A traveling wave with a buffer zone for asymptotic behavior of an asymmetric fixed credit migration model

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Abstract

In this paper, we introduce a new traveling wave with a buffer zone, which is approached by an asymmetric credit migration model with fixed migration boundary. The asymptotic behavior of the solution of the model is discussed. By constructing two sets of sub and super solutions sequences, it is proved that the solution of the credit rating migration model approaches the new traveling wave with buffer zone as time goes to infinite in a direction. Additionally, some numerical results are presented.

Keywords traveling wave, asymptotic behavior, monotonic iteration, asymmetric migration threshold, buffer zone, hysteresis

1 Introduction

With the rapid development of credit products, such as defaultable bonds and credit derivatives, more accurate risk management and valuation methods

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are needed. In addition to the default risk, the credit migration risk caused by credit quality transformation is also considered in a relatively complete credit model. In recent years, the structural approach for measuring the credit migration risk have become popular. Compared with the traditional reduced-form model (e.g., Jarrow, Lando and Turnbull ([6], 1997), Arvanitis, Gregory and Laurent ([1], 1999), Hurd and Kuznetsov ([5], 2007)), the structural model is more essential since it directly links the credit ratings with firm's assets and debts.

Liang and Zeng ([7], 2015) built the first structural model for measuring credit migration risk, who took the asset value as the determinant of the credit grade migration. A fixed credit migration boundary divides asset value into high and low rating regions, where the value subjects to geometric Brownian motion with different volatility. Further, Hu, Liang and Wu ([4], 2015) turned to considering the liability-asset ratio as a driving factor for the credit rating migration, and deduced the migration boundary as a free boundary. With further research, the extensions of the structural model to more general cases are discussed and more theoretical and empirical results are obtained (see [10, 12, 15–17]).

A significant improvement has occurred in this kind of models. Liang and Lin ([8]) and Chen and Liang ([2], 2021) replaced the one critical threshold for migration by two fixed and free asymmetry thresholds, respectively. That is, they set one threshold for upgrades and the other slight lower threshold for downgrades, to generate a buffer zone, where the credit rating has not changed; i.e., low ratings remain low and high ratings remain high. The introduction of the buffer area avoids the high frequent changes of credit ratings, which may appear in the previous models due to the assumption that the asset value follows geometric Brownian motion. The existence and uniqueness of the solution has been obtained in both cases. Indeed, the asymmetry models can describe not only the buffer zone in credit rating migration, but also the deadband in automatic control, backlash-like hysteresis, etc. Further research on this kind of models in theory would benefit the related fields in both finance and engineering.

A traveling wave is a kind of solution of partial differential equation problems that move forward in a particular direction with a constant speed, while preserving its pattern. It is a common phenomenon in many fields, such as in physics, chemistry and biology etc., see [11, 13, 14]. Except for the areas of natural science, there are also traveling waves in finance. Liang, Wu and Hu ([9], 2016) found an asymptotic traveling wave solution in credit migration model with single free boundary. The discounted value solution of a debt with credit migration risk approaches the traveling wave in a certain direction.

Asymptotic behavior of a solution is always an interest problem, especially for a new model. As a credit rating migration model with single migration boundary approaches a traveling wave ([9]), and a traveling wave always consists a lot of information, we wonder if the new asymmetric credit migration model has a similar behavior as time goes large. Also, if the traveling wave has an explicit expression, and our model's solution converges it in some direction, it will be greatly helpful for us to understand our asymmetric credit rating migration model when time is large.

This article attempts to respond to this concern on the model with fixed migration boundary ([8]). At first, we find a traveling wave solution and solve it explicitly, though this traveling wave does not fit the traditional shape, but has a buffer zone. Next, we establish monotonic and convergent sequences as sub and super solutions of our model. The difficulty here is that we should arrange the sequences properly such that they are monotone and converging. In detail, we establish two sets of sub and super solution sequences, one for our credit rating model, the other for the new traveling wave. Define a decreasing sequence, by using a supersolution sequence of the model's solution minus a subsolution sequence of the traveling wave, and define an increasing sequence in a symmetric manner. By analyzing the asymptotic behavior of each element in these sequences, we finally obtain that the discounted pricing solution converges to the traveling wave along a particular direction. Moreover, we show the approaching process through numerical simulation. As far as we know, it is the first time that the asymptotic traveling wave with buffer zone has been studied in an asymmetric credit migration problem with fixed boundary.

The paper is organized as follows. In Section 2, the asymmetric fixed migration model is expressed. The existence and uniqueness of the solution obtained in [8] is presented in Section 3. Traveling wave solution is discussed in Section 4. In Section 5, we prove that the bond value function is convergent to the traveling wave by constructing monotonic sequences. Section 6 shows the numerical results. Section 7 is a summary of this paper.

2 Model

2.1 Assumption

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. We assume that the firm issues one corporate bond, which is a contingent claim of its asset value on the space $\mathscr{F}$.

Assumption 2.1 (Firm's asset with credit rating migration) *Let S_t denote the firm's asset value in the risk neutral world. It satisfies*

$$dS_t = \begin{cases} rS_t dt + \sigma_H S_t dW_t, & \text{in the high rating region,} \\ rS_t dt + \sigma_L S_t dW_t, & \text{in the low rating region,} \end{cases}$$

where r is the risk-free interest rate, and

$$\sigma_H < \sigma_L \tag{2.1}$$

represent volatilities of the firm under the high and low credit grades respectively. They are assumed to be positive constants. W_t is the standard Brownian motion which generates the filtration $\{\mathscr{F}_t\}$.

Assumption 2.2 (Debt obligation) *The bond issued by the company is zero-coupon and has a face value of F. On the maturity time T, the bond value is $\Phi_T = \min\{S_T, F\}$.*

Assumption 2.3 (Risk discount rate) *We introduce a nonnegative constant δ,*

$$r - \frac{1}{2}\sigma_L^2 < \delta < r - \frac{1}{2}\sigma_H^2, \tag{2.2}$$

which represents the risk discount rate on the threshold of the credit migration.

Assumption 2.4 (Credit rating migration) *High and low rating regions are determined by the firm's asset value. Let S^H, S^L represent the downgrade threshold and the upgrade threshold respectively and*

$$F < S^H < S^L. \tag{2.3}$$

At time t, if the asset value rises to $S^L e^{-\delta(T-t)}$, the firm goes from a low credit rating to a high credit rating; if the asset value drops to $S^H e^{-\delta(T-t)}$, the firm goes from a high credit rating to a low credit rating.

