lemma 5.2 *Suppose that the* ***Case (c)*** *and conditions* $(V_1)-(V_3)$ *and* $(A_1)-(A_2)$ *hold. Then for each* $a>0$ *and* $\lambda>0$,

$$I^{\mu}_{a,\lambda}(u)\geq\frac{1}{4}\|u\|^2_{\mu}-M_{a,\lambda}$$

for μ *sufficiently large, where the number* $M_{a,\lambda}>0$ *is independent of* u.

Proof. By condition (A_2), the Hölder and Young's inequalities, we have

$$\frac{\lambda}{2}\int_{\mathbb{R}^N}f(x)u^2dx\leq\frac{\lambda}{2}\|f\|_{L^{N/2}}\mathcal{S}^{-2}\|\nabla u\|^2_{L^2}\leq\frac{a}{12}\|\nabla u\|^4_{L^2}+\frac{3\lambda^2\|f\|_{L^{N/2}}}{4a\mathcal{S}^4}.$$

Then we have

$$I^{\mu}_{a,\lambda}(u)\geq\frac{1}{2}\|u\|^2_{\mu}+\frac{a}{6}\|\nabla u\|^4_{L^2}-\frac{3\lambda^2\|f\|_{L^{N/2}}}{4a\mathcal{S}^4}-\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*Q|u|^p)Q|u|^pdx.$$

By using Hardy-Littlewood-Sobolev inequality, Hölder and Caffarelli-Kohn-Nirenberg inequalities and condition (V_3) gives

$$\begin{aligned}
&\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*Q|u|^p)Q|u|^pdx\\
\leq\ &\frac{\mathcal{C}_{HLS}}{2p}\left(\int_{\mathbb{R}^N}|Q|^{\frac{4}{N+\alpha-(N-2)p}}u^2dx\right)^{\frac{N+\alpha-(N-2)p}{2}}\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{\frac{Np-N-\alpha}{2^*}}\\
\leq\ &\frac{\mathcal{C}_{HLS}\|\nabla u\|^{Np-N-\alpha}_{L^2}}{2p\mathcal{S}^{Np-N-\alpha}}\left(\int_{\{|x|>R_*\}}|Q|^{\frac{4}{N+\alpha-(N-2)p}}u^2dx+\int_{B_{R^*}(0)}|Q|^{\frac{4}{N+\alpha-(N-2)p}}u^2dx\right)^{\frac{N+\alpha-(N-2)p}{2}}\\
\leq\ &\frac{\mathcal{C}_{HLS}\|\nabla u\|^{Np-N-\alpha}_{L^2}}{2p\mathcal{S}^{Np-N-\alpha}}\left[C_*^{\frac{4}{N+\alpha-(N-2)p}}\left(\int_{\mathbb{R}^N}V(x)u^2dx\right)^{\frac{2(2-p)}{N+\alpha-(N-2)p}}\left(\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}dx\right)^{\frac{N+\alpha-4-(N-4)p}{N+\alpha-(N-2)p}}\right.\\
&\left.+\frac{\|Q\|^{\frac{4}{N+\alpha-(N-2)p}}_{L^\infty}|B_{R_*}(0)|^{2/N}\|\nabla u\|^2_{L^2}}{\mathcal{S}^2}\right]^{\frac{N+\alpha-(N-2)p}{2}}\\
\leq\ &\frac{\mathcal{C}_{HLS}\|\nabla u\|^{Np-N-\alpha}_{L^2}}{2p\mathcal{S}^{Np-N-\alpha}}\left[C_*^{\frac{4}{N+\alpha-(N-2)p}}\mathcal{C}_{CKN}\left(\int_{\mathbb{R}^N}V(x)u^2dx\right)^{\frac{2(2-p)}{N+\alpha-(N-2)p}}\|\nabla u\|_{L^2}^{\frac{2(N+\alpha-4-(N-4)p)}{N+\alpha-(N-2)p}}\right.\\
&\left.+\frac{\|Q\|^{\frac{4}{N+\alpha-(N-2)p}}_{L^\infty}|B_{R_*}(0)|^{2/N}\|\nabla u\|^2_{L^2}}{\mathcal{S}^2}\right]^{\frac{N+\alpha-(N-2)p}{2}}\\
\leq\ &\frac{\mathcal{C}_{HLS}\|\nabla u\|^{Np-N-\alpha}_{L^2}}{2p\mathcal{S}^{Np-N-\alpha}}\left[(2\mathcal{C}_{CKN})^{\frac{N+\alpha-(N-2)p}{2}}C_*^2\left(\int_{\mathbb{R}^N}V(x)u^2dx\right)^{2-p}\|\nabla u\|^{N+\alpha-4-(N-4)p}_{L^2}\right.\\
&\left.+\frac{2^{\frac{N+\alpha-(N-2)p}{2}}\|Q\|^2_{L^\infty}|B_{R_*}(0)|^{\frac{N+\alpha-(N-2)p}{N}}\|\nabla u\|^{N+\alpha-(N-2)p}_{L^2}}{\mathcal{S}^{N+\alpha-(N-2)p}}\right]\\
=\ &\frac{(2\mathcal{C}_{CKN})^{\frac{N+\alpha-(N-2)p}{2}}\mathcal{C}_{HLS}C_*^2\left(\int_{\mathbb{R}^N}V(x)u^2dx\right)^{2-p}\|\nabla u\|^{4(p-1)}_{L^2}}{2p\mathcal{S}^{Np-N-\alpha}}\\
&+\frac{2^{\frac{N+\alpha-(N-2)p}{2}}\mathcal{C}_{HLS}\|Q\|^2_{L^\infty}|B_{R_*}(0)|^{\frac{N+\alpha-(N-2)p}{N}}\|\nabla u\|^{2p}_{L^2}}{2p\mathcal{S}^{2p}},
\end{aligned}$$

where $\mathcal{C}_{CKN}$ is the sharp constant of Caffarelli-Kohn-Nirenberg inequality.

