Long time behavior of stochastic evolution equations

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Abstract

In this paper, we consider the long time behavior of stochastic evolution equations. The exponential, polynomial and logarithmic decay for stochastic equations are considered. Sufficient conditions are given to obtain these exponents. All the results show the noise (time diffusion) will prevent the solutions to decay in p-th moment, which coincides with the fact that the noise is a diffusion process but it will be different in the sense of almost surely, and the partial diffusion operator (spatial diffusion) will accelerate the decay of solutions.

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1 Introduction

The stability of solutions is an important issue in the theory of partial differential equations, which has been studied by many authors [23]. In the stability theory, the Lyapunov exponent is to judge the decay (increase) velocity of the solution. Thus there are a lot of works to calculate the value. We will only give a partial and incomplete survey of some parts that we feel more relevant for this paper. In view of probability theory, stability of SDEs is also important, which covers p-th moment stable, stochastic stable and almost surely stable. In our paper [19], we considered the p-th moment stable and stochastic stable, and in this paper, we consider the stochastic stable and almost surely stable.

The exponential stability of SDEs has been studied by many authors, see [1, 10, 12, 20, 21, 22], and for delayed dynamic system, one consults to [7, 14, 18, 25]. Recently, the exponential stability of stochastic partial differential equations (SPDEs) is also studied by many authors [3, 4, 5, 8, 9, 15, 17]. Liu-Mao [15] considered the following equation

$$\begin{cases} dX = f(t,X)dt + g(t,X)dW(t), & t \in [0,T], \\ X(0) = x_0, \end{cases} \tag{1.1}$$

where $T > 0$, $f(t,\cdot)$ and $g(t,\cdot)$ are families of (non-linear) operators in Hilbert spaces and $W(t)$ is a Hilbert space-valued Wiener process. They established the exponential stability of (1.1). Caraballo et al. [5] studied the exponential behaviour and stabilizability of stochastic 2D-Navier-Stokes equations. In the book [8], the long time behavior of solutions was considered in Chapter 11 and sufficient condition of mean square stable is given, see Theorem 11.14. More precisely, Da Prato- Zabczyk studied the following equation

$$\begin{cases} dX = AXdt + B(X)dW(t), \\ X(0) = x \in H, \end{cases} \tag{1.2}$$

where H is a Hilbert space, A generates a C_0 semigroup and $B \in L(H; L_2^0)$ (see p309 of [8] for more details). They proved that the following statements

(i) there exists $M > 0$, $\gamma > 0$ such that

$$\mathbb{E}|X(t,x)|^2 \leq Me^{-\gamma t}|x|^2, \quad t \geq 0;$$

(ii) for any $x \in H$ we have

$$\mathbb{E}\int_0^\infty |X(t,x)|^2 dt < \infty,$$

are equivalent.

On the other hand, not all the stochastic systems are exponentially stable, such as polynomial or logarithmic one. In [16], Liu-Chen considered the moment decay rates of solutions of SDEs, where the polynomial decay and logarithmic decay are studied, also see [13]. In this paper, we will give some sufficient conditions to make sure the solutions decay polynomially to a class of stochastic evolution equations (SEEs) with nonlinear terms and partial differential operators, which will cover the cases (1.1) and (1.2). Moreover, we will prove that the additive noise will prevent the decay of solutions and the partial diffusion operator (spatial diffusion) will accelerate the decay of solutions. Some examples are given to illustrate our main results.

2 Main Results

In the section, we will use the Lyapunov functional method to consider the following SPDE:

$$\begin{cases} dX(t) = [AX(t) + f(t,X(t))]dt + g(t,X(t))dW(t), & t > 0, \\ X(0) = x \in H, \end{cases} \tag{2.1}$$

where $f : V \to V'$, $g : [0,\infty) \times V \to L(K,H)$ are continuous functions. The spaces V, H are Hilbert spaces satisfying

$$V \subset H \equiv H' \subset V',$$

where injections are dense, continuous, and compact. Let $\beta_n(t)$ $(n = 1, 2, 3, \cdots)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$. Set

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \quad t \geq 0,$$

where $\lambda_n \geq 0$ $(n = 1, 2, 3, \cdots)$ are nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\{e_n\}$ $(n = 1, 2, 3, \cdots)$ is a complete orthonormal basis in the real and separable Hilbert space K. Let $Q \in L(K, K)$ be the operator defined by $Qe_n = \lambda_n e_n$. The above K-valued stochastic process $W(t)$ is called a Q-Wiener process. And set

$$\|g(t,x)\|^2_{L_2^0} = tr(g(t,x)Qg(t,x)^*).$$

Denote $|u|^2 = (u,u)_H$ and $\|u\|^2 = (u,u)_V$. Firstly, we give the notion on strong solutions for (2.1).

Definition 2.1 *A stochastic process $X(t)$, $t \geq 0$ is said to be a strong solution of (2.1) if*
(i) $X(t)$ is $\mathcal{F}_t$-adapted;
(ii) $X(\cdot) \in C([0,T]; H) \cap L^2(0,T;V)$ almost surely for all $T > 0$;
(iii) As an identity in V', the following equation

$$X(t) = X(0) + \int_0^t [AX(s) + f(s, X(s))]ds + \int_0^t g(s, X(s))dW(s)$$

holds almost surely for $t \in [0,T]$.

Assume $\lambda_1 > 0$ is the first eigenvalue of A satisfying $|v|^2 \leq \lambda_1^{-1}\|v\|^2$, $\forall\ v \in V$. Since the injections $V \subset H \equiv H' \subset V'$ are dense, continuous, and compact, under some assumptions, the existence of strong solution of (2.1) has been obtained in [6, 8], for example [6, Theorem 6.5 Chapter 3].

Let $C^{1,2}([0,\infty) \times H, \mathbb{R}_+)$ denote the space of all $\mathbb{R}_+$-valued functions Ψ defined on $[0,\infty) \times H$ with the following properties:

(i) $\Psi(t,x)$ is differentiable in $t \in [0,\infty)$ and twice Frechet differential in x with $\partial_t\Psi(t,\cdot)$, $\partial_x\Psi(t,\cdot)$ and $\partial_{xx}\Psi(t,\cdot)$ are locally bounded on H, and $\Psi(t,\cdot)$, $\partial_t\Psi(t,\cdot)$, $\partial_x\Psi(t,\cdot)$ are continuous on H;

(ii) For all trace class operators R, $tr(\partial_{xx}\Psi(t,\cdot)R)$ is continuous from H into $\mathbb{R}$;

(iii) $\partial_x\Psi(t,v) \in V$ for all $v \in V$, and $v \to \langle\partial_x\Psi(t,v), v^*\rangle$ is continuous for each $v^* \in V'$;

(iv) $\|\partial_x\Psi(t,v)\| \leq C_0(t)(1 + \|v\|)$ for all $v \in V$, where $C_0(t) > 0$.

