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Local existence-uniqueness of positive solutions for tempered fractional differential equations with p-Laplacian operators

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Abstract: In this paper, we are concerned with a kinds of tempered fractional differential equation Riemann-Stieltjes integral boundary values problem involving $p-$Laplacian operator. By means of the sum-type mixed monotone operators fixed point theorem based on the cone P_h, not only the local existence of unique positive solution is obtained, but also two successively monotone iterative sequences are constructed for approximating the unique positive solution. Finally, we present an example to illustrate our main results.

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1 Introduction

In this paper, we devoted to study the nonlinear tempered fractional differential equation involving $p-$Laplacian operator as follows:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big)\Big) = F(t,u(t)),\ 0\le t\le 1; \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0; \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)dA(s); \\ \varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u\big)(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}\big(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u)\big)(0) = 0; \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}\big(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u)\big)(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}\Big[\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(s)\big)\Big]dA(s); \end{cases} \tag{1.1}$$

where $2<\alpha\le 3,\ 0<\beta_2<\beta_1<\alpha-1,\ 1<\alpha-\gamma<2,\ F(t,u(t)) = f\big(t,u(t),u(t)\big)+g\big(t,u(t)\big),\ a\in C(0,1)$ and φ_p is $p-$Laplacian operator. ${}_0^R\mathbb{D}_t^{\alpha,\lambda}u$, ${}_0^R\mathbb{D}_t^{\gamma,\lambda}u$ and ${}_0^R\mathbb{D}_t^{\beta_i,\lambda}u(i=1,2)$ are tempered fractional derivatives defined by

$${}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) = e^{-\lambda t}{}_0^RD_t^\alpha(e^{\lambda t}u(t)),\quad \lambda\ge 0.$$

Here, ${}_0^RD_t^\alpha$ is the standard Riemann-Liouville fractional derivative defined by

$${}_0^RD_t^\alpha u(t) = \frac{d^n}{dt^n}({}_0I_t^{n-\alpha}u(t)),$$

where ${}_0I_t^\beta$ is β-order fractional integral operator defined by:

$${}_0I_t^\beta\psi = \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\psi(s)ds.$$

A is a function of bounded variation, $\int_0^1 a(s){}_0^R\mathbb{D}_t^{\beta_2}u(s)dA(s)$ denote Riemann-Stieltjes integral with respect to A. By using the sum-type mixed monotone fixed point theorem based on cone P_h, we investigate the existence-uniqueness and monotone iteration of positive solutions for $p-$Laplacian differential system (1.1).

In the past decades, fractional calculus and all kinds of fractional differential equations have been proved to be power tools in the modeling of various phenomena in a great deal of fields of science and engineering, such as chemical physics, fluid mechanics, heat conduction, control theory, economics, etc.; see[1-4] for example. In fact, a standard Riemann-Liouville (or Caputo) fractional derivative is a convolution with a power law, so does fractional integration, the difference between the two fractional derivatives only lies in the order of derivation and integration. Moreover, based on the definition of classical fractional derivative, the tempered fractional derivative multiplies the power law kernel by exponential factor, and the various differential equation models based on tempered fractional derivative open up a new possibility for robust mathematical modeling of anomalous phenomena and complex multi-scal problems, readers can refer to [5-9]. In [9], we studied two kinds of tempered fractional differential systems involving Riemann-Stieltjes integral boundary values condtions as follows:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)+f(t,u(t),u(t))+g(t,u(t))=0, \quad t\in(0,1),\\ u(0)={}_0^R\mathbb{D}_t^{\gamma_1,\lambda}u(0)={}_0^R\mathbb{D}_t^{\gamma_2,\lambda}u(0)=\cdots={}_0^R\mathbb{D}_t^{\gamma_{n-2},\lambda}u(0)=0,\\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1)=\int_0^\eta b(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)dA(s)+\int_0^1 a(s){}_0^R\mathbb{D}_t^{\beta_3,\lambda}u(s)dA(s) \end{cases} \tag{1.2}$$