Let

$$K_{1,p} := \frac{(2\mathcal{C}_{CKN})^{\frac{N+\alpha-(N-2)p}{2}} \mathcal{C}_{HLS} C_*^2}{2p\mathcal{S}^{Np-N-\alpha}}$$

and

$$K_{2,p} := \frac{2^{\frac{N+\alpha-(N-2)p}{2}} \mathcal{C}_{HLS} \|Q\|_{L^\infty}^2 |B_{R_*}(0)|^{\frac{N+\alpha-(N-2)p}{N}}}{2p\mathcal{S}^{2p}}.$$

Hence, by using Young's inequality yields

$$\begin{aligned} & K_{1,p} \left(\int_{\mathbb{R}^N} V(x) u^2 dx \right)^{2-p} \|\nabla u\|_{L^2}^{4(p-1)} \\ \leq \;& \frac{a}{12} \|\nabla u\|_{L^2}^4 + \frac{K_{1,p}^{1/(2-p)}}{2-p} \left(\frac{12(p-1)}{a} \right)^{\frac{p-1}{2-p}} \int_{\mathbb{R}^N} V(x) u^2 dx \end{aligned} \tag{33}$$

and

$$K_{2,p} \|\nabla u\|_{L^2}^{2p} \leq \frac{a}{12} \|\nabla u\|_{L^2}^4 + \frac{(2-p) K_{2,p}^{2/(2-p)}}{2} \left(\frac{6p}{a} \right)^{\frac{p}{2-p}}. \tag{34}$$

It follows from (33)-(34) that

$$\begin{aligned} I_{a,\lambda}^\mu(u) &\geq \frac{1}{2} \|u\|_\mu^2 - \frac{K_{1,p}^{1/(2-p)}}{2-p} \left(\frac{12(p-1)}{a} \right)^{\frac{p-1}{2-p}} \int_{\mathbb{R}^N} V(x) u^2 dx - M_{a,\lambda} \\ &\geq \frac{1}{4} \|u\|_\mu^2 - M_{a,\lambda} \end{aligned}$$

for all $\mu \geq \widehat{\mu}$, where

$$\widehat{\mu} := \frac{4K_{1,p}^{1/(2-p)}}{2-p} \left(\frac{12(p-1)}{a} \right)^{\frac{p-1}{2-p}}$$

and

$$M_{a,\lambda} := \frac{(2-p) K_{2,p}^{2/(2-p)}}{2} \left(\frac{6p}{a} \right)^{\frac{p}{2-p}}.$$

Thus, this completes the proof. ■

Now, we give the proof of Theorem 1.5-1.6: By Lemma 5.1 , for each $0 < a < a_*(p)$ and $0 < \lambda < \lambda_1(f_\Omega)$, the functional $I_{a,\lambda}^\mu$ has the mountain pass geometry for μ sufficiently large. Let

$$\kappa_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{a,\lambda}^\mu(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], E) | \gamma(0) = 0, \gamma(1) = e_0\}.$$

Let $\{u_n\}$ be a $(PS)_{\kappa_\mu}$ sequence, that is $I_{a,\lambda}^\mu(u_n) \to \kappa_\mu$ and $(I_{a,\lambda}^\mu)'(u_n) \to 0$ as $n \to \infty$. Moreover, by Lemma 5.2, we deduce that $(PS)_{\kappa_\mu}$ sequence $\{u_n\}$ is bounded for μ sufficiently large, which implies that the functional $I_{a,\lambda}^\mu$ satisfies the $(PS)_{\kappa_\mu}$-condition by Proposition 2.10. Therefore, there exists $0 \leq u^{(1)} \in E$ such that $I_{a,\lambda}^\mu(u^{(1)}) = \kappa_\mu > 0$ for $u^{(1)}$ sufficiently large, and this implies that $u^{(1)}$ is a nontrivial nonnegative solution of Eq. $(K^{\mu}_{a,\lambda})$. The strong maximum principle implies that $u^{(1)} > 0$ in $\mathbb{R}^N$.

Next, we consider the infimum of $I^{\mu}_{a,\lambda}$ on the set $\{u \in E : \|u\|_{\mu} \geq \rho_{a,\lambda}\}$ with $\rho_{a,\lambda}$ as given in Lemma 5.1. Set

$$\overline{\kappa_{\mu}} = \inf_{\|u\|_{\mu} \geq \rho_{a,\lambda}} I^{\mu}_{a,\lambda}(u).$$

By virtue of $\|e_0\|_{\mu} \geq \rho_{a,\lambda}$, $I^{\mu}_{a,\lambda}(e_0) < 0$ and Lemma 5.2, we conclude that $-M_{a,\lambda} < \overline{\kappa_{\mu}} < 0$. By using the Ekeland variational principle and Lemma 5.2, there exists a bounded (PS)-sequence $\{u_n\} \subset E$. Hence, by Proposition 2.10, there exists $0 \leq u^{(2)} \in E$ with $\|u^{(2)}\|_{\mu} \geq \rho_{a,\lambda}$ such that $I^{\mu}_{a,\lambda}(u^{(2)}) = \overline{\kappa_{\mu}} < 0$ for μ sufficiently large, and this implies that $u^{(2)}$ is a nontrivial nonnegative solution of Eq. $(K^{\mu}_{a,\lambda})$. The strong maximum principle implies that $u^{(2)} > 0$ in $\mathbb{R}^N$. Consequently, we complete the proof of Theorem 1.5.

By using the Ekeland variational principle and Lemma 5.2, for each $a > 0$ and $\lambda \geq \lambda_1(f_{\Omega})$, there exists a bounded (PS)-sequence $\{u_n\} \subset E$. Hence, by Proposition 2.10, there exists $0 \leq \overline{u} \in E$ such that $I^{\mu}_{a,\lambda}(\overline{u}) = \overline{\kappa_{\mu}} < 0$ for μ sufficiently large, and this implies that $\overline{u}$ is a nontrivial nonnegative solution of Eq. $(K^{\mu}_{a,\lambda})$. The strong maximum principle implies that $\overline{u} > 0$ in $\mathbb{R}^N$. Consequently, we complete the proof of Theorem 1.6.