We recall the well-known Itô formula, see [5, Theorem 2.1]. If the stochastic process $X(t)$ is a strong solution of (2.1), then it holds that

$$\Psi(t, X(t)) = \Psi(0, X(0)) + \int_0^t L\Psi(s, X(s))ds + \int_0^t (\partial_x\Psi(s, X(s)), g(s, X(s)))dW(s),$$

where

$$\begin{aligned} L\Psi(s, X(s)) \quad &= \quad \partial_t\Psi(s, X(s)) + \langle AX(s) + f(s, X(s)), \partial_x\Psi(s, X(s))\rangle \\ &\quad +\frac{1}{2}tr(\partial_{xx}\Psi(s, X(s))g(s, X(s))Qg(s, X(s))^*). \end{aligned}$$

Throughout this paper, we assume x_∞ is a solution of limited equation (2.1). That is to say, x_∞ satisfies

$$AX + f^*(X) = 0, \tag{2.2}$$

where $f(t,x) \to f^*(x)$ in V' as $t \to \infty$ for any $x \in V$.

Definition 2.2 *Assume that $\lambda(t) \uparrow +\infty$, as $t \to +\infty$, and satisfies $\lambda(t+s) \leq \lambda(t)\lambda(s)$ for $s, t \in \mathbb{R}_+$ largely enough. We say a strong solution $X(t)$ of (2.1) converges to $x_\infty \in V$ in p-th moment with decay $\lambda(t)$ if there exist positive constants μ and C such that*

$$\mathbb{E}|X(t) - x_\infty|^p \leq C(x_0)\lambda(t)^{-\mu}, \quad t \geq 0$$

holds for any $X_0 = x_0 \in H$, an $\mathcal{F}_0$-measurable random vector, or equivalently,

$$\limsup_{t\to+\infty} \frac{\log\left(\mathbb{E}|X(t) - x_\infty|^p\right)}{\log \lambda(t)} \leq -\mu.$$

Apart from the decay of p-th moment, we will consider the convergence almost surely.

Definition 2.3 *Assume that $\lambda(t) \uparrow +\infty$, as $t \to +\infty$, and satisfies $\lambda(t+s) \leq \lambda(t)\lambda(s)$ for $s, t \in \mathbb{R}_+$ largely enough. We say a strong solution $X(t)$ of (2.1) converges to $x_\infty \in V$ almost surely with decay $\lambda(t)$ if there exist positive constant μ such that*

$$\lim_{t\to\infty} \frac{\log|X(t) - x_\infty|}{\log \lambda(t)} \leq -\mu, \quad t \geq 0, \quad \text{almost surely}.$$

In order to get the main results, we need the following Gronwall lemma, which is a generalization of the well-known Gronwall lemma [11, Page 9].

Lemma 2.1 *Let x, ψ and ϕ be real continuous positive functions defined on $[a,b]$. If*

$$x(t) \leq \psi(t) + \int_a^t \phi(s)x(s)ds, \quad t \in [a,b],$$

then

$$x(t) \leq \psi(t) + \int_a^t \phi(s)\psi(s)\exp\left(\int_s^t \phi(r)dr\right)ds, \quad t \in [a,b].$$

Assume further that ψ is differentiable and increasing function and $\phi \geq 0$, then

$$x(t) \leq \psi(t)\exp\left(\int_a^t \phi(r)dr\right)ds, \quad t \in [a,b].$$

Proof. The former part is classical and we omit the proof. Now we prove the latter part. We first claim

$$\begin{aligned}
&\psi(t) + \int_a^b \phi(s)\psi(s)\exp\left(\int_s^t \phi(r)dr\right)ds \\
= \;& \psi(a)\exp\left(\int_a^t \phi(r)dr\right) + \int_a^t \psi'(s)\exp\left(\int_s^t \phi(r)dr\right)ds.
\end{aligned}$$

Indeed, by integrating by parts, we have

$$\begin{aligned} & \int_a^t \psi'(s) \exp\left(\int_s^t \phi(r)dr\right) ds \\ = \;& \psi(s) \exp\left(\int_s^t \phi(r)dr\right)\Big|_a^t - \int_a^t \psi(s) \frac{d}{ds} \exp\left(\int_s^t \phi(r)dr\right) ds \\ = \;& \psi(t) - \psi(a) \exp\left(\int_a^t \phi(r)dr\right) + \int_a^t \psi(s) \exp\left(\int_s^t \phi(r)dr\right) \phi(s)ds. \end{aligned}$$

Thus we prove the claim. And thus we obtain

$$x(t) \leq \psi(a) \exp\left(\int_a^t \phi(r)dr\right) + \int_a^t \psi'(s) \exp\left(\int_s^t \phi(r)dr\right) ds, \quad t \in [a, b].$$

It follows from the assumptions that

$$\begin{aligned} x(t) \;&\leq\; \psi(a) \exp\left(\int_a^t \phi(r)dr\right) + \exp\left(\int_a^t \phi(r)dr\right) \int_a^t \psi'(s)ds \\ &=\; \psi(t) \exp\left(\int_a^t \phi(r)dr\right), \quad t \in [a, b]. \end{aligned}$$

The proof is complete. □

The first result is similar to [16, Theorem 1.1], where the stability of stochastic ordinary differential equations is obtained and here we consider the stability of stochastic partial differential equations.

Theorem 2.1 *Let $\Psi \in C^{1,2}([0, \infty) \times H, \mathbb{R}_+)$ and let ψ_1, ψ_2 be two continuous non-negative functions on $\mathbb{R}_+$. Assume that for all $x \in H$ and $t \in \mathbb{R}_+$, there exist positive constants $p > 0$, $m > 0$ and real numbers ν, θ such that*

(1) $\lambda^m(t)|x|^p \leq \Psi(t, x)$, for all $t \in \mathbb{R}_+$ and $x \in H$;

(2) $L\Psi(t, x) \leq \psi_1(t) + \psi_2(t)\Psi(t, x)$, for all $t \in \mathbb{R}_+$ and $x \in H$;

(3) $\limsup_{t\to\infty} \dfrac{\log\left(\int_0^t \psi_1(s)ds\right)}{\log \lambda(t)} \leq \nu$, $\limsup_{t\to\infty} \dfrac{\int_0^t \psi_2(s)ds}{\log \lambda(t)} \leq \theta$.