Remark 2.1 *In [8], we assume that risk discount rate $\delta = 0$.*

2.2 Cash Flow

Starting from time t and a high credit rating, the first credit downgrade time is

$$\tau_d = \inf\{\tau > t | S_\tau < S^H e^{-\delta(T-\tau)}, S_t \geqslant S^H e^{-\delta(T-t)}\}. \tag{2.4}$$

Starting from time t and a low credit rating, the first credit upgrade time is

$$\tau_u = \inf\{\tau > t | S_\tau > S^L e^{-\delta(T-\tau)}, S_t \leqslant S^L e^{-\delta(T-t)}\}. \tag{2.5}$$

When the company is in the high credit rating, the bond has two possible benefits in the future. The first is the principal income, the second is a virtual benefit: the bond value after the credit rating migration. Therefore, the value of the debt in high grades, $\Phi^H(s,t)$, is an conditional expectation as follows:

$$\begin{aligned}\Phi^H(s,t) =& E[e^{-r(T-t)} \min\{S_T, F\} \cdot \mathbf{1}_{\{\tau_d \geqslant T\}}. \\ &+ e^{-r(\tau_d - t)} \Phi^L(S_{\tau_d}, \tau_d) \cdot \mathbf{1}_{\{\tau_d < T\}} | S_t = s \geqslant S^H e^{-\delta(T-t)}].\end{aligned} \tag{2.6}$$

We can write out the value of debt with the low credit rating in the same way:

$$\begin{aligned}\Phi^L(s,t) =& E[e^{-r(T-t)} \min\{S_T, F\} \cdot \mathbf{1}_{\{\tau_u \geqslant T\}} \\ &+ e^{-r(\tau_u - t)} \Phi^H(S_{\tau_u}, \tau_u) \cdot \mathbf{1}_{\{\tau_u < T\}} | S_t = s \leqslant S^L e^{-\delta(T-t)}].\end{aligned} \tag{2.7}$$

2.3 PDE problem

By Feynman-Kac formula (see e.g. [3]), it is not difficult to drive the partial differential equations that Φ^H and Φ^L satisfy in their respective regions:

$$\frac{\partial \Phi^H}{\partial t} + \frac{1}{2}\sigma_H^2 S^2 \frac{\partial^2 \Phi^H}{\partial S^2} + rS\frac{\partial \Phi^H}{\partial S} - r\Phi^H = 0, \quad \text{for } S > S^H e^{-\delta(T-t)}, 0 < t < T,$$
$$\frac{\partial \Phi^L}{\partial t} + \frac{1}{2}\sigma_L^2 S^2 \frac{\partial^2 \Phi^L}{\partial S^2} + rS\frac{\partial \Phi^L}{\partial S} - r\Phi^L = 0, \quad \text{for } 0 < S < S^L e^{-\delta(T-t)}, 0 < t < T,$$

with the terminal condition:

$$\Phi^H(S,T) = \Phi^L(S,T) = \min\{S, F\}.$$

The formulas (2.6) and (2.7) imply that the value of the bond is continuous when it passes the rating thresholds, i.e.,

$$\Phi^H(S^H e^{-\delta(T-t)}, t) = \Phi^L(S^H e^{-\delta(T-t)}, t),$$
$$\Phi^L(S^L e^{-\delta(T-t)}, t) = \Phi^H(S^L e^{-\delta(T-t)}, t).$$

The above is a complete PDE problem. Without losing generality, we assume $F = 1$. Using the standard change of variables $x = \log S$ and renaming $T - t$ as t, and defining

$$\phi^i(x,t) = \Phi^i(e^x, T-t), i = H, L,$$

we can derive the following equations and conditions:

$$\begin{cases} \dfrac{\partial \phi^H}{\partial t} - \dfrac{1}{2}\sigma_H^2 \dfrac{\partial^2 \phi^H}{\partial x^2} - (r - \dfrac{1}{2}\sigma_H^2)\dfrac{\partial \phi^H}{\partial x} + r\phi^H = 0, & \text{for } x > X^H - \delta t, t > 0, \\ \phi^H(x,0) = \min\{e^x, 1\}, & \text{for } x > X^H - \delta t, \\ \phi^H(X^H - \delta t, t) = \phi^L(X^H - \delta t, t), & \text{for } t > 0, \\ \dfrac{\partial \phi^L}{\partial t} - \dfrac{1}{2}\sigma_L^2 \dfrac{\partial^2 \phi^L}{\partial x^2} - (r - \dfrac{1}{2}\sigma_L^2)\dfrac{\partial \phi^L}{\partial x} + r\phi^L = 0, & \text{for } x < X^L - \delta t, t > 0, \\ \phi^L(x,0) = \min\{e^x, 1\}, & \text{for } \xi < X^L - \delta t, \\ \phi^L(X^L - \delta t, t) = \phi^H(X^L - \delta t, t), & \text{for } t > 0. \end{cases} \tag{2.8}$$

where $X^L = \log S^L$ and $X^H = \log S^H$.

However, as we shall establish convergence to a traveling wave solution, it will be more convenient for us to work on

$$u^i(\xi,t) = e^{rt}\phi^i(x,t),\ \xi = x + ct,\ i = H, L,$$

where $c = \delta$. Then equations (2.8) become

$$\begin{cases} \dfrac{\partial u^H}{\partial t} - \dfrac{1}{2}\sigma_H^2 \dfrac{\partial^2 u^H}{\partial \xi^2} - (r - \delta - \dfrac{1}{2}\sigma_H^2)\dfrac{\partial u^H}{\partial \xi} = 0, & \text{for } \xi > \xi^H, t > 0, \\ u^H(\xi, 0) = \min\{e^\xi, 1\}, & \text{for } \xi > \xi^H, \\ u^H(\xi^H, t) = u^L(\xi^H, t), & \text{for } t > 0, \\ \dfrac{\partial u^L}{\partial t} - \dfrac{1}{2}\sigma_L^2 \dfrac{\partial^2 u^L}{\partial \xi^2} - (r - \delta - \dfrac{1}{2}\sigma_L^2)\dfrac{\partial u^L}{\partial \xi} = 0, & \text{for } \xi < \xi^L, t > 0, \\ u^L(\xi, 0) = \min\{e^\xi, 1\}, & \text{for } \xi < \xi^L, \\ u^L(\xi^L, t) = u^H(\xi^L, t), & \text{for } t > 0, \end{cases} \tag{2.9}$$

where $\xi^H = X^H$ and $\xi^L = X^L$.

3 Existence and uniqueness

[8] established the existence and uniqueness of the solution for this kind of problem.

Theorem 3.1 (Existence) *The problem* (2.9) *admits a solution* (u^L, u^H) *that satisfies*

$$u^L(\xi,t) \in C^\infty(\overline{Q^L}\backslash(0,0)), \quad u^H(\xi,t) \in C^\infty(\overline{Q^H}). \tag{3.1}$$

Theorem 3.2 (Uniqueness) *The solution of problem* (2.9) *is unique.*

Theorem 3.2 is proved by establishing a special comparison principle. And the main idea of proving Theorem 3.1 is to establish a supersolution sequence by monotonic iteration and take the limit as a solution. For the later application in demonstrating the solution's asymptotic behavior in this paper, we still explain these super and sub solution sequence.

Define $\mathcal{L}^i = \frac{\partial}{\partial t} - \frac{1}{2}{\sigma_i}^2\frac{\partial^2}{\partial \xi^2} - (r - \delta - \frac{1}{2}{\sigma_i}^2)\frac{\partial}{\partial \xi}$, $i = H, L$; $Q^L = (-\infty, \xi^L) \times (0, +\infty)$ and $Q^H = (\xi^H, \infty) \times (0, +\infty)$. Suppose $k \geqslant 0$ is an integer.