Considering the **Case (d)**, we set $\Phi_p(u) = \int_{\mathbb{R}^N} (I_{\alpha} * Q|u|^p)Q|u|^p$ in (28). Then we have following results.

Lemma 5.3 *Suppose that the* ***Case (d)*** *holds. If* $u \in E_{\mu} \setminus \{0\}$, *then*
(i) if $\frac{u}{\|u\|_{\mu}}$ *lies in* $\overline{\Sigma^{-}_{\mu,\lambda}} \cap \Psi^{+}_{\mu,\lambda}$ *or* $\Sigma^{-}_{\mu,\lambda} \cap \overline{\Psi^{-}_{\mu,\lambda}}$, *then a multiple of* u *lies in* $\mathbf{N}^{+}_{\mu,\lambda}$;
(ii) if $u \in \Sigma^{+}_{\mu,\lambda} \cap \Psi^{-}_{\mu,\lambda}$, *no multiple of* u *lies in* $\mathbf{N}_{\mu,\lambda}$.

Next, we need to verify that non-empty of submanifold $\mathbf{N}^{+}_{\mu,\lambda}$. Firstly, we consider the following nonlinear Schrödinger equation

$$-\Delta u = \lambda f_{\Omega} u + \int_{\Omega} (I_{\alpha} * Q_{\Omega}|u|^p) Q_{\Omega}|u|^{p-2}u, \quad u \in H^1_0(\Omega), \tag{K_{∞}}$$

where f_{Ω} and Q_{Ω} are restriction of f and Q on $\overline{\Omega}$. It is easy to find the positive ground state solution ω_{∞} of Eq. (K_{∞}) for $0 \leq \lambda < \lambda_1(f_{\Omega})$. Moreover, we have

$$\beta = \inf_{u \in \mathbf{N}_{\infty}} J(u) = J(\omega_{\infty}) > 0,$$

and

$$\begin{aligned}
\beta &= \frac{1}{2}\left(\int_{\Omega} |\nabla \omega_{\infty}|^2 dx - \lambda \int_{\Omega} f_{\Omega} \omega_{\infty}^2 dx\right) - \frac{1}{2p}\int_{\Omega} (I_{\alpha} * Q_{\Omega} |\omega_{\infty}|^p) Q_{\Omega} |\omega_{\infty}|^p dx \\
&= \frac{p-1}{2p}\left(\int_{\Omega} |\nabla \omega_{\infty}|^2 dx - \lambda \int_{\Omega} f_{\Omega} \omega_{\infty}^2 dx\right),
\end{aligned}$$

where $J(u)$ is the energy functional related to Eq. (K_{∞}) in $H^1_0(\Omega)$ given by

$$J(u) = \frac{1}{2}\left(\int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} f_{\Omega} u^2 dx\right) - \frac{1}{2p}\int_{\Omega} (I_{\alpha} * Q_{\Omega} |u|^p) Q_{\Omega} |u|^p dx$$

and

$$\mathbf{N}_{\infty}=\left\{u\in H_0^1(\Omega)\setminus\{0\}\ \mid\ \langle J'(u),u\rangle=0\right\}.$$

Let

$$T(\omega_\infty)=\left(\frac{\int_\Omega|\nabla\omega_\infty|^2dx-\lambda\int_\Omega f_\Omega\omega_\infty^2dx}{\int_{\mathbb{R}^N}(I_\alpha*Q_\Omega|\omega_\infty|^p)Q_\Omega|\omega_\infty|^pdx}\right)^{1/(2p-2)}.$$

Lemma 5.4 *Suppose that the* ***Case (d)*** *and conditions $(V_1)-(V_2)$ and $(A_1)-(A_2)$ hold. Then we have the following results.*
(i) For each $0<\lambda<\lambda_1(f_\Omega)$, there exists $a_{**}(p)>0$ independent of λ,μ such that for every

$$0<a<\overline{a}_{**}(p):=\frac{p-1}{2(2-p)}\left(\frac{2p\beta(2-p)}{p(p-1)}\right)^{\frac{1}{p-1}},$$

there exists a positive constant $t_a^+(\omega_\infty)$ such that $t_a^+(\omega_\infty)\omega_\infty\in\mathbf{N}_{\mu,\lambda}^+$ and

$$I_{a,\lambda}^\mu(t_a^+(\omega_\infty)\omega_\infty)=\inf_{t\geq0}I_{a,\lambda}^\mu(t\omega_\infty)<0;$$

(ii) For each $a>0$ and $\lambda\geq\lambda_1(f_\Omega)$ there exists $t_a^+(\phi_1)$ such that $t_a^+(\phi_1)\phi_1\in\mathbf{N}_{\mu,\lambda}^+$ and

$$I_{a,\lambda}^\mu(t_a^+(\phi_1)\phi_1)=\inf_{t\geq0}I_{a,\lambda}^\mu(t\phi_1)<0.$$

Proof. (i) Since $0<a<a_{**}(p)$, we have

$$\int_\Omega(I_\alpha*Q_\Omega|\omega_\infty|^p)Q_\Omega|\omega_\infty|^pdx>\frac{p}{2-p}\left(\frac{2a(2-p)}{p-1}\right)^{p-1}\left(\int_\Omega|\nabla\omega_\infty|^2dx-\lambda\int_\Omega f_\Omega\omega_\infty^2dx\right)^p.\quad(35)$$

Define

$$g(t)=t^{-2}\left(\int_\Omega|\nabla\omega_\infty|^2dx-\lambda\int_\Omega f_\Omega\omega_\infty^2dx\right)-t^{2p-4}\int_\Omega(I_\alpha*Q_\Omega|\omega_\infty|^p)Q_\Omega|\omega_\infty|^pdx\quad\text{for}\quad t>0.$$