Then the solution to (2.1) decays in the p-th moment to zero with decay $\lambda(t)$. Moreover, we have

$$\limsup_{t\to+\infty} \frac{\log\left(\mathbb{E}|X(t)|^p\right)}{\log \lambda(t)} \leq -\mu,$$

where $\mu = m - \theta - \nu > 0$.

Proof. The proof is similar to that of [16, Theorem 1.1] and we only give the outline of proof. By Itô's formula and the definition of L, we can derive that

$$\Psi(t, X(t)) = \Psi(0, X(0)) + \int_0^t L\Psi(s, X(s))ds + \int_0^t (\partial_x \Psi(s, X(s)), g(s, X(s)))dW(s). \tag{2.3}$$

Note that the last term of right hand side of (2.3) is a continuous martingale, thus taking the expectation on both sides of (2.3) yields that

$$\begin{aligned}\mathbb{E}\Psi(t, X(t)) &= \mathbb{E}\Psi(0, X(0)) + \int_0^t \mathbb{E}[L\Psi(s, X(s))]ds \\ &\leq \mathbb{E}\Psi(0, X(0)) + \int_0^t [\psi_1(s) + \psi_2(s)\mathbb{E}\Psi(s, X(s))]ds.\end{aligned}$$

The Lemma 2.1 implies that

$$\mathbb{E}\Psi(t, X(t)) \leq \left[\mathbb{E}\Psi(0, X(0)) + \int_0^t \psi_1(s)ds\right] \exp\left(\int_0^t \psi_2(s)ds\right).$$

Consequently, we have

$$\log(\mathbb{E}\Psi(t, X(t))) \leq \log\left[\mathbb{E}\Psi(0, X(0)) + \int_0^t \psi_1(s)ds\right] + \int_0^t \psi_2(s)ds.$$

By using the assumptions (2) and (3) and the property of limit, we have for any $\varepsilon > 0$, $t > 0$ large enough implies that

$$\log(\mathbb{E}\Psi(t, X(t))) \leq \log\left[\mathbb{E}\Psi(0, X(0)) + \lambda(t)^{\nu+\varepsilon}\right] + \log[\lambda(t)^{\theta+\varepsilon}].$$

That is to say,

$$\limsup_{t\to+\infty} \frac{\log(\mathbb{E}\Psi(t, X(t)))}{\log \lambda(t)} \leq \nu + \varepsilon + \theta.$$

Letting $\varepsilon \to 0$ and using assumption (1), we have

$$\limsup_{t\to+\infty} \frac{\log(\mathbb{E}|X(t)|^p)}{\log \lambda(t)} \leq \limsup_{t\to+\infty} \frac{\log(\lambda(t)^{-m}\mathbb{E}\Psi(t, X(t)))}{\log \lambda(t)} \leq -[m - \nu - \theta].$$

The proof is complete. □

The above result can be regarded as the stability of trivial solution 0. Next, we consider the stability of non-trivial solution. Assume that x_∞ is a solution of limited equation (2.2).

Theorem 2.2 *Assume that there exist positive constants ν, θ and $m > \nu + \theta$ such that f and g satisfy the following conditions*

$$\begin{aligned}&(x - x_\infty, f(t, x) - f^*(x_\infty)) \leq \beta_0(t)|x - x_\infty| + \beta_1(t)|x - x_\infty|^2, \quad t \geq 0,\ x \in H; \\ &\|g(t, x)\|^2_{L_2^0} \leq \gamma_1(t) + \gamma_2(t)|x - x_\infty|^2, \quad t \geq 0,\ x \in H,\end{aligned}$$

where $\beta_i(t)$ and $\gamma_i(t)$, $i = 1, 2$ are positive functions satisfying

$$\limsup_{t\to\infty} \frac{\log\left(\int_0^t \lambda(s)^m(\beta_0(s) + \gamma_1(s))ds\right)}{\log \lambda(t)} \leq \nu,$$

$$\limsup_{t\to\infty} \frac{\int_0^t \left(-2\lambda_1 + 2\beta_1(s) + \beta_0(s) + \gamma_2(s) + \frac{m\lambda'(s)}{\lambda(s)}\right) ds}{\log \lambda(t)} \leq \theta.$$

Then the solution of (2.1) converges to x_∞ in the mean square with decay $\lambda(t)$, i.e.

$$\limsup_{t\to+\infty} \frac{\log\left(\mathbb{E}|X(t) - x_\infty|^2\right)}{\log \lambda(t)} \leq -\mu,$$

where $\mu = m - \theta - \nu > 0$.

Proof. By using the Itô formula, we have

$$\begin{aligned}\lambda(t)^m\mathbb{E}|X(t)-x_\infty|^2 &= \lambda(0)^m\mathbb{E}|x-x_\infty|^2 + m\int_0^t \lambda(s)^{m-1}\lambda'(s)\mathbb{E}|X(s)-x_\infty|^2 ds\\ &\quad +2\int_0^t \lambda(s)^m\mathbb{E}(X(s)-x_\infty, AX(s)-Ax_\infty)ds\\ &\quad +2\int_0^t \lambda(s)^m\mathbb{E}(X(s)-x_\infty, f(s,X(s))-f^*(x_\infty))ds\\ &\quad +\int_0^t \lambda(s)^m\mathbb{E}\|g(s,X(s))\|^2_{L_2^0}ds.\end{aligned}$$