Starting from $\overline{u}_0^L(\xi^L, t) \equiv 1$, we define successively a sequence of supersolutions $\{\bar{u}_k^H, \bar{u}_k^L\}_{k=0}^{\infty}$ which satisfy

$$\mathcal{L}^H\overline{u}_k^H = 0 \text{ in } Q^H,\ \mathcal{L}^L\overline{u}_k^L = 0 \text{ in } Q^L,$$

with the initial value $\min\{e^{\xi}, 1\}$. By the induction assumption $\overline{u}_k^H(\xi^H, t) = \overline{u}_k^L(\xi^H, t)$ and $\overline{u}_{k+1}^L(\xi^L, t) = \overline{u}_k^H(\xi^L, t)$, we have completed the definition of the sequence. If we start from $\underline{u}_0^L(\xi^L, t) \equiv 0$, a sequence of subsolutions $\{\underline{u}_k^H, \underline{u}_k^L\}_{k=0}^{\infty}$ can be defined in the same manner.

Proposition 1 *The sequence $\{\bar{u}_k^H, \bar{u}_k^L\}_{k=0}^{\infty}$ is bounded and monotonically decreasing in k in the sense that for each $k \geqslant 0$,*

$$\begin{aligned} 0 \leqslant \bar{u}_{k+1}^H \leqslant \bar{u}_k^H \leqslant 1, \quad & \text{in } \overline{Q^H}, \\ 0 \leqslant \bar{u}_{k+1}^L \leqslant \bar{u}_k^L \leqslant 1, \quad & \text{in } \overline{Q^L}. \end{aligned}$$

Consequently, for each $(\xi, t) \in \overline{Q^H}$ or $(\xi, t) \in \overline{Q^L}$ there is the limit

$$u^H(\xi, t) = \lim_{k\to\infty} \bar{u}_k^H(\xi, t), \quad u^L(\xi, t) = \lim_{k\to\infty} \bar{u}_k^L(\xi, t), \tag{3.2}$$

where $(u^H(\xi, t), u^L(\xi, t))$ is a solution of the problem (2.9).

Proposition 2 *The sequence $\{\underline{u}_k^H, \underline{u}_k^L\}_{k=0}^{\infty}$ is bounded and monotonically increasing in k in the sense that for each $k \geqslant 0$,*

$$\begin{aligned} 0 \leqslant \underline{u}_k^H \leqslant \underline{u}_{k+1}^H \leqslant 1, \quad & \text{in } \overline{Q^H}, \\ 0 \leqslant \underline{u}_k^L \leqslant \underline{u}_{k+1}^L \leqslant 1, \quad & \text{in } \overline{Q^L}. \end{aligned}$$

Consequently, for each $(\xi, t) \in \overline{Q^H}$ or $(\xi, t) \in \overline{Q^L}$ there is the limit

$$\underline{u}^H(\xi, t) = \lim_{k\to\infty} \underline{u}_k^H(\xi, t), \quad \underline{u}^L(\xi, t) = \lim_{k\to\infty} \underline{u}_k^L(\xi, t), \tag{3.3}$$

where $(u^H(\xi, t), u^L(\xi, t))$ is a solution of the problem (2.9).

The proof of Proposition 1 can be found in [8] and we also put it in the Appendix 7. Proposition 2 can be proved in a similar way.

4 Traveling wave solution

Let $\mathcal{L}_0^i = \frac{1}{2}{\sigma_i}^2\frac{\partial^2}{\partial\xi^2} + (r - \delta - \frac{1}{2}{\sigma_i}^2)\frac{\partial}{\partial\xi}$, $i = H, L$.

Lemma 4.1 *For any given δ satisfying (2.2), the problem*

$$\begin{cases} \mathcal{L}_0^H U^H = 0, \quad \textit{for } \xi > \xi^H, \\ U^H(+\infty) = 1, \\ U^H(\xi^H) = U^L(\xi^H), \\ \mathcal{L}_0^L U^L = 0, \quad \textit{for } \xi < \xi^L, \\ U^L(-\infty) = 0, \\ U^L(\xi^L) = U^H(\xi^L). \end{cases} \tag{4.1}$$

admits a unique solution,

$$\begin{aligned} U^H &= \frac{b-d}{ad-cb}e^{(1-\frac{2(r-\delta)}{\sigma_H^2})\xi} + 1, \quad \xi > \xi^H, \\ U^L &= \frac{a-c}{ad-cb}e^{(1-\frac{2(r-\delta)}{\sigma_L^2})\xi}, \quad \xi < \xi^L, \end{aligned}$$

where

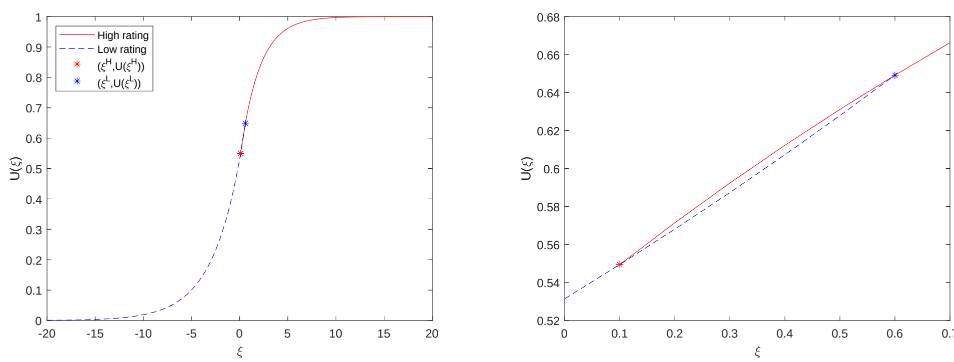
$$\begin{aligned} a &= e^{(1-\frac{2(r-\delta)}{\sigma_H^2})\xi^H}, b = e^{(1-\frac{2(r-\delta)}{\sigma_L^2})\xi^H}, \\ c &= e^{(1-\frac{2(r-\delta)}{\sigma_H^2})\xi^L}, d = e^{(1-\frac{2(r-\delta)}{\sigma_L^2})\xi^L}. \end{aligned}$$

Proof. It is easy to obtain the general solution of the ODE problem

$$\begin{aligned} U^H &= C_1 e^{(1-\frac{2(r-\delta)}{\sigma_H^2})\xi} + C_2, \quad \xi > \xi^H, \\ U^L &= C_3 e^{(1-\frac{2(r-\delta)}{\sigma_L^2})\xi} + C_4, \quad \xi < \xi^L. \end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1) yields that C_1, C_2, C_3, C_4 are uniquely determined, therefore the unique solution of the problem is obtained. □

This traveling wave is different from traditional one, as it has a "bulge" which is buffer zone, see Figure 1.



(a) Entire area

(b) Buffer area

Figure 1. Traveling wave solution. The lines plot the traveling wave solution, in which the left shows the overview and the right highlights the buffer area. It is assumed that the risk-free interest rate $r = 3.5\%$, the risk discount rate $\delta = 0.5\%$, the asset volatility in the low rating $\sigma_L = 30\%$ and in the high rating $\sigma_H = 20\%$, and the migration boundary for upgrades $\xi^L = 0.6$ and for downgrades $\xi^H = 0.1$.

Even though the problem (4.1) has explicit solutions, in order to analysis the asymptotic behavior, we still need to construct sub-super solution sequences which converge to $(U^H(\xi), U^L(\xi))$ by monotonic iteration.