Clearly, $tu\in\mathbf{N}_{\mu,\lambda}$ if and only if $g(t)+a\|\nabla\omega_\infty\|_{L^2}^4=0$. A straightforward evaluation gives

$$g(T(u))=0,\ \lim_{t\to0^+}g(t)=\infty\text{ and }\lim_{t\to\infty}g(t)=0.$$

Since $p<2$ and

$$g'(t)=t^{-3}\left[-2\left(\int_\Omega|\nabla\omega_\infty|^2dx-\lambda\int_\Omega f_\Omega\omega_\infty^2dx\right)+(4-2p)t^{2p-2}\int_\Omega(I_\alpha*Q_\Omega|\omega_\infty|^p)Q_\Omega|\omega_\infty|^pdx\right],$$

we obtain that $g(t)$ is decreasing when $0<t<\left(\frac{1}{2-p}\right)^{1/(2p-2)}T(\omega_\infty)$ and is increasing when $t>\left(\frac{1}{2-p}\right)^{1/(2p-2)}T(\omega_\infty)$. This gives

$$\inf_{t>0}g(t)=g\left(\left(\frac{1}{2-p}\right)^{1/(2p-2)}T(\omega_\infty)\right).$$

By (35), we have

$$g\left(\left(\frac{1}{2-p}\right)^{1/(2p-2)}T\left(\omega_{\infty}\right)\right)<-a\|\nabla u\|_{L^{2}}^{4}.$$

By the above calculation, there exists a constant $t_{a}^{+}(\omega_{\infty})$ satisfying

$$0<\left(\frac{1}{2-p}\right)^{1/(2p-2)}T\left(\omega_{\infty}\right)<t_{a}^{+}(\omega_{\infty})$$

such that

$$g\left(t_{a}^{+}(\omega_{\infty})\right)+a\|\nabla\omega_{\infty}\|_{L^{2}}^{4}=0.$$

Namely, $t_{a}^{+}(\omega_{\infty})\omega_{\infty}\in\mathbf{N}_{\lambda}$. By a calculation on the second order derivatives, we have

$$\begin{aligned}h''_{t_{a}^{+}(\omega_{\infty})\omega_{\infty}}(1) &= -2\left(\int_{\Omega}\left|\nabla t_{a}^{+}(\omega_{\infty})\omega_{\infty}\right|^{2}dx-\lambda\int_{\Omega}f_{\Omega}(t_{a}^{+}(\omega_{\infty})\omega_{\infty})^{2}dx\right)\\ &\quad +2\left(2-p\right)\int_{\Omega}\left(I_{\alpha}*Q_{\Omega}\left|t_{a}^{+}(\omega_{\infty})\omega_{\infty}\right|^{p}\right)Q_{\Omega}\left|t_{a}^{+}(\omega_{\infty})\omega_{\infty}\right|^{p}dx\\ &= \left(t_{a}^{+}(\omega_{\infty})\right)^{5}g'\left(t_{a}^{+}(\omega_{\infty})\right)>0.\end{aligned}$$

This implies that $t_{a}^{+}(\omega_{\infty})\omega_{\infty}\in\mathbf{N}_{\mu,\lambda}^{+}$.

Let

$$l\left(t\right)=\frac{t^{-2}}{2}\left(\int_{\Omega}\left|\nabla\omega_{\infty}\right|^{2}dx-\lambda\int_{\Omega}f_{\Omega}\omega_{\infty}^{2}dx\right)-\frac{t^{2p-4}}{2p}\int_{\Omega}\left(I_{\alpha}*Q_{\Omega}\left|\omega_{\infty}\right|^{p}\right)Q_{\Omega}\left|\omega_{\infty}\right|^{p}dx.$$

Clearly, $I_{a,\lambda}^{\mu}\left(t\omega_{\infty}\right)=0$ if and only if $l\left(t\right)+\frac{a}{4}\|\nabla\omega_{\infty}\|_{L^{2}}^{4}=0$. Observe that

$$l\left(t_{0}\right)=0,\ \lim_{t\to0^{+}}l\left(t\right)=\infty\text{ and }\lim_{t\to\infty}l\left(t\right)=0.$$

where $t_{0}=p^{\frac{1}{2p-2}}T\left(u\right)$. Considering the derivative of $l\left(t\right)$, we obtain

$$\begin{aligned}l'\left(u\right) &= -t^{-3}\left(\int_{\Omega}\left|\nabla\omega_{\infty}\right|^{2}dx-\lambda\int_{\Omega}f_{\Omega}\omega_{\infty}^{2}dx\right)+\frac{2-p}{p}t^{2p-5}\int_{\Omega}\left(I_{\alpha}*Q_{\Omega}\left|\omega_{\infty}\right|^{p}\right)Q_{\Omega}\left|\omega_{\infty}\right|^{p}dx\\ &= t^{-3}\left[\frac{\left(2-p\right)t^{2p-2}}{p}\int_{\Omega}\left(I_{\alpha}*Q_{\Omega}\left|\omega_{\infty}\right|^{p}\right)Q_{\Omega}\left|\omega_{\infty}\right|^{p}dx-\left(\int_{\Omega}\left|\nabla\omega_{\infty}\right|^{2}dx-\lambda\int_{\Omega}f_{\Omega}\omega_{\infty}^{2}dx\right)\right],\end{aligned}$$

which implies that $l\left(t\right)$ is decreasing when $0<t<\left(\frac{p}{2-p}\right)^{1/(2p-2)}T\left(\omega_{\infty}\right)$ and is increasing when $t>\left(\frac{p}{2-p}\right)^{1/(2p-2)}T\left(\omega_{\infty}\right)$. By using (35), we have