By using $|v|^2 \le \lambda_1^{-1}\|v\|^2$ and the assumptions, we get

$$\begin{aligned}\lambda(t)^m\mathbb{E}|X(t)-x_\infty|^2 &\le \lambda(0)^m\mathbb{E}|x-x_\infty|^2 + m\int_0^t \lambda(s)^{m-1}\lambda'(s)\mathbb{E}|X(s)-x_\infty|^2 ds\\ &\quad -2\int_0^t \lambda(s)^m\mathbb{E}\|X(s)-x_\infty\|^2 ds + 2\int_0^t \beta_1(s)\lambda(s)^m\mathbb{E}|X(s)-x_\infty|^2 ds\\ &\quad +2\int_0^t \beta_0(s)\lambda(s)^m\mathbb{E}|X(s)-x_\infty| ds\\ &\quad +\int_0^t \lambda(s)^m\mathbb{E}[\gamma_1(s)+\gamma_2(s)|X(s)-x_\infty|^2]ds\\ &\le \lambda(0)^m\mathbb{E}|x-x_\infty|^2 + \int_0^t \lambda(s)^m\gamma_1(s)ds + 2\int_0^t \beta_0(s)\lambda(s)^m\mathbb{E}|X(s)-x_\infty| ds\\ &\quad +\int_0^t \lambda(s)^m\left(\frac{m\lambda'(s)}{\lambda(s)} - 2\lambda_1 + 2\beta_1(s)+\gamma_2(s)\right)\mathbb{E}|X(s)-x_\infty|^2 ds.\end{aligned}$$

By using Hölder's inequality, we have

$$\begin{aligned}&2\int_0^t \beta_0(s)\lambda(s)^m\mathbb{E}|X(s)-x_\infty| ds\\ \le\ & 2\int_0^t \beta_0(s)\lambda(s)^m\left(\mathbb{E}|X(s)-x_\infty|^2\right)^{\frac{1}{2}} ds\\ \le\ & 2\left(\int_0^t \beta_0(s)\lambda(s)^m ds\right)^{\frac{1}{2}}\left(\int_0^t \beta_0(s)\lambda(s)^m\mathbb{E}|X(s)-x_\infty|^2 ds\right)^{\frac{1}{2}}\\ \le\ & \int_0^t \beta_0(s)\lambda(s)^m ds + \int_0^t \beta_0(s)\lambda(s)^m\mathbb{E}|X(s)-x_\infty|^2 ds.\end{aligned}$$

The Lemma 2.1 and the above inequality imply that

$$\begin{aligned}\lambda(t)^m\mathbb{E}|X(t)-x_\infty|^2 &\le \left(\lambda(0)^m\mathbb{E}|x-x_\infty|^2 + \int_0^t \lambda(s)^m(\beta_0(s)+\gamma_1(s))ds\right)\\ &\quad \times\exp\left(\int_0^t \left[\frac{m\lambda'(s)}{\lambda(s)} - 2\lambda_1 + 2\beta_1(s) + \beta_0(s) + \gamma_2(s)\right]ds\right).\end{aligned}$$

Consequently, we have

$$\log(\lambda(t)^m\mathbb{E}|X(t)-x_\infty|^2)$$

$$
\begin{aligned}
&\leq \ \log\left[\lambda(0)^m\mathbb{E}|x-x_\infty|^2+\int_0^t\lambda(s)^m(\beta_0(s)+\gamma_1(s))ds\right]\\
&\quad+\int_0^t\left[\frac{m\lambda'(s)}{\lambda(s)}-2\lambda_1+2\beta_1(s)+\beta_0(s)+\gamma_2(s)\right]ds.
\end{aligned}
$$

By using the assumptions and the property of limit, we have for any $\varepsilon>0$, $t>0$ large enough implies that

$$
\log(\lambda(t)^m\mathbb{E}|X(t)-x_\infty|^2)\leq\log\left[\lambda(0)^m\mathbb{E}|x-x_\infty|^2+\lambda(t)^{\nu+\varepsilon}\right]+\log[\lambda(t)^{\theta+\varepsilon}].
$$

That is to say,

$$
\limsup_{t\to+\infty}\frac{\log(\lambda(t)^m\mathbb{E}|X(t)-x_\infty|^2)}{\log\lambda(t)}\leq\nu+\varepsilon+\theta+\varepsilon.
$$

Letting $\varepsilon\to 0$, we have

$$
m+\limsup_{t\to+\infty}\frac{\log(\mathbb{E}|X(t)-x_\infty|^2}{\log\lambda(t)}\leq\limsup_{t\to+\infty}\frac{\log(\lambda(t)^m\mathbb{E}|X(t)-x_\infty|^2)}{\log\lambda(t)}\leq\nu+\theta.
$$

Thus we get the desired result. $\square$

Remark 2.1 *In Theorem 2.2, the function $\beta_1(t)<0$ will play an important role. Obviously, if we take $\beta_0=0$ and $\beta_1(t)\equiv\beta=\lambda_1$ (constant), then*

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{\int_0^t\left(-2\lambda_1+2\beta_1(s)+\beta_0(s)+\gamma_2(s)+\frac{m\lambda'(s)}{\lambda(s)}\right)ds}{\log\lambda(t)}\\
=\ &\limsup_{t\to\infty}\frac{\int_0^t\left(\gamma_2(s)+\frac{m\lambda'(s)}{\lambda(s)}\right)ds}{\log\lambda(t)}\\
\geq\ &\limsup_{t\to\infty}\frac{\int_0^t\left(\frac{m\lambda'(s)}{\lambda(s)}\right)ds}{\log\lambda(t)}=m.
\end{aligned}
$$

Consequently, $\mu=0$. If $\beta_1(t)<0$, there will help the stability for x_∞, see Example 2.3. This theorem covers the results of SDEs.

Theorem 2.3 *Assume the hypotheses in Theorem 2.2 hold. Assume further that the function $\gamma_1(t)$ is a positive bounded function on any finite interval and*

$$
\int_0^\infty\gamma_2(t)dt<\infty,\quad\int_0^\infty\lambda(t)^{-\frac{\mu}{4}}dt<\infty,\quad\int_0^\infty\frac{-2\lambda_1+2\beta_1(t)+\beta_0(t)}{\lambda(t)^\mu}dt<\infty. \tag{2.4}
$$

Then there exist positive constants M, $\varepsilon>0$ and a subset $\Omega_0\subset\Omega$ with $\mathbb{P}(\Omega_0)=0$ such that, for each $\omega\notin\Omega_0$ there exists a positive random number $T(\omega)$ such that

$$
|X(t)-x_\infty|\leq\frac{M}{\lambda(t)^\varepsilon},\quad\forall\ t\geq T(\omega).
$$

Proof. It follows from the assumptions of Theorem 2.2 that

$$\limsup_{t\to\infty}\frac{\int_0^t \lambda(s)^m\gamma_1(s)ds}{\lambda(t)^\nu}\le C.$$