Starting from $\bar{U}_0^L(\xi^L) = 1$, we first construct a sequence of supersolutions $\{\bar{U}_k^L(\xi),\, \bar{U}_k^H(\xi)\}_{k=0}^{\infty}$ which satisfy

$$\begin{aligned}
&\mathcal{L}_0^L \bar{U}_k^L = 0, \text{ for } \xi < \xi^L, \ \mathcal{L}_0^H \bar{U}_k^H = 0, \text{ for } \xi > \xi^H, \\
&\bar{U}_k^L(-\infty) = 0, \ \bar{U}_k^H(\infty) = 1.
\end{aligned}$$

By the induction assumption $\bar{U}_k^H(\xi^H) = \bar{U}_k^L(\xi^H)$ and $\bar{U}_{k+1}^L(\xi^L) = \bar{U}_k^H(\xi^L)$, we have completed the definition of the sequence. The sequence is decreasing with k, thus we can get the limit as a solution of problem (4.1). The proof is similar to the one of Proposition 1.

Lemma 4.2 *The sequence $\{\bar{U}_k^H(\xi), \bar{U}_k^L(\xi)\}_{k=0}^{\infty}$ is bounded and monotonically decreasing in k in the sense that for each $k \geqslant 0$,*

$$\begin{aligned}
&0 \leqslant \bar{U}_{k+1}^H(\xi) \leqslant \bar{U}_k^H(\xi) \leqslant 1, \quad \forall \xi \in [\xi^H, \infty), \\
&0 \leqslant \bar{U}_{k+1}^L(\xi) \leqslant \bar{U}_k^L(\xi) \leqslant 1, \quad \forall \xi \in (-\infty, \xi^L].
\end{aligned}$$

Consequently, for each $\xi \in [\xi^H, \infty)$ or $\xi \in (-\infty, \xi^L]$ there is the limit

$$U^H(\xi) = \lim_{k \to \infty} \bar{U}_k^H(\xi), \quad U^L(\xi) = \lim_{k \to \infty} \bar{U}_k^L(\xi), \tag{4.3}$$

where $(U^H(\xi), U^L(\xi))$ is the solution of the problem (4.1).

Proof. First, we show $\bar{U}_1^L < \bar{U}_0^L$ for $\xi \in (-\infty, \xi^L)$. Since $\bar{U}_0^L(\xi^L) = 1$, the maximum principle gives

$$\bar{U}_0^L(\xi) < 1, \quad \forall \xi \in (-\infty, \xi^L).$$

Thus, $\bar{U}_0^H(\xi^H) = \bar{U}_0^L(\xi^H) < 1$. By maximum principle, we get

$$\bar{U}_0^H(\xi) < 1, \quad \forall \xi \in (\xi^H, \infty).$$

Therefore, $\bar{U}_1^L(\xi^L) = \bar{U}_0^H(\xi^L) < 1$. And this implies $\bar{U}_1^L(\xi^L) < \bar{U}_0^L(\xi^L)$. Using the comparison principle, we find

$$\bar{U}_1^L(\xi) < \bar{U}_0^L(\xi), \quad \forall \xi \in (-\infty, \xi^L). \tag{4.4}$$

Then we make an inductive assumption that $\bar{U}_k^L < \bar{U}_{k-1}^L$ for $\xi \in (-\infty, \xi^L)$, $k \geqslant 1$. According to the iterative relationship $\bar{U}_k^H(\xi^H) = \bar{U}_k^L(\xi^H), \bar{U}_{k-1}^H(\xi^H) = \bar{U}_{k-1}^L(\xi^H)$, we get $\bar{U}_k^H(\xi^H) < \bar{U}_{k-1}^H(\xi^H)$. By the comparison principle, it follows that

$$\bar{U}_k^H(\xi) < \bar{U}_{k-1}^H(\xi), \quad \forall \xi \in (\xi^H, \infty). \tag{4.5}$$

Also the iterative condition $\bar{U}_{k+1}^L(\xi^L) = \bar{U}_k^H(\xi^L), \bar{U}_k^L(\xi^L) = \bar{U}_{k-1}^H(\xi^L)$ implies $\bar{U}_{k+1}^L(\xi^L) < \bar{U}_k^L(\xi^L)$. And using the comparison principle shows that

$$\bar{U}_{k+1}^L(\xi) < \bar{U}_k^L(\xi), \quad \forall \xi \in (-\infty, \xi^L). \tag{4.6}$$

This completes the induction argument for the monotonicity of the sequence.

Similarly, by the extremum principle, we derive that for each integer $k \geqslant 0$, $0 < \bar{U}_k^H < 1$ for $\xi \in (\xi^H, \infty)$ and $0 < \bar{U}_k^L < 1$ for $\xi \in (-\infty, \xi^L)$.

Thus, we can have a limit in (4.3), and check that it is a solution of the problem (4.1). $\square$

We also start from $\underline{U}_0^L(\xi^L) = 0$ to construct successively a sequence of subsolutions $\{\underline{U}_k^H(\xi), \underline{U}_k^L(\xi)\}_{k=0}^{\infty}$ in the same manner.

Lemma 4.3 *The sequence $\{\underline{U}_k^H(\xi), \underline{U}_k^L(\xi)\}_{k=0}^{\infty}$ is bounded and monotonically increasing in k in the sense that for each $k \geqslant 0$,*

$$\begin{aligned} 0 \leqslant \underline{U}_{k+1}^H(\xi) \leqslant \underline{U}_k^H(\xi) \leqslant 1, \quad \forall \xi \in [\xi^H, \infty), \\ 0 \leqslant \underline{U}_{k+1}^L(\xi) \leqslant \underline{U}_k^L(\xi) \leqslant 1, \quad \forall \xi \in (-\infty, \xi^L]. \end{aligned}$$

Consequently, for each $\xi \in [\xi^H, \infty)$ or $\xi \in (-\infty, \xi^L]$ there is the limit

$$U^H(\xi) = \lim_{k \to \infty} \underline{U}_k^H(\xi), \quad U^L(\xi) = \lim_{k \to \infty} \underline{U}_k^L(\xi), \tag{4.7}$$

where $(U^H(\xi), U^L(\xi))$ is the solution of the problem (4.1).

5 Asymptotic behavior

We want to show that $u^i(\xi, t)$ tends to $U^i(\xi)$ as $t \to \infty$ for $i = H, L$. If we define $v^i(\xi, t) = u^i(\xi, t) - U^i(\xi), i = H, L$, we only need to show that $v^i(\xi, t)$ converges to zero as $t \to \infty$. The difficulty of this argument is that the behavior of $v^i(\xi, t)$ on the asymmetric boundary is unknown. Therefore, we construct monotonic sequences which converge to $v^L(\xi,t)$ and $v^H(\xi,t)$. The asymptotic behavior of $v^i(\xi,t)$ can be obtained by analyzing the asymptoticity of each subsolution or supersolution in the sequences.

5.1 An asymptotic lemma

As a preparatory work, we first introduce an asymptotic lemma for ordinary semi-unbounded problems, of which values on the boundary are explicit and converge with respect to t.