$$\begin{aligned}\inf_{t>0}l\left(t\right) &= -\frac{p-1}{4-2p}\left[\frac{p\left(\int_{\Omega}\left|\nabla\omega_{\infty}\right|^{2}dx-\lambda\int_{\Omega}f_{\Omega}\omega_{\infty}^{2}dx\right)}{\left(2-p\right)\int_{\Omega}\left(I_{\alpha}*Q_{\Omega}\left|\omega_{\infty}\right|^{p}\right)Q_{\Omega}\left|\omega_{\infty}\right|^{p}dx}\right]^{-\frac{1}{p-1}}\\ &\quad\cdot\left(\int_{\Omega}\left|\nabla\omega_{\infty}\right|^{2}dx-\lambda\int_{\Omega}f_{\Omega}\omega_{\infty}^{2}dx\right)<-\frac{a}{4}\|\nabla\omega_{\infty}\|_{L^{2}}^{4},\end{aligned}$$

which indicates that there exist $\hat{t}^{(1)}$ and $\hat{t}^{(2)}$ satisfying

$$0<\hat{t}^{(1)}<\left(\frac{p}{2-p}\right)^{1/(2p-2)}T\left(\omega_\infty\right)<\hat{t}^{(2)}$$

such that

$$l\left(\hat{t}^{(i)}\right)+\frac{a}{4}\|\nabla\omega_\infty\|_{L^2}^4=0\quad\text{for}\quad i=1,2.$$

That is

$$I_{a,\lambda}^{\mu}\left(\hat{t}^{(i)}u\right)=0\quad\text{for}\quad i=1,2.$$

Then we have

$$I_{a,\lambda}^{\mu}\left[\left(\frac{p}{2-p}\right)^{1/(2p-2)}T\left(\omega_\infty\right)\omega_\infty\right]<0,\text{ and }I_{a,\lambda}^{\mu}\left(t_a^+(\omega_\infty)\omega_\infty\right)=\inf_{t\geq 0}I_{a,\lambda}^{\mu}\left(t\omega_\infty\right)<0.$$

(ii) Since $\lambda\geq\lambda_1(f_\Omega)$, we have $\|\phi_1\|_\mu^2-\lambda\int_\Omega f_\Omega\phi_1^2dx\leq 0$, which implies that $\frac{\phi_1}{\|\phi_1\|_\mu}\in\overline{\Sigma_\mu^-}$. Then by Lemma 5.3 (i), for each $a>0$ there exists $t_a^+(\phi_1)>0$ such that $t_a^+(\phi_1)\phi_1\in\mathbf{N}_{\mu,\lambda}^+$. Moreover, $h_{\phi_1}'(t)<0$ for all $t\in(0,t_a^+(\phi_1))$ and $h_{\phi_1}'(t)>0$ for all $t>t_a^+(\phi_1)$, which leads to

$$I_{a,\lambda}^{\mu}(t_a^+(\phi_1)\phi_1)=\inf_{t\geq 0}I_{a,\lambda}^{\mu}(t\phi_1)<0.$$

This completes the proof. ■

Set

$$\gamma_{\mu,\lambda}^+=\inf_{u\in\mathbf{N}_{\mu,\lambda}^+}I_{a,\lambda}^{\mu}(u).$$

Lemma 5.5 *Suppose that the **Case (d)** and conditions $(V_1)-(V_2)$ and $(A_1)-(A_2)$ hold. Then the following statements are true.*
(i) For each $\lambda>0$ and $a>0$, the manifold $\mathbf{N}_{\mu,\lambda}^+$ is uniformly bounded for $\mu>0$ sufficiently large;
(ii) For each $\lambda>0$ and $a>0$, there exist two numbers D_0,D_1 such that

$$\inf_{u\in\mathbf{N}_{\mu,\lambda}^-\cup\mathbf{N}_{\mu,\lambda}^0}I_{a,\lambda}^{\mu}(u)\geq 0>-D_0>\gamma_{\mu,\lambda}^+>-D_1,$$

for $\mu>0$ sufficiently large.

Proof. (i) Let $u\in\mathbf{N}_{\mu,\lambda}^+$. By using the Hölder inequality gives

$$\begin{aligned}
1&=\frac{\int_{\mathbb{R}^N}\left(I_\alpha*Q\left|u\right|^p\right)Q\left|u\right|^pdx+\int_{\mathbb{R}^N}fu^2dx}{\|u\|_\mu^2+a\|\nabla u\|_{L^2}^4}\\
&<\frac{\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2\left(\int_{\mathbb{R}^N}|u|^{\frac{2Np}{N+\alpha}}dx\right)^{\frac{N+\alpha}{N}}+\int_{\mathbb{R}^N}fu^2dx}{a\|\nabla u\|_{L^2}^4}\\
&\leq\frac{\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2\|\nabla u\|_{L^2}^{2(N+\alpha-(N-2)p)}\|u\|_{L^{2^*}}^{Np-N-\alpha}}{a\|\nabla u\|_{L^2}^4}+\frac{\lambda\|f\|_{L^{N/2}}}{a\mathcal{S}^2\|\nabla u\|_{L^2}^2}\\
&\leq\frac{\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2\|\nabla u\|_{L^2}^{2(N+\alpha-(N-2)p)}\|\nabla u\|_{L^2}^{Np-N-\alpha}}{a\mathcal{S}^{Np-N-\alpha}\|\nabla u\|_{L^2}^4}+\frac{\lambda\|f\|_{L^{N/2}}}{a\mathcal{S}^2\|\nabla u\|_{L^2}^2}\\
&=\frac{\mathcal{C}_{HLS}\mathcal{S}^{-(Np-N-\alpha)}\|Q\|_{L^\infty}^2}{a\|\nabla u\|_{L^2}^{(N-4)p-(N+\alpha-4)}}+\frac{\lambda\|f\|_{L^{N/2}}}{a\mathcal{S}^2\|\nabla u\|_{L^2}^2}.
\end{aligned}$$

Since $1+\frac{\alpha}{N-4}<p<\frac{N+\alpha}{N-2}$, there exists a constant $d_0>0$ such that

$$\|\nabla u\|_{L^2}\le d_0 \quad \text{for all} \quad u\in \mathbf{N}^+_{\mu,\lambda}. \tag{36}$$

Thus, by (27) and (36), we have

$$\|u\|^2_\mu<\frac{a(2-p)}{p-1}\|\nabla u\|^4_{L^2}+\frac{\lambda\|f\|_{L^{N/2}}}{S^2}\|\nabla u\|^2_{L^2}\le\frac{a(2-p)}{p-1}d_0^4+\frac{\lambda\|f\|_{L^{N/2}}}{S^2}d_0^2 \tag{37}$$

for all $u\in\mathbf{N}^+_{\mu,\lambda}$.