Noting that $\lambda(t+s)\le\lambda(t)\lambda(s)$ for $s,t\in\mathbb{R}_+$ largely enough, we have there exists $T_0>0$ such that

$$\lambda(nT_0)\le\lambda((n-1)T_0)\lambda(T_0)\le\lambda((n-2)T_0)\lambda^2(T_0)\le\lambda^n(T_0).$$

It follows from $\lambda(t)\to\infty$ as $t\to\infty$ that $\lambda(T_0)>1$.

$$\lim_{t\to\infty}\frac{\int_0^t \lambda(s)^m\gamma_1(s)ds}{\lambda(t)^\nu}=\lim_{t\to\infty}\frac{\lambda(t)^m\gamma_1(t)}{\nu\lambda(t)^{\nu-1}\lambda'(t)},$$

thus without loss of generality we assume that the following inequality holds for $t\ge T_0$,

$$\gamma_1(t)\le C\nu\frac{\lambda'(t)}{\lambda(t)^{m-\nu+1}}.$$

Consequently, the property of $\gamma_1(t)$ implies that

$$\int_0^\infty\gamma_1(t)dt=\int_0^{T_0}\gamma_1(t)dt+\int_{T_0}^\infty\gamma_1(t)dt\le C<\infty.$$

Together with the assumption (2.4), it is not hard to prove that

$$\int_0^T\mathbb{E}\|g(t,X(t))\|^2_{L_2^0}dt\le C<\infty,\quad\forall\,T>0.$$

Next, we prove that there exists a positive constant $M>0$ such that

$$\mathbb{E}\left(\sup_{0\le t<\infty}|X(t)-x_\infty|^2\right)\le M. \tag{2.5}$$

Itô's formula implies that

$$\begin{aligned}|X(t)-x_\infty|^2 &= |x-x_\infty|^2+2\int_0^t(X(s)-x_\infty,AX(s)-Ax_\infty)ds\\ &\quad+2\int_0^t(X(s)-x_\infty,f(s,X(s))-f(x_\infty))ds\\ &\quad+\int_0^t\|g(s,X(s))\|^2_{L_2^0}ds+2\int_0^t(X(s)-x_\infty,g(s,X(s))dW(s)).\end{aligned}$$

Following Burkholder-Davis-Gundy's inequality, we get for any $T>0$

$$\begin{aligned}&2\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t(X(s)-x_\infty,g(s,X(s))dW(s))\right|\right]\\ \le\;& C_1\mathbb{E}\left[\left(\int_0^T|X(s)-x_\infty|^2\|g(s,X(s))\|^2_{L_2^0}ds\right)^{1/2}\right]\\ \le\;& C_1\mathbb{E}\left\{\sup_{0\le s\le T}|X(s)-x_\infty|\left[\int_0^T\|g(s,X(s))\|^2_{L_2^0}ds\right]^{1/2}\right\}\\ \le\;& \frac12\mathbb{E}\left[\sup_{0\le s\le T}|X(s)-x_\infty|^2\right]+C_2\int_0^T\mathbb{E}\|g(s,X(s))\|^2_{L_2^0}ds,\end{aligned}$$

where C_i, $i=1,2$ are positive constants. The above inequality and the assumptions yield that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le T}|X(s)-x_\infty|^2\right] &\le \mathbb{E}|x-x_\infty|^2+\int_0^T(\beta_0(s)+\gamma_1(s))ds\\
&\quad+\int_0^T\left(-2\lambda_1+2\beta_1(s)+\beta_0(s)+\gamma_2(s)\right)\mathbb{E}|X(s)-x_\infty|^2ds\\
&\quad+\frac{1}{2}\mathbb{E}\left[\sup_{0\le s\le T}|X(s)-x_\infty|^2\right]+C_2\int_0^T\mathbb{E}\|g(s,X(s))\|^2_{L_2^0}ds.
\end{aligned}
$$

This proves (2.5). Meanwhile, we have

$$
\begin{aligned}
|X(T)-x_\infty|^2 &\le |X(N)-x_\infty|^2+\int_N^T(\beta_0(s)+\gamma_1(s))ds\\
&\quad+\int_N^T\left(-2\lambda_1+2\beta_1(s)+\beta_0(s)+\gamma_2(s)\right)|X(s)-x_\infty|^2ds\\
&\quad+\sup_{N\le t\le T}\left|\int_N^t(X(s)-x_\infty,g(s,X(s))dW(s))\right|,
\end{aligned}
$$

for $T\ge N$, where N is a natural number. Taking $N\in\mathbb{N}$ large enough, we obtain

$$
\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[N,N+1]}|X(t)-x_\infty|^2\ge\varepsilon_N^2\right\}\\
\le\ &\mathbb{P}\{|X_N-x_\infty|^2\ge\varepsilon_N^2/4\}\\
&+\mathbb{P}\left\{2\int_N^{N+1}\left(-2\lambda_1+2\beta_1(s)+\beta_0(s)+\gamma_2(s)\right)|X(s)-x_\infty|^2ds\ge\varepsilon_N^2/4\right\}\\
&+\mathbb{P}\left\{\sup_{N\le t\le N+1}\left|\int_N^t(X(s)-x_\infty,g(s,X(s))dW(s))\right|\ge\varepsilon_N^2/4\right\},
\end{aligned}
$$

where $\varepsilon_N^2=C\lambda(N)^{-\frac{\mu}{4}}$. Kolomogorov's inequality [2] and (2.4) imply that

$$
\mathbb{P}\{|X_N-x_\infty|^2\ge\varepsilon_N^2/4\}\le C\frac{\mathbb{E}|X_N-x_\infty|^2}{\varepsilon_N^2}\le C\lambda(N)^{-\frac{\mu}{4}},
$$

and

$$
\begin{aligned}
&\mathbb{P}\left\{2\int_N^{N+1}\left(-2\lambda_1+2\beta_1(s)+\beta_0(s)+\gamma_2(s)\right)|X(s)-x_\infty|^2ds\ge\varepsilon_N^2/4\right\}\\
\le\ &\frac{C}{\varepsilon_N^2}\int_N^{N+1}\left(-2\lambda_1+2\beta_1(s)+\beta_0(s)+\gamma_2(s)\right)\mathbb{E}|X(s)-x_\infty|^2ds\\
\le\ &C\lambda(N)^{-\frac{\mu}{4}}.
\end{aligned}
$$