Lemma 5.1 *1) Suppose $u_*^L(\xi,t)$ is a solution of*

$$\begin{cases} \mathcal{L}^L u_*^L(\xi,t) = 0, & \text{for } x < \xi^L, t > 0, \\ u_*^L(\xi,0) = f^L(\xi), & \text{for } x < \xi^L, \\ u_*^L(\xi^L,t) = g^L(t), & \text{for } t > 0. \end{cases}$$

Assume that $\lim_{\xi\to-\infty} f^L(\xi) = f^L$, *and* $\lim_{t\to\infty} g^L(t) = g^L$, *then*

$$\lim_{t\to\infty} u_*^L(\xi,t) = U_*^L(\xi), \quad \text{uniformly for} \quad \xi \in (-\infty, \xi^L],$$

and $U_*^L(\xi)$ is the solution of

$$\begin{cases} \mathcal{L}_0^L U_*^L(\xi) = 0, & \text{for } x < \xi^L, \\ U_*^L(-\infty) = f^L, U_*^L(\xi^L) = g^L. \end{cases}$$

2) Suppose $u_*^H(\xi,t)$ is a solution of

$$\begin{cases} \mathcal{L}^H u_*^H = 0, & \text{for } x > \xi^H, t > 0, \\ u_*^H(\xi,0) = f^H(\xi), & \text{for } x > \xi^H, \\ u_*^H(\xi^L,t) = g^H(t), & \text{for } t > 0. \end{cases}$$

Assume that $\lim_{\xi\to\infty} f^H(\xi) = f^H$, *and* $\lim_{t\to\infty} g^H(t) = g^H$, *then*

$$\lim_{t\to\infty} u_*^H(\xi,t) = U_*^H(\xi), \quad \text{uniformly for} \quad \xi \in [\xi^H, \infty),$$

and $U_*^H(\xi)$ is the solution of

$$\begin{cases} \mathcal{L}_0^H U_*^H(\xi) = 0, & \text{for } x > \xi^H, \\ U_*^H(\infty) = f^H, U_*^H(\xi^H) = g^H. \end{cases}$$

Proof. Define $v_*^i(\xi,t)=u_*^i(\xi,t)-U_*^i(\xi), i=H,L$. We only need to prove

$$\lim_{t\to\infty} v_*^L(\xi,t)=0, \quad \text{uniformly for} \quad \xi\in(-\infty,\xi^L], \tag{5.1}$$

and

$$\lim_{t\to\infty} v_*^H(\xi,t)=0, \quad \text{uniformly for} \quad \xi\in[\xi^H,\infty). \tag{5.2}$$

1) Let α and β be positive numbers to be determined later. Define

$$w(\xi,t)=e^{\alpha\xi-\beta t},$$

then

$$\mathcal{L}^L w=(-\frac{1}{2}\sigma_L^2\alpha^2-(r-\delta-\frac{1}{2}\sigma_L^2)\alpha-\beta)e^{\alpha\xi-\beta t},$$

Since (2.2), it is easy to choose α to be sufficiently small so that

$$-\frac{1}{2}\sigma_L^2\alpha^2-(r-\delta-\frac{1}{2}\sigma_L^2)\alpha>0.$$

Next we choose β small enough such that $\mathcal{L}^L w\geqslant 0$.

We introduce the function

$$z_\pm(\xi,t)=Mw(\xi,t)+\epsilon\pm v_*^L(\xi,t),$$

where M be a positive number to be determined later. Obviously, $\mathcal{L}^L z_\pm\geqslant 0$. Now let ϵ be an arbitrary positive number. Using $lim_{\xi\to-\infty}v_*^L(\xi,0)=0$ it follows that we can choose $N(\epsilon)>0$ such that $v_*^L(\xi,0)\leqslant\epsilon$ when $\xi\leqslant -N$. Next, we choose $M=M_1$ sufficiently large so that $v_*^L(\xi,0)\leqslant M_1e^{-\alpha N}$ for $-N\leqslant\xi\leqslant\xi^L$. We conclude that

$$z_\pm(\xi,0)\geqslant 0 \quad \text{for} \quad \xi\leqslant\xi^L.$$

Since $\lim_{t\to\infty}v_*^L(\xi^L,t)=0$, we can choose $\tau(\epsilon)$ such that $v_*^L(\xi^L,t)\leqslant\epsilon$ for $t\geqslant\tau$. Next, we reselect $M=M_2(M_2>M_1)$ sufficiently large so that $v_*^L(\xi^L,t)\leqslant M_2e^{\alpha\xi^L-\beta\tau}$ when $0\leqslant t\leqslant\tau$. We obtain

$$z_\pm(\xi^L,t)\geqslant 0 \quad \text{for} \quad t\geqslant 0.$$

Applying the maximum principle, we get

$$z_\pm(\xi,t)\geqslant 0, \quad \text{for all} \quad \xi\leqslant\xi^L, t\geqslant 0,$$

that is

$$|v_*^L(\xi,t)| \leqslant M_2 w(\xi,t) + \epsilon.$$

Since ϵ is arbitrary, it is easy to get (5.1) holds.

2) A slight change in the proof above shows that (5.2) holds. Actually, we only need to redefine

$$w(\xi,t) = e^{-\alpha\xi-\beta t}.$$

□

5.2 A sequence of supersolutions

We now carry out the construction of a sequence of supersolutions $\{\bar{v}_k^H(\xi,t), \bar{v}_k^L(\xi,t)\}_{k=0}^{\infty}$ by defining $\bar{v}_k^H(\xi,t) = \bar{u}_k^H(\xi,t) - \underline{U}_k^H(\xi)$ and $\bar{v}_k^L(\xi,t) = \bar{u}_k^L(\xi,t) - \underline{U}_k^L(\xi)$, since $\bar{u}_k^i(\xi,t)$ decreases and $\underline{U}_k^i(\xi)$ increases with respect to k for $i = H, L$.

Lemma 5.2 *The sequence $\{\bar{v}_k^H(\xi,t), \bar{v}_k^L(\xi,t)\}_{k=0}^{\infty}$ is decreasing with k and converges to $(v^H(\xi,t), v^L(\xi,t))$.*

Proof. Noting by Proposition 1 that $\{\bar{u}_k^H(\xi,t), \bar{u}_k^L(\xi,t)\}_{k=0}^{\infty}$ converges to $(u^H(\xi,t), u^L(\xi,t))$ and by Lemma 4.3 that $\{\underline{U}_k^H(\xi), \underline{U}_k^L(\xi)\}_{k=0}^{\infty}$ converges to $(U^H(\xi), U^L(\xi))$, we can obtain the result of this lemma at once. □

For later analysis, we also establish a sequence $\{\bar{V}_k^H(\xi), \bar{V}_k^L(\xi)\}_{k=0}^{\infty}$ in the method of monotonic iteration, which satisfies

$$\begin{aligned}
&\mathcal{L}_0^L \bar{V}_k^L = 0, \text{ for } \xi < \xi^L, \ \mathcal{L}_0^H \bar{V}_k^H = 0, \text{ for } \xi > \xi^H, \\
&\bar{V}_k^L(-\infty) = 0, \ \bar{V}_k^H(\infty) = 0.
\end{aligned}$$

The sequence is defined by the induction assumption $\bar{V}_k^H(\xi^H) = \bar{V}_k^L(\xi^H)$ and $\bar{V}_{k+1}^L(\xi^L) = \bar{V}_k^H(\xi^L)$, starting from $\bar{V}_0^L(\xi^L) = 1$.