(ii) By Lemma 5.4, there exists D_0 such that $\gamma^+_{\mu,\lambda}<-D_0$. Next, we prove that there exist constants $D_1>0$ such that $\gamma^+_{\mu,\lambda}>-D_1$.

Let $u\in\mathbf{N}^+_{\mu,\lambda}$. Using (36) gives

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\left(\|u\|^2_\lambda-\lambda\int_{\mathbb{R}^N}fu^2dx\right)+\frac{a}{4}\|\nabla u\|^4_{L^2}-\frac{1}{2p}\int_{\mathbb{R}^N}\left(I_\alpha * Q\,|u|^p\right)Q\,|u|^p\,dx\\
&= \frac{p-1}{2p}\left(\|u\|^2_\lambda-\lambda\int_{\mathbb{R}^N}fu^2dx\right)-\frac{a(2-p)}{4p}\|\nabla u\|^4_{L^2}\\
&> -\frac{a(2-p)}{4p}\|\nabla u\|^4_{L^2}-\frac{\lambda}{2}\int_{\mathbb{R}^N}fu^2dx\\
&\ge -\frac{a(2-p)}{4p}\|\nabla u\|^4_{L^2}-\frac{\lambda\|f\|_{L^{N/2}}}{2S^2}\|\nabla u\|^2_{L^2}\\
&\ge -\left(\frac{a(2-p)}{4p}d_0^2+\frac{\lambda\|f\|_{L^{N/2}}}{2S^2}\right)d_0^2,
\end{aligned}$$

which indicates that there exists a constant $D_1>0$ such that $\gamma^+_{\mu,\lambda}>-D_1$ for μ sufficiently large. Furthermore, for $u\in\mathbf{N}^-_{\mu,\lambda}\cup\mathbf{N}^0_{\mu,\lambda}$, by (27),

$$\begin{aligned}
I^\mu_{a,\lambda}(u) &= \frac{1}{4}\left(\|u\|^2_\lambda-\lambda\int_{\mathbb{R}^N}fu^2dx\right)-\frac{2-p}{4p}\int_{\mathbb{R}^N}\left(I_\alpha * Q\,|u|^p\right)Q\,|u|^p\,dx\\
&\ge \frac{(p-1)(2-p)}{4p}\int_{\mathbb{R}^N}\left(I_\alpha * Q\,|u|^p\right)Q\,|u|^p\,dx>0
\end{aligned}$$

Therefore,

$$\inf_{u\in\mathbf{N}^-_{\mu,\lambda}\cup\mathbf{N}^0_{\mu,\lambda}} I^\mu_{a,\lambda}(u)\ge 0>-D_0>\gamma^+_{\mu,\lambda}>-D_1,$$

for $\mu>0$ sufficiently large. Consequently, we complete the proof. ■

Lemma 5.6 *Suppose that the **Case (d)** and conditions $(V_1)-(V_2)$ and $(A_1)-(A_2)$ hold. Then for $0<\lambda<\lambda_1(f_\Omega)$ and $\mu\ge\overline{\mu_*}(\lambda)$, the functional $I^\mu_{a,\lambda}(u)$ is coercive and bounded below on $\mathbf{N}^-_{\mu,\lambda}$.*

Proof. Note that $u\in\mathbf{N}_{\mu,\lambda}$ if and only if

$$\|u\|^2_\mu+a\|\nabla u\|^4_{L^2}=\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^pdx+\lambda\int_{\mathbb{R}^N}fu^2dx. \tag{38}$$

By (8) and (29), for $u \in \mathbf{N}_{\mu,\lambda}$ and $\mu \geq \overline{\mu_*}(\lambda)$, we have

$$
\begin{aligned}
\frac{\lambda_{1,\mu}(f)-\lambda}{\lambda_{1,\mu}(f)}\|u\|_\mu^2 &\leq \|u\|_\mu^2 + a\|\nabla u\|_{L^2}^4 - \lambda\int_{\mathbb{R}^N} fu^2dx \\
&= \int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p dx \\
&\leq \mathcal{C}_{HLS}\|Q\|_{L^\infty}^2 |\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}} \mathcal{S}^{-2p}\|u\|_\mu^{2p}.
\end{aligned}
$$

Then we get

$$
\frac{\lambda_{1,\mu}(f)}{\lambda_{1,\mu}(f)-\lambda}\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p dx \geq \|u\|_\mu^2 \geq \left(\frac{(\lambda_{1,\mu}(f)-\lambda)\mathcal{S}^{2p}}{\lambda_{1,\mu}(f)\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2 |\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}\right)^{\frac{1}{p-1}}
$$

for all $u \in \mathbf{N}_{\mu,\lambda}$ and $\mu \geq \overline{\mu_*}(\lambda)$. Moreover, for $u \in \mathbf{N}_{\mu,\lambda}^-$ and $\mu \geq \overline{\mu_*}(\lambda)$, we have

$$
\begin{aligned}
I_{a,\lambda}^\mu(u) &= \frac{1}{4}\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) - \frac{2-p}{4p}\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p dx \\
&\geq \frac{(2-p)(p-1)}{4p}\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p dx \\
&\geq \frac{(2-p)(p-1)(\lambda_{1,\mu}(f)-\lambda)}{4p\lambda_{1,\mu}(f)}\|u\|_\mu^2 \\
&\geq \frac{(2-p)(p-1)}{4p}\left(\frac{\lambda_{1,\mu}(f)-\lambda}{\lambda_{1,\mu}(f)}\right)^{\frac{p}{p-1}}\left(\frac{\mathcal{S}^{2p}}{\mathcal{C}_{HLS}\|Q\|_{L^\infty}^2 |\{V<b\}|^{\frac{N+\alpha-p(N-2)}{N}}}\right)^{\frac{1}{p-1}}.
\end{aligned}
$$