Direct calculus shows that

$$
\int_N^{N+1}(\beta_0(s)+\gamma_1(s))ds\le C\nu\int_N^{N+1}\frac{\lambda'(t)}{\lambda(t)^{m-\nu+1}}dt\le C\lambda(N)^{-\mu}.
$$

Similar to the proof of $\int_0^T \mathbb{E}\|g(t,X(t))\|_{L_2^0}^2 dt \le C$, we obtain

$$\mathbb{P}\left\{\sup_{N\le t\le N+1}\Big|\int_N^t (X(s)-x_\infty, g(s,X(s))dW(s))\Big| \ge \varepsilon_N^2/4\right\} \le C\lambda(N)^{-\frac{\mu}{4}}.$$

Combining the above discussions, we have

$$\mathbb{P}\left\{\sup_{t\in[N,N+1]}|X(t)-x_\infty|^2 \ge \varepsilon_N^2\right\} \le C\lambda(N)^{-\frac{\mu}{4}}.$$

Finally, a Bore-Cantelli's lemma-type argument completes the proof. □

Remark 2.2 *We can not get (2.4) under the assumptions of Theorem 2.2. For example, if $\lambda_1 > \beta$ then we can let $\gamma_2(t) = \frac{1}{1+t}$, $\mu = 1$ and $\lambda(t) = \log(1+t)$. Consequently, we have*

$$\int_0^\infty \frac{\gamma_2(t)}{\lambda(t)^\mu}dt = \int_0^\infty \frac{1}{(1+t)\log(1+t)}dt = \infty.$$

Even though $\lambda(t) = t$ and $x_\infty = 0$, the assumptions of Theorem 2.3 is weaker than those of [4, Theorem 2.3].

It follows from the assumptions of Theorems 2.2 and 2.3 that noise may be not stabilize the solution. In the following, we want to show the noise can stabilize the solution. For simplicity and inspired by [5], we assume that (I): $x_\infty = 0$, $f(t,0) = g(t,0) = 0$, $W(t)$ is a one-dimensional Wiener process and

$$|f(t,x)| \le \beta(t)|x|, \quad g(t,x) = \gamma(t)x, \quad t \ge 0, \ x \in H.$$

Theorem 2.4 *Assume that the condition (I) holds. If*

$$\frac{\int_0^t(-2\lambda_1 + 2\beta(s) - \gamma(s)^2)ds}{\log\lambda(t)} \le -2\mu, \quad \frac{\int_0^t \gamma(s)^2 ds}{(\log\lambda(t))^2} \to 0, \quad \text{as } t\to\infty,$$

then there exist positive constants M, $\varepsilon > 0$ and a subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 0$ such that, for each $\omega \notin \Omega_0$ there exists a positive random number $T(\omega)$ such that

$$|X(t)|^2 \le \frac{M}{\lambda(t)^\mu}, \quad \forall\ t \ge T(\omega).$$

Proof. The Itô formula and the assumptions imply that

$$\begin{aligned}
\log|X(t)|^2 &= \log|x|^2 + 2\int_0^t \frac{1}{|X(s)|^2}[(X(s),AX(s)) + (X(s),f(X(s)))]ds \\
&\quad - \int_0^t \frac{\gamma(s)^2|X(s)|^4}{|X(s)|^4}ds + 2\int_0^t \frac{(X(s),g(s,X(s)))}{|X(s)|^2}dW(s) \\
&\le \log|x|^2 + \int_0^t(-2\lambda_1 + 2\beta(s) - \gamma(s)^2)ds + 2\int_0^t \gamma(s)dW(s) \\
&\le \log|x|^2 + \log[\lambda(t)^{-2\mu}] + 2\int_0^t \gamma(s)dW(s).
\end{aligned}$$

Note that for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{P}\left\{\frac{2}{\log \lambda(t)}\left|\int_0^t \gamma(s)dW(s)\right| \geq \varepsilon\right\} \\
\leq\ & \frac{1}{\varepsilon^2}\frac{4}{(\log \lambda(t))^2}\mathbb{E}\left[\int_0^t \gamma(s)dW(s)\right]^2 \\
\leq\ & \frac{4}{\varepsilon^2}\frac{\int_0^t \gamma(s)^2 ds}{(\log \lambda(t))^2} \to 0.
\end{aligned}$$

Therefore, we can find a subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 0$ such that, for each $\omega \notin \Omega_0$ there exists a positive random number $T(\omega)$ such that for all $t \geq T(\omega)$

$$2\int_0^t \gamma(s)dW(s) \leq \log[\lambda(t)^{\mu}].$$

Consequently, it holds that for all $t \geq T(\omega)$

$$\log |X(t)|^2 \leq \log |x|^2 + \log[\lambda^{-\mu}(t)].$$

This completes the proof. $\square$

Remark 2.3 *Theorem 2.4 shows that the noise has stabilizing effect on stability. For example, let*

$$\beta(t) = \lambda_1 + \frac{1}{t+1}, \quad \gamma(t) = \frac{2}{\sqrt{t+1}}.$$

Then it is to see that if there is no noise, we have

$$\begin{aligned}
\log |X(t)|^2 &= \log |x|^2 + 2\int_0^t \frac{1}{|X(s)|^2}[(X(s), AX(s)) + (X(s), f(X(s)))]ds \\
&\leq \log |x|^2 + \int_0^t (-2\lambda_1 + 2\beta(s))ds \\
&= \log |x|^2 + 2\log[1+t].
\end{aligned}$$

Hence we can not get stability of null solution. However, direct calculation shows that for $\lambda(t) = 1+t$

$$\frac{\int_0^t(-2\lambda_1 + 2\beta(s) - \gamma(s)^2)ds}{\log \lambda(t)} = -2, \quad \frac{\int_0^t \gamma(s)^2 ds}{(\log \lambda(t))^2} = \frac{4}{\log(1+t)} \to 0, \quad \text{as } t \to \infty,$$

But it follows from Theorem 2.4 that the solution of (2.1) is exponentially stable almost surely.

Moreover, if we consider the following stochastic differential equation

$$\begin{cases} dX(t) = \frac{1}{1+t}X(t)dt + \frac{2p}{\sqrt{1+t}}X(t)dW(t), \\ X(0) = x, \end{cases}$$

where p is a constant, then we have the following results:

(i) if $p = 0$, then $X(t) = x(1+t)$, which shows that the solution of ordinary differential equation does not decay;

(ii) if $p \geq 1$, it follows from Theorem 2.4 that the solution of stochastic ordinary differential equations will decay polynomially almost surely. In fact, simple calculations show that

$$X(t) = \frac{x}{(1+t)^{2p^2-1}} \exp\left(2p \int_0^t \frac{1}{\sqrt{1+s}} dW(s)\right),$$

which verifies Theorem 2.4.