Lemma 5.3 *$\bar{V}_k^H(\xi)$ converges to zero uniformly for $\xi \in [\xi^H, \infty)$, and $\bar{V}_k^L(\xi)$ converges to zero uniformly for $\xi \in (-\infty, \xi^L]$, as $k \to \infty$.*

Proof. By the same method as in the proof of Lemma 4.2, we can obtain that the sequence $\{\bar{V}_k^H(\xi), \bar{V}_k^L(\xi)\}_{k=0}^{\infty}$ decreasing with k and tends to $(V^H(\xi), V^L(\xi))$, which is the solution of the equations

$$\begin{cases} \mathcal{L}_0^H V^H = 0, \quad \text{for } \xi > \xi^H, \\ V^H(+\infty) = 0, \\ V^H(\xi^H) = V^L(\xi^H), \\ \mathcal{L}_0^L V^L = 0, \quad \text{for } \xi < \xi^L, \\ V^L(-\infty) = 0, \\ V^L(\xi^L) = V^H(\xi^L). \end{cases} \tag{5.3}$$

This problem (5.3) admits a unique solution 0. Actually, the ODE problem has a general solution in the same form as the formula (4.2). Substituting it into (5.3) gives that $C_1 = C_2 = C_3 = C_4 = 0$. Thus, $\{\bar{V}_k^H(\xi), \bar{V}_k^L(\xi)\}_{k=0}^{\infty}$ tends to zero uniformly on any compact set of $[X^H, \infty)$ and $(-\infty, X^L]$, respectively.

Combining this with the estimates for each $k \geqslant 0$,

$$\begin{aligned} &0 \leqslant \bar{V}_k^H \leqslant e^{(1-\frac{2(r-\delta)}{\sigma_H^2})(\xi-\xi^H)}, \quad \forall \xi \in [\xi^H, \infty), \\ &0 \leqslant \bar{V}_k^L \leqslant e^{(1-\frac{2(r-\delta)}{\sigma_L^2})(\xi-\xi^L)}, \quad \forall \xi \in (-\infty, \xi^L], \end{aligned}$$

we find that the convergence is uniform for $\xi \in (-\infty, \xi^L]$ and $\xi \in [\xi^H, \infty)$, respectively. □

We now discuss the asymptotic behavior of $\bar{v}_k^i(\xi, t), i = H, L$ for each $k \geqslant 0$. Lemma 5.1 presents an asymptotic result for ordinary semi-unbounded problems and it will be used here.

Lemma 5.4 *For all $k \geqslant 0$, $\bar{v}_k^H(\xi, t)$ tends to $\bar{V}_k^H(\xi)$ uniformly for $\xi \in [\xi^H, \infty)$ and $\bar{v}_k^L(\xi, t)$ tends to $\bar{V}_k^L(\xi)$ uniformly for $\xi \in (-\infty, \xi^L]$, as $t \to \infty$.*

Proof. First observe that for all $k \geqslant 0$, $\bar{v}_k^i(\xi, 0) = \bar{u}_k^i(\xi, 0) - \underline{U}_k^i(\xi), i = H, L$. Since $\bar{u}_k^L(\xi, 0) = \min\{e^{\xi}, 1\}$ and $\underline{U}_k^L(-\infty) = 0$, we obtain

$$\lim_{\xi \to -\infty} \bar{v}_k^L(\xi, 0) = 0. \tag{5.4}$$

Since $\bar{u}_k^H(\xi,0)=1$ and $\underline{U}_k^H(\infty)=1$, we have

$$\lim_{\xi\to\infty}\bar{v}_k^H(\xi,0)=0. \tag{5.5}$$

We now proceed by induction. When $k=0$, by $\bar{u}_0^L(\xi^L,t)=1$ and $\underline{U}_0^L(\xi^L)=0$, it is obvious that $\lim_{t\to\infty}\bar{v}_0^L(\xi^L,t)=1$. Combined with (5.4) and by Lemma 5.1, it is sufficient to show

$$\lim_{t\to\infty}\bar{v}_0^L(\xi,t)=\bar{V}_0^L(\xi),\quad \text{uniformly for } \xi\in(-\infty,\xi^L]. \tag{5.6}$$

Assume the convergence holds for $\bar{v}_k^L(\xi,t)$; we get $\bar{v}_k^L(\xi^H,t)$ tends to $\bar{V}_k^L(\xi^H)$. Hence $\bar{v}_k^H(\xi^H,t)$ tends to $\bar{V}_k^H(\xi^H)$. Combined with (5.5) and applying Lemma 5.1, we get

$$\lim_{t\to\infty}\bar{v}_k^H(\xi,t)=\bar{V}_k^H(\xi),\quad \text{uniformly for } \xi\in[\xi^H,\infty).$$

Then, $\bar{v}_k^H(\xi^H,t)$ tends to $\bar{V}_k^H(\xi^H)$. It follows that $\bar{v}_{k+1}^L(\xi^H,t)$ tends to $\bar{V}_{k+1}^L(\xi^H)$. Combined with (5.4) and using Lemma 5.1, we conclude that

$$\lim_{t\to\infty}\bar{v}_{k+1}^L(\xi,t)=\bar{V}_{k+1}^L(\xi),\quad \text{uniformly for } \xi\in(-\infty,\xi^L].$$

□

Lemma 5.5 *For any given $\epsilon>0$, there exists $t_1>0$ such that when $t>t_1$, $v^L(\xi,t)\leqslant\epsilon$ holds uniformly for $\xi\in(-\infty,\xi^L]$; there exists $t_2>0$ such that when $t>t_2$, $v^H(\xi,t)\leqslant\epsilon$ holds uniformly for $\xi\in[\xi^H,\infty)$.*

Proof. For any given $\epsilon>0$, from Lemma 5.3 we conclude that there exits a integer K such that

$$\bar{V}_K^L(\xi)<\frac{1}{2}\epsilon \quad \text{for} \quad \xi\in(-\infty,\xi^L]. \tag{5.7}$$

The Lemma 5.4 implies that there exits a number t_1 such that

$$\bar{v}_K^L(\xi,t)-\bar{V}_K^L(\xi)<\frac{1}{2}\epsilon \quad \text{for} \quad t\geqslant t_1 \quad \text{and} \quad \xi\in(-\infty,\xi^L]. \tag{5.8}$$

Noting by Lemma 5.2 that $v^L(\xi,t)\leqslant\bar{v}_k^L(\xi,t)$ in Q^L, we obtain that

$$v^L(\xi,t)\leqslant\bar{v}_K^L(\xi,t)<\frac{1}{2}\epsilon+\frac{1}{2}\epsilon=\epsilon \tag{5.9}$$

holds for all $\xi \in (-\infty, \xi^L], t \geqslant t_1$.