■

For $u \in \mathbf{N}_{\mu,\lambda}$ with $I_{a,\lambda}^\mu(u) < \frac{1}{2}\left(\frac{1}{2-p}\right)^{\frac{1}{p-1}}\beta$, we deduce that

$$
\begin{aligned}
\frac{1}{2}\left(\frac{1}{2-p}\right)^{\frac{1}{p-1}}\beta &> I_{a,\lambda}^\mu(u) = \frac{p-1}{2p}\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^N} fu^2dx\right) - \frac{(2-p)a}{4p}\|\nabla u\|_{L^2}^4 \\
&\geq \frac{(p-1)(\lambda_{1,\mu}(f)-\lambda)}{2p\lambda_{1,\mu}(f)}\|u\|_\mu^2 - \frac{a(2-p)}{4p}\|u\|_\mu^4.
\end{aligned}
$$

Using the above inequality, together with condition $0 < a < a_{**}(p,\lambda)$, we obtain that there exist two constants $\widehat{M}_1, \widehat{M}_2 > 0$ independent of λ with

$$
\left(\frac{p\beta(\lambda_{1,\mu}(f)-\lambda)}{(p-1)\lambda_{1,\mu}(f)}\right)^{\frac{1}{2}}\left(\frac{1}{2-p}\right)^{\frac{1}{2(p-1)}} < \widehat{M}_1 < \left(\frac{2p\beta(\lambda_{1,\mu}(f)-\lambda)}{(p-1)\lambda_{1,\mu}(f)}\right)^{\frac{1}{2}}\left(\frac{1}{2-p}\right)^{\frac{1}{2(p-1)}} < \widehat{M}_2
$$

such that

$$
\|u\|_\mu < \widehat{M}_1 \quad \text{or} \quad \|u\|_\mu > \widehat{M}_2.
$$

Thus, there holds

$$
\begin{aligned}
\mathbf{N}_{\mu,\lambda}\left[\frac{1}{2}\left(\frac{1}{2-p}\right)^{\frac{1}{p-1}}\beta\right] &:= \left\{u \in \mathbf{N}_{\mu,\lambda} \mid I_{a,\lambda}^\mu(u) < \frac{1}{2}\left(\frac{1}{2-p}\right)^{\frac{1}{p-1}}\beta\right\} \\
&= \mathbf{N}_{\mu,\lambda}^{(1)} \cup \mathbf{N}_{\mu,\lambda}^{(2)},
\end{aligned}
$$

where

$$\mathbf{N}_{\mu,\lambda}^{(1)}=\left\{u\in\mathbf{N}_{\mu,\lambda}\left[\frac{1}{2}\left(\frac{1}{2-p}\right)^{\frac{1}{p-1}}\beta\right]\mid\|u\|_{\mu}<\widehat{M}_{1}\right\}$$

and

$$\mathbf{N}_{\mu,\lambda}^{(2)}=\left\{u\in\mathbf{N}_{\mu,\lambda}\left[\frac{1}{2}\left(\frac{1}{2-p}\right)^{\frac{1}{p-1}}\beta\right]\mid\|u\|_{\mu}>\widehat{M}_{2}\right\}.$$

In addition, by the definition of submanifold $\mathbf{N}_{\lambda}^{(1)}$, we get

$$\|u\|_{\mu}<\widehat{M}_{1}<\left(\frac{2p\beta(\lambda_{1,\mu}(f)-\lambda)}{(p-1)\lambda_{1,\mu}(f)}\right)^{\frac{1}{2}}\left(\frac{1}{2-p}\right)^{\frac{1}{2(p-1)}}\quad\text{for all}\quad u\in\mathbf{N}_{\lambda}^{(1)}.\tag{39}$$

By using (27) and (39), we have

$$\begin{aligned}h_{u}^{\prime\prime}(1) &\leq -2(p-1)\left(\|u\|_{\mu}^{2}-\lambda\int_{\mathbf{R}^{N}}fu^{2}dx\right)+2a(2-p)\|u\|_{\mu}^{4}\\ &\leq \frac{-2(p-1)(\lambda_{1,\mu}(f)-\lambda)}{\lambda_{1,\mu}(f)}\|u\|_{\mu}^{2}+\frac{(p-1)^{2}(2-p)^{1/(p-1)}(\lambda_{1,\mu}(f)-\lambda)^{2}}{p\beta\lambda_{1,\mu}^{2}(f)}\|u\|_{\mu}^{4}\\ &< 0,\quad\text{for all}\quad u\in\mathbf{N}_{\lambda}^{(1)}.\end{aligned}$$

Morover, by the similar proof of Lemma 5.4, there exist $t_{a}^{-}(\omega_{\infty})>0$ such that $t_{a}^{-}(\omega_{\infty})\omega_{\infty}\in\mathbf{N}_{\mu,\lambda}^{-}$. A direct calculation shows that