Comparing with Theorems 2.1 and 2.4, we see that the decay index of the solution to the stochastic evolutional equations will be lower than that of deterministic evolutional equations in p-th moment sese, but in the sense of almost surely, the results will be entirely different. Similar to Theorem 2.4, Zhu et al. [24] obtained the results of stochastic delayed differential equations.

Example 2.1: Consider the following stochastic partial differential equations

$$\begin{cases} du = [\Delta u + (\beta - \frac{p}{1+t})u]dt + (1+t)^{-p} dW(t), \ x \in D, \ t > 0, \\ u|_{\partial D} = 0, \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in D, \end{cases} \tag{2.6}$$

where $D \subset \mathbb{R}^n$ and $W(t)$ is a one-dimensional Brownian motion. It follows from the results of [6] that (2.6) admits a unique strong solution. Let

$$\Psi(t,x) = (1+t)^{2p}|x|^2,$$

then it is easy to show that

$$L\Psi(t,u) \leq 2(-\lambda_1 + \beta)(1+t)^{2p}|u|^2 + 1.$$

Theorem 2.1 implies that if $-\lambda_1 + \beta \leq 0$ and $p > 1/2$, then the solution of (2.6) is the second moment stable with polynomial decay. Moreover, we have

$$\limsup_{t\to\infty} \frac{1}{\log t} \log \mathbb{E}|u(t,\cdot)|^2 \leq -(2p-1).$$

Example 2.2: Consider the following stochastic partial differential equations

$$\begin{cases} du = [\alpha \Delta u - \frac{p}{1+t}u]dt + (1+t)^{-p} u dW(t), \ x \in D \ t > 0, \\ u|_{\partial D} = 0, \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in D, \end{cases} \tag{2.7}$$

where D and $W(t)$ are the same as in (2.6). It also follows from the results of [6] that (2.7) admits a unique strong solution. If $\alpha = 0$, that is to say, equation (2.7) becomes the stochastic ordinary differential equation, then the solution of (2.7) is the second moment stable with polynomial decay. In fact, let

$$\Psi(t,x) = (1+t)^{2p}|x|^2,$$

then it is easy to show that

$$L\Psi(t,u) = |u|^2 = \frac{1}{(1+t)^{2p}}\Psi(t,u).$$

If $p > 1/2$, then the solution of (2.7) with $\alpha = 0$ is the second moment stable with polynomial decay by using the result of [16, Theorem 1.1]. Moreover,

$$\limsup_{t\to\infty} \frac{\int_0^t \frac{1}{(1+s)^{2p}} ds}{\log \lambda(t)} = 0.$$

Therefore, we have

$$\limsup_{t\to\infty} \frac{1}{\log(1+t)} \log \mathbb{E}|u(t,\cdot)|^2 \leq -2p.$$

However, if $\alpha > 0$, then the solution (2.7) is the second moment stable with exponential decay. Indeed, let

$$\Psi(t,x) = e^{\alpha\lambda_1 t}|x|^2,$$

then it is easy to show that

$$L\Psi(t,u) \leq \left(-\alpha\lambda_1 - \frac{2p}{1+t} + \frac{1}{(1+t)^{2p}}\right) e^{\alpha\lambda_1 t}|u|^2.$$

Then there exists a positive constant T_0 such that for any $t > T_0$, $L\Psi(t,u) \leq 0$. Using the Lyapunov functional method, it is easy to obtain that for any $t > T_0$,

$$\mathbb{E}|u(t,\cdot)|^2 \leq Ce^{-\alpha\lambda_1 t}.$$

Example 2.3: Consider the following stochastic 2D-Navier-Stokes

$$\begin{cases} du = [\kappa\Delta u - \langle u, \nabla\rangle u + f(u) + \nabla p]dt + g(t,u)dW(t), \\ divu = 0, \quad t \geq 0, \ x \in D, \\ u|_{\partial D} = 0, \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in D, \end{cases} \tag{2.8}$$

where the meanings of parameters are the same as in [5, Page 715]. As in (2.2) of [5], the stochastic 2D-Navier-Stokes equation can be written as

$$du = [\kappa\Delta u - B(u) + f(u)]dt + g(t,u)dW(t),$$

where $B(u) = B(u,u)$.

Condition (A): There exists $\beta > 0$ such that

$$\|f(u) - f(v)\|_{V'} \leq \beta\|u - v\|_V, \quad \beta > 0, u, v \in V,$$

where $V =$ the closure of the set $\{u \in C_0^\infty(D,\mathbb{R}^2) : \ divu = 0\}$ in $H_0^1(D,\mathbb{R}^2)$ with the norm $\|u\| = ((u,v))^{1/2}$ and $H =$ the closure of the set $\{u \in C_0^\infty(D,\mathbb{R}^2) : \ divu = 0\}$ in $L^2(D,\mathbb{R}^2)$ with the norm $|u| = (u,v)^{1/2}$.

Under the condition (A) and $\kappa > \beta$, they first proved the equation

$$\kappa\Delta u - B(u) = -f(u) \ \ (\text{equality in } V') \tag{2.9}$$

admits a stationary solution $u_\infty \in V$. Moreover, if $\kappa > c_1\|f(0)\|_{V'}/(\sqrt{\lambda_1}(\nu-\beta))+\beta$, then the stationary solution is unique, where $c_1>0$ satisfies

$$|\langle B(u,v),w\rangle| = |b(u,v,w)| \le c_1|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|\|w\|^{\frac{1}{2}}, \quad \forall\, u,v,w\in V.$$

Condition B. $\|g(t,u)\|^2_{L_2^0} \le \gamma(t)+(\xi+\delta(t))|u-u_\infty|^2$, where $\xi>0$ is a constant and $\gamma(t)$, $\delta(t)$ are nonnegative integrable functions such that there exist real numbers $\vartheta>0$, M_γ, $M_\delta \ge 1$ with

$$\gamma(t) \le M_\gamma e^{-\vartheta t}, \quad \delta(t) \le M_\delta e^{-\vartheta t}, \quad t\ge 0. \tag{2.10}$$