Similarly, we can obtain that there exits a number t_2 such that

$$v^H(\xi, t) < \epsilon \tag{5.10}$$

holds for all $\xi \in [\xi^H, \infty), t \geqslant t_2$. $\square$

5.3 A sequence of subsolutions

We can also construct a sequence of subsolutions $\{\underline{v}_k^H(\xi,t), \underline{v}_k^L(\xi,t)\}_{k=0}^{\infty}$ by defining $\underline{v}_k^H(\xi,t) = \underline{u}_k^H(\xi,t) - \bar{U}_k^H(\xi)$ and $\underline{v}_k^L(\xi,t) = \underline{u}_k^L(\xi,t) - \bar{U}_k^L(\xi)$, since $\underline{u}_k^i(\xi,t)$ increases and $\bar{U}_k^L(\xi)$ decreases with respect to k for $i = H, L$.

Lemma 5.6 *The sequence $\{\underline{v}_k^H(\xi,t), \underline{v}_k^L(\xi,t)\}_{k=0}^{\infty}$ is increasing with k and converges to $(v^H(\xi,t), v^L(\xi,t))$.*

We also establish a sequence $\{\underline{V}_k^H(\xi), \underline{V}_k^L(\xi)\}_{k=0}^{\infty}$ starting from $\underline{V}_0^L(\xi) = -1$, by the same method as in the construction of $\{\bar{V}_k^H(\xi), \bar{V}_k^L(\xi)\}_{k=0}^{\infty}$.

Lemma 5.7 *$\underline{V}_k^H(\xi)$ converges to zero uniformly for $\xi \in [\xi^H, \infty)$ and $\underline{V}_k^L(\xi)$ converges to zero uniformly for $\xi \in (-\infty, \xi^L]$, as $k \to \infty$.*

We now discuss the asymptotic behavior of $\underline{v}_k^i(\xi,t), i = H, L$ for each $k \geqslant 0$.

Lemma 5.8 *For all $k \geqslant 0$, $\underline{v}_k^H(\xi,t)$ tends to $\underline{V}_k^H(\xi)$ uniformly for $\xi \in [\xi^H, \infty)$ and $\underline{v}_k^L(\xi,t)$ tends to $\underline{V}_k^L(\xi)$ uniformly for $\xi \in (-\infty, \xi^L]$, as $t \to \infty$.*

Lemma 5.9 *For any given $\epsilon > 0$, there exists $t_1 > 0$ such that when $t > t_1$, $v^L(\xi,t) \geqslant -\epsilon$ holds uniformly for $\xi \in (-\infty, \xi^L]$; there exists $t_2 > 0$ such that when $t > t_2$, $v^L(\xi,t) \geqslant -\epsilon$ holds uniformly for $\xi \in [\xi^H, \infty)$.*

Proof. For any given $\epsilon > 0$, from Lemma 5.7 we conclude that there exits a integer K such that

$$\underline{V}_K^L(\xi) > -\frac{1}{2}\epsilon \quad \text{for} \quad \xi \in (-\infty, \xi^L]. \tag{5.11}$$

And the Lemma 5.8 implies that there exits a number t_1 such that

$$\underline{v}_K^L(\xi,t) - \underline{V}_K^L(\xi) > -\frac{1}{2}\epsilon \quad \text{for} \quad t \geqslant t_1 \quad \text{and} \quad \xi \in (-\infty, \xi^L]. \tag{5.12}$$

Noting by Lemma 5.6 that $v^L(\xi,t) \geqslant \underline{v}_k^L(\xi,t)$ in Q^L, we obtain that

$$v^L(\xi,t) \geqslant \underline{v}_K^L(\xi,t) > -\frac{1}{2}\epsilon - \frac{1}{2}\epsilon = -\epsilon \tag{5.13}$$

holds for all $\xi \in (-\infty, \xi^L], t \geqslant t_1$. □

Similarly, we can obtain that there exits a number t_2 such that

$$v^H(\xi,t) > -\epsilon \tag{5.14}$$

holds for all $\xi \in [\xi^H, \infty), t \geqslant t_2$.

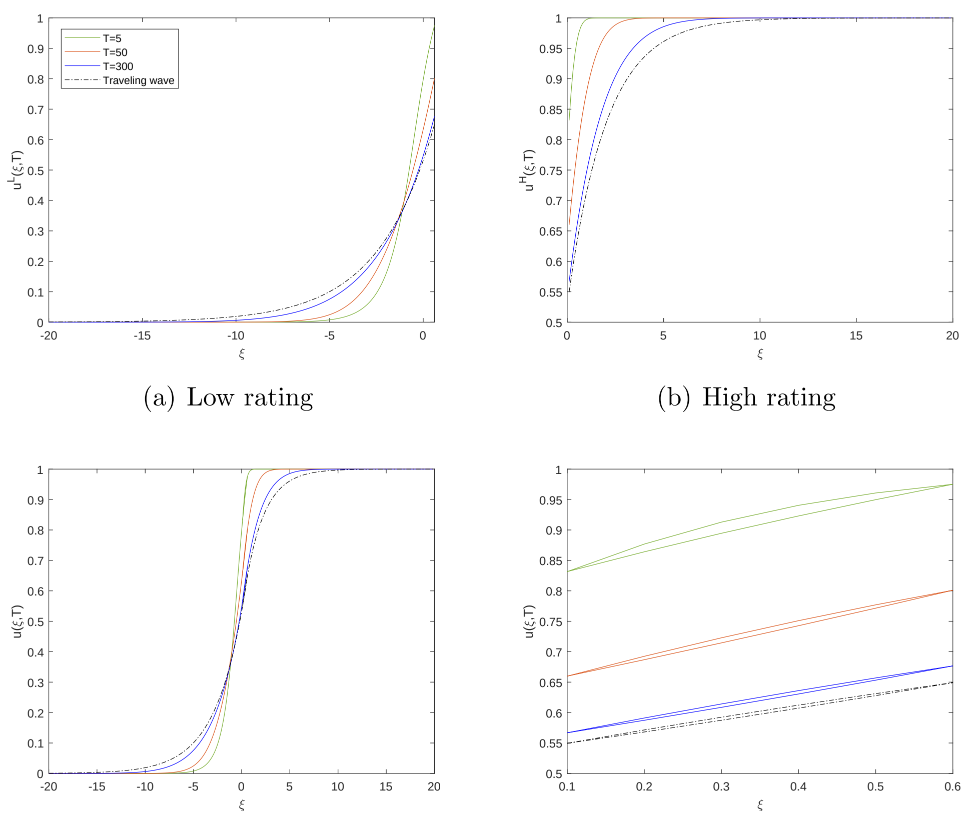
5.4 Main result

Combining Lemma (5.5) and (5.9) gives the following asymptotic behavior.

Theorem 5.1 (Main theorem) *The solution of the problem (2.9) converges the corresponding branch of the solution of traveling wave with buffer zone (4.1) in their region. That is, in the low rating, the backwards discounted pricing solution of a corporate debt with asymmetric credit migration risk $e^{rt}\phi^L(x,t)$ tends uniformly to the traveling wave $U^L(\xi)$ on $(-\infty, \xi^L]$; in the high rating, the backwards discounted pricing solution of a corporate debt with asymmetric credit migration risk $e^{rt}\phi^H(x,t)$ tends uniformly to the traveling wave $U^H(\xi)$ on $[\xi^H, \infty)$. Wherein, $\xi = x + \delta t$ with δ satisfying* (2.2).