$$\begin{aligned}I_{a,\lambda}^{\mu}\left(t_{a}^{-}(\omega_{\infty})\omega_{\infty}\right) &= \frac{1}{4}\left(\left\|t_{a}^{-}(\omega_{\infty})\omega_{\infty}\right\|_{\mu}^{2}-\lambda\int_{\mathbb{R}^{N}}f(t_{a}^{-}(\omega_{\infty})\omega_{\infty})^{2}dx\right)\\ &\quad -\frac{2-p}{4p}\int_{\mathbb{R}^{N}}\left(I_{\alpha}*Q\left|t_{a}^{-}(\omega_{\infty})\omega_{\infty}\right|^{p}\right)Q\left|t_{a}^{-}(\omega_{\infty})\omega_{\infty}\right|^{p}dx\\ &= \frac{\left(t_{a}^{-}(\omega_{\infty})\right)^{2}}{4p}\left[p-(2-p)\left(t_{a}^{-}(\omega_{\infty})\right)^{2p-2}\right]\left(\|\omega_{\infty}\|_{\mu}^{2}-\lambda\int_{\mathbb{R}^{N}}f\omega_{\infty}^{2}dx\right)\\ &< \frac{1}{2}\left(\frac{1}{2-p}\right)^{\frac{1}{p-1}}\beta,\end{aligned}$$

which implies that $t_{a}^{-}(\omega_{\infty})\omega_{\infty}\in\mathbf{N}_{\mu,\lambda}^{(1)}$. This tells us that $\mathbf{N}_{\mu,\lambda}^{(1)}$ is nonempty.

Lemma 5.7 *Suppose that the **Case (d)** and conditions $(V_1)-(V_2)$, $(A_1)-(A_2)$ and $0<a<a_{**}(p,\lambda)$ hold. Then there holds $\mathbf{N}_{\lambda}^{(1)}\subset\mathbf{N}_{\lambda}^{-}$ is C^1 sub-manifolds. Furthermore, each local minimizer of the functional $I_{a,\lambda}^{\mu}$ in the sub-manifolds $\mathbf{N}_{\lambda}^{(1)}$ is a critical point of $I_{a,\lambda}^{\mu}$ in E.*

Set

$$\gamma_{\mu,\lambda}^{-}=\inf_{u\in\mathbf{N}_{\mu,\lambda}^{(1)}}I_{a,\lambda}^{\mu}(u)=\inf_{u\in\mathbf{N}_{\mu,\lambda}^{-}}I_{a,\lambda}^{\mu}(u).$$

Now, we give the proof of Theorem 1.7: (i) By using the Ekeland variational principle, Lemma 5.5 and Lemma 5.6, there exists a bounded (PS)-sequence $\{u_n\}\subset E$. Hence, by Proposition 2.10, there exists $0\leq u^{(1)},u^{(2)}\in E$ such that $I_{a,\lambda}^{\mu}(u^{(2)})=\gamma_{\mu,\lambda}^{+}<0<I_{a,\lambda}^{\mu}(u^{(1)})=\gamma_{\mu,\lambda}^{-}$ for μ

sufficiently large, and this implies that $u^{(1)}$, $u^{(2)}$ are nontrivial nonnegative solutions of Eq. $(K^{\mu}_{a,\lambda})$. The strong maximum principle implies that $u^{(1)}, u^{(2)} > 0$ in $\mathbb{R}^N$.

(ii) By using the Ekeland variational principle and Lemma 5.5, for each $a > 0$ and $\lambda \geq \lambda_1(f_\Omega)$, there exists a bounded (PS)-sequence $\{u_n\} \subset E$. Hence, by Proposition 2.10, there exists $0 \leq \overline{u} \in E$ such that $I^{\mu}_{a,\lambda}(\overline{u}) = \gamma^{+}_{\mu,\lambda} < 0$ for μ sufficiently large, and this implies that $\overline{u}$ is a nontrivial nonnegative solution of Eq. $(K^{\mu}_{a,\lambda})$. The strong maximum principle implies that $\overline{u} > 0$ in $\mathbb{R}^N$. Consequently, we complete the proof of Theorem 1.7.

Lemma 5.8 *Suppose that the* ***Case (e)*** *and conditions* $(V_1) - (V_2)$ *and* $(A_2) - (A_3)$ *hold. Then for* $0 < \lambda < \lambda_1(f_\Omega)$ *and* $\mu \geq \overline{\mu_*}(\lambda)$*, the functional* $I^{\mu}_{a,\lambda}(u)$ *is coercive and bounded below on* $\mathbf{N}_{\mu,\lambda} = \mathbf{N}^{+}_{\mu,\lambda}$.

Proof. By (27), it is easy to see that $\mathbf{N}_{\mu,\lambda} = \mathbf{N}^{+}_{\mu,\lambda}$. Moreover, by (38), we have

$$\begin{aligned} I^{\mu}_{a,\lambda}(u) &\geq \frac{\lambda_{1,\mu}(f) - \lambda}{4\lambda_{1,\mu}(f)}\|u\|^2_{\mu} - \frac{2-p}{4p}\int_{\mathbb{R}^N}(I_\alpha * Q|u|^p)Q|u|^p dx \\ &\geq \frac{\lambda_{1,\mu}(f) - \lambda}{4\lambda_{1,\mu}(f)}\|u\|^2_{\mu} - \frac{2-p}{4p}\|Q\|^2_{L^{\frac{2N}{N+\alpha-Np}}}\|u\|^{2p}_{\mu}. \end{aligned}$$

Thus the functional $I^{\mu}_{a,\lambda}(u)$ is coercive and bounded below on $\mathbf{N}_{\mu,\lambda}$. ∎

Now, we give the proof of Theorem 1.8: (i) By using the Ekeland variational principle and Lemma 5.8, for each $a > 0$ and $0 < \lambda < \lambda_1(f_\Omega)$, there exists a bounded (PS)-sequence $\{u_n\} \subset E$. Hence, by Proposition 2.10, there exists $0 \leq \overline{u} \in E$ such that $I^{\mu}_{a,\lambda}(\overline{u}) = \gamma^{+}_{\mu,\lambda} = \inf_{u \in \mathbf{N}_{\mu,\lambda}} I^{\mu}_{a,\lambda}(u) < 0$ for μ sufficiently large, and this implies that $\overline{u}$ is a nontrivial nonnegative solution of Eq. $(K^{\mu}_{a,\lambda})$. The strong maximum principle implies that $\overline{u} > 0$ in $\mathbb{R}^N$. Consequently, we complete the proof of Theorem 1.8.

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