Proposition 2.1 *[5, Theorem 3.2] Let $u_\infty\in V$ be the unique solution to (2.9) and let $2\kappa > \lambda_1^{-1}\xi+2\beta+(2c_1/\sqrt{\lambda_1})\|u_\infty\|$. Suppose that Conditions A and B are satisfied. Then any weak solution $u(t,x)$ to (2.8) converges to the stationary solution u_∞ to (2.9) exponentially in the mean square. That is, there exist real numbers $a\in(0,\vartheta)$, $M_0=M_0(u_0)>0$ such that*

$$\mathbb{E}|u(t,\cdot)-u_\infty|^2 \le M_0 e^{-at}, \quad t\ge 0.$$

The assumption (2.10) is critical for Proposition 2.1, and now we want to release the assumptions. Suppose u_∞ is a solution of limited equation (2.8), where $f(t,u)$ is replaced by $f^*(u)$ (as in Theorem 2.2, $f(t,x)\to f^*(x)$ in V' as $t\to\infty$ for any $x\in V$). In order to generalize the exponential decay to polynomial decay, we replace $f(u)$ by $f(t,u)$ in (2.8). Consequently, Condition (A) will be changed into (A'): There exists $\beta_0(t)>0$ and $\beta_1(t)$ such that

$$\langle u-u_\infty, f(t,u)-f^*(u_\infty)\rangle \le \beta_0(t)\|u-u_\infty\|+\beta_1(t)\|u-u_\infty\|^2, \quad \beta>0, \quad u\in V.$$

Theorem 2.5 *Let $u_\infty\in V$ be the unique solution to (2.9). Suppose that Conditions (A') and B are satisfied, where $\gamma(t)$ and $\delta(t)$ satisfy*

$$\limsup_{t\to\infty}\frac{\log\left(\int_0^t\lambda(s)^m(\beta_0(s)+\gamma(s))ds\right)}{\log\lambda(t)} \le \nu,$$

$$\limsup_{t\to\infty}\frac{\int_0^t\left(-2\kappa+(2c_1/\sqrt{\lambda_1})\|u_\infty\|+\beta_0(s)+2\beta_1(s)+\lambda_1^{-1}\xi+\delta(s)+\frac{m\lambda_1^{-1}\lambda'(s)}{\lambda(s)}\right)ds}{\log\lambda(t)} \le \theta.$$

Then any weak solution $u(t,x)$ to (2.8) converges to the stationary solution u_∞ to (2.9) polynomially in the mean square. That is, there exist real numbers $a\in(0,\theta)$, $M_0=M_0(u_0)>0$ such that

$$\mathbb{E}|u(t,\cdot)-u_\infty(\cdot)|^2 \le M_0\lambda(t)^\mu, \quad t\ge 0,$$

where $\mu=m-\nu-\theta<0$.

The proof is exactly same as Theorem 2.2. We only remark that when $\beta_0(t)\equiv 0$, $\beta_1(t)\equiv\beta$, one can take $\lambda(t)=e^t$, the index $\mu\in(0,\vartheta-\theta)$ satisfies that $2\kappa>\lambda_1^{-1}\xi+\mu+2\beta+(2c_1/\sqrt{\lambda_1})\|u_\infty\|$. It follows from $\gamma(t)\le M_\gamma e^{-\vartheta t}$ that

$$\limsup_{t\to\infty}\frac{\log\left(\int_0^t\lambda(s)^m\gamma(s)ds\right)}{\log\lambda(t)} \le m-\vartheta.$$

Meanwhile, the assumption on $\delta(t)$ can be rewritten as

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\int_0^t\left(-2\kappa+(2c_1/\sqrt{\lambda_1})\|u_\infty\|+\beta_0(s)+2\beta_1(s)+\lambda_1^{-1}\xi+\delta(s)+\frac{m\lambda_1^{-1}\lambda'(s)}{\lambda(s)}\right)ds}{\log\lambda(t)}\\
\le\ &\limsup_{t\to\infty}\frac{\int_0^t\left(-\mu\lambda_1^{-1}+\delta(s)+\frac{m\lambda_1^{-1}\lambda'(s)}{\lambda(s)}\right)ds}{\log\lambda(t)}\\
=\ &(\mu-m)\lambda_1^{-1}+\limsup_{t\to\infty}\frac{\int_0^t\delta(s)ds}{t}.
\end{aligned}$$

Clearly, if $\delta(t)\le M_\delta e^{-\vartheta t}$ and $\lambda(t)=e^t$ then $\limsup_{t\to\infty}\dfrac{\int_0^t\delta(s)ds}{\log\lambda(t)}=0$. Thus we have the convergence rate is $e^{-\mu t}$. Actually, we can release the assumption of $\delta(t)$, for example, we can take $\delta(t)=\frac{1}{(1+t)^2}$. Similar to Theorem 2.3, one can establish that $u(t,x)$ converges to $u_\infty\in H$ almost surely with decay $\lambda(t)$.

If $\lambda(t)=1+t$, we can take the nonlinear term as

$$f(t,u)=-\frac{p}{1+t}u+f_1(u),$$

where $f_1(u)$ satisfies the condition B. That is to say, we take

$$\beta_0(t)=\frac{p}{1+t},\quad \beta_1(t)=-\frac{p}{1+t}.$$

Apart from that, we assume that

$$2\kappa=\lambda_1^{-1}\xi+\mu+2\beta+(2c_1/\sqrt{\lambda_1})\|u_\infty\|,\quad \gamma(t)=\frac{\epsilon_1}{1+t},\quad \delta(t)=\frac{\epsilon_2}{1+t}.$$

Direct calculus shows that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\log\left(\int_0^t\lambda(s)^m(\beta_0(s)+\gamma(s))ds\right)}{\log\lambda(t)}=(p+\epsilon_1)(2-m);\\
&\limsup_{t\to\infty}\frac{\int_0^t\left(-2\kappa+(2c_1/\sqrt{\lambda_1})\|u_\infty\|+\beta_0(s)+2\beta_1(s)+\lambda_1^{-1}\xi+\delta(s)+\frac{m\lambda_1^{-1}\lambda'(s)}{\lambda(s)}\right)ds}{\log\lambda(t)}\\
=\ &-p+\epsilon_2+m\lambda_1^{-1}.
\end{aligned}$$

Hence we can take suitable $m,p,\epsilon_1,\epsilon_2$ such that Theorem 2.5 holds.

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