6 Numerical results

Figure 2 presents the numerical results about the traveling wave solution. It can be seen that the backwards discounted bond value function approaches the traveling wave solution as T goes larger. Besides, we can find from Figure 2(a) that in the low rating, when ξ is smaller, the bond value $u^L(\xi,T)$ rises as the time to maturity T becomes longer; when ξ is larger, the situation is



(a) Low rating

(b) High rating

(c) Entire area

(d) Buffer area

Figure 2. Asymptotic behavior of bond pricing solution in different rating areas. The lines plot the backwards discounted bond pricing solution at different time to maturity and the traveling wave solution. It is assumed that the parameters are the same as in Figure 1.

just the opposite. From Figure 2(b), $u^H(\xi, T)$ decreases as T grows in the high rating. Figure 2(c) shows the overall trend of the value function across the entire region. Noting from Figure 2(d) that in the buffer area, the value of high-grade bonds is always greater than that of low-grade bonds, and the gap seems to narrow with the increase of T.

7 Conclusion

In this article, we analyzes a traveling wave with a buffer zone, which is approached by the solution of an asymmetric credit migration model when time goes larger.

We first introduce and solve a new traveling wave with a buffer zone, if and only if the risk discount rate is in a specific range determined by the volatilities and risk-free interest rate. Then, it is proved that the solution of the asymmetric credit migration model converges to the traveling wave as time goes larger in a particular direction, which is the main result of this article. The main technique is to construct two sets of sub and super solution sequences by monotonic iteration. As the traveling wave has an explicit solution, we can use it to approximate the bond value when time is long enough before the maturity.

The asymmetric models with a buffer are also seen in other financial studies, such as the high frequency trades with different bid and ask prices, and even in the area of engineering, e.g., the temperature control of thermostats. This study about the traveling wave with a buffer may help us to understand the widespread hysteresis phenomenon, especially in steady state, and further refine the mathematical foundation of the asymmetric models.

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Appendix A. A proof of Proposition 1

We prove the sequence is monotonic in k by induction. First, we claim that $u_1^L < u_0^L$ in Q^L. By the definition of u_0^L and the maximum principle, it is obvious that

$$u_0^L(\xi,t) < \max\{u_0^L(\xi,0), u_0^L(\xi^L,t)\} = 1, \quad \forall(\xi,t)\in Q^L.$$

Thus, $u_0^H(\xi^H,t) = u_0^L(\xi^H,t) < 1$ for $t>0$. By the definition of u_0^H and the maximum principle, we get

$$u_0^H(\xi,t) < \max\{u_0^H(\xi,0), u_0^H(\xi^H,t)\} = 1, \quad \forall(\xi,t)\in Q^H.$$

Therefore, $u_1^L(\xi^L,t) = u_0^H(\xi^L,t) < 1$ for $t>0$. And this implies $u_1^L(\xi^L,t) < u_0^L(\xi^L,t)$ for $t>0$. Using the comparison principle, we find

$$u_1^L(\xi,t) < u_0^L(\xi,t), \quad \forall(\xi,t)\in Q^L.$$

Then we make an inductive assumption that $u_k^L < u_{k-1}^L$ in $Q^L, k \geqslant 1$. According to the iterative relationship $u_k^H(\xi^H,t) = u_k^L(\xi^H,t), u_{k-1}^H(\xi^H,t) = u_{k-1}^L(\xi^H,t)$, we get $u_k^H(\xi^H,t) < u_{k-1}^H(\xi^H,t)$. By the comparison principle,

$$u_k^H(\xi,t) < u_{k-1}^H(\xi,t), \quad \forall(\xi,t)\in Q^H.$$

Also the iterative condition $u_{k+1}^L(\xi^L,t) = u_k^H(\xi^L,t), u_k^L(\xi^L,t) = u_{k-1}^H(\xi^L,t)$ implies $u_{k+1}^L(\xi^L,t) < u_k^L(\xi^L,t)$. And the comparison principle shows that

$$u_{k+1}^L(\xi,t) < u_k^L(\xi,t), \quad \forall(\xi,t)\in Q^L.$$

This completes the induction argument for the monotonicity of the sequences.

Similarly, by the minimum principle, we derive that for each integer $k \geqslant 0$, $u_k^H > 0$ in Q^H and $u_k^L > 0$ in Q^L. Thus, for each $(\xi,t)\in \overline{Q^H}$ or $(\xi,t)\in\overline{Q^L}$, the limit in 3.2 exits. One can check that the limit is a solution of the problem (2.9):

i) $\mathcal{L}^H u^H = 0$ in Q^H and $\mathcal{L}^L u^L = 0$ in Q^L.

For any given $(\xi_0,t_0)\in Q^H$, there is a neighborhood $q_T = (\xi_0-\eta,\xi_0+\eta)\times(0,T)$ in Q^H, such that $(\xi_0,t_0)\in q_T$. $\forall\varphi(\xi,t)\in C_0^\infty(q_T)$,

$$\int_{q_T}(\mathcal{L}^H u_k^H)\varphi \mathrm{d}\xi\mathrm{d}t = 0.$$

Integration by parts gives

$$\begin{aligned}
&\int_{q_T} (\mathcal{L}^H u_k^H)\varphi \mathrm{d}\xi \mathrm{d}t \\
=&\int_{\xi_0-\eta}^{\xi_0+\eta} u_k^H(\xi,T)\varphi(\xi,T) - u_k^H(\xi,0)\varphi(\xi,0)\mathrm{d}\xi - \int_{q_T} u_k^H \varphi_t \mathrm{d}\xi \mathrm{d}t \\
&- \frac{1}{2}\sigma_H^2 \int_{q_T} u_k^H \varphi_{\xi\xi}\mathrm{d}\xi \mathrm{d}t - (r-\delta-\frac{1}{2}\sigma_H^2)\int_{q_T} u_k^H \varphi_\xi \mathrm{d}\xi \mathrm{d}t.
\end{aligned}$$

Sending $k \to \infty$, we get

$$\begin{aligned}
&\int_{\xi_0-\eta}^{\xi_0+\eta} u^H(\xi,T)\varphi(\xi,T) - u^H(\xi,0)\varphi(\xi,0)\mathrm{d}\xi - \int_{q_T} u^H \varphi_t \mathrm{d}\xi \mathrm{d}t \\
&- \frac{1}{2}\sigma_H^2 \int_{q_T} u^H \varphi_{\xi\xi}\mathrm{d}\xi \mathrm{d}t - (r-\delta-\frac{1}{2}\sigma_H^2)\int_{q_T} u^H \varphi_\xi \mathrm{d}\xi \mathrm{d}t
\end{aligned}$$

Since for any $k \geqslant 0$, u_k^H satisfy the same equation with constant coefficients, by interior estimates in PDE theory, u_k^H has uniform bounded estimates for any finite derivatives in the given interior region q_T. Then it is easy to obtain that the limit u^H is smooth enough in q_T. Thus, we have

$$\int_{q_T} (\mathcal{L}^H u^H)\varphi \mathrm{d}\xi \mathrm{d}t = 0.$$

From the arbitraryness of φ and (ξ_0, t_0), it follows that $\mathcal{L}^H u^H = 0$ in Q^H. Similarly, we derive that $\mathcal{L}^L u^L = 0$ in Q^L.

ii) u^H and u^L satisfy the initial and boundary conditions.

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