and

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)+\psi(t,u(t))=c, \quad t\in(0,1),\\ u(0)={}_0^R\mathbb{D}_t^{\gamma_1,\lambda}u(0)={}_0^R\mathbb{D}_t^{\gamma_2,\lambda}u(0)=\cdots={}_0^R\mathbb{D}_t^{\gamma_{n-2},\lambda}u(0)=0,\\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1)=\int_0^\eta b(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)dA(s)+\int_0^1 a(s){}_0^R\mathbb{D}_t^{\beta_3,\lambda}u(s)dA(s), \end{cases} \tag{1.3}$$

where ${}_0^R\mathbb{D}_t^{\alpha,\lambda}u$, ${}_0^R\mathbb{D}_t^{\gamma_k,\lambda}u(k=1,2,\cdot\cdot\cdot,n-2)$ and ${}_0^R\mathbb{D}_t^{\beta_i,\lambda}u(i=1,2,3)$ are the tempered fractional derivatives. By using a class of sum-type mixed monotone operators fixed point theorems and increasing $\varphi-(h,\sigma)-$concave operators fixed point theorems, respectively, we constructed the sufficient conditions to guarantee the existence-uniqueness of positive solutions for Riemann-Stieltjes integral boundary value problems (1.2) and (1.3), respectively.

It is well known that p-Laplacian operator is used in analyzing various complex problems in physics, mechanics and the related fields of mathematical modeling, see[10-14]. In [10], for studying the turbulent flow in a kind of porous media, Leibenson introduced the p-Laplacian differential equation as follow:

$$\big(\varphi_p\big(u'(t)\big)\big)'=f\big(t,u(t),u'(t)\big),\ t\in(0,1), \tag{1.4}$$

where $\varphi_p(s)=|s|^{p-2}s,\ p>1$. Motivated by the Leibenson's work in [10], Ren, Li and Zhang [11] studied the existence of maximum and minimum solutions for nonlocal p-Laplacian fractional differential systems as follows:

$$\begin{cases} -D_t^{\beta_1}\Big(\varphi_{p_1}\big(-D_t^{\alpha_1}x_1\big)\Big)(t)=f_1\big(x_1(t),x_2(t)\big), \\ -D_t^{\beta_2}\Big(\varphi_{p_2}\big(-D_t^{\alpha_2}x_2\big)\Big)(t)=f_2\big(x_1(t),x_2(t)\big), \\ x_1(0)=0, \quad D_t^{\alpha_1}x_1(0)=D_t^{\alpha_1}x_1(1)=0, \quad x_1(1)=\int_0^1 x_1(t)dA_1(t), \\ x_2(0)=0, \quad D_t^{\alpha_2}x_2(0)=D_t^{\alpha_2}x_2(1)=0, \quad x_2(1)=\int_0^1 x_2(t)dA_2(t), \end{cases} \tag{1.5}$$

where φ_{p_i} denotes p-Laplacian operator, $D_t^{\alpha_i}, D_t^{\beta_i}$ are all the standard Riemann-Liouville derivatives, which satisfies $1<\alpha_i,\beta_i<2$, $\int_0^1 x_i(t)dA_i(t)$ denotes the Riemann-Stieltjes integral, A_i is a function of bounded variation. By employing the cone theory and monotone iterative technique, some new existence results on maximal and minimal solutions were established. Furthermore, the estimation of the bounds of maximum and minimum solutions was derived.

In [13], we investigated the existence results of multiple positive solutions for p-Laplacian fractional differential equations two points boundary value problems as follows:

$$\begin{cases} {}_0^RD_t^{\alpha}\Big(\varphi_p\big({}_0^RD_t^{\alpha}u(t)\big)\Big)=f\big(t,u(t),{}_0^RD_t^{\alpha}u(t)\big), \ 0\le t\le 1; \\ u^{(i)}(0)=0, \ [\varphi_p({}_0^RD_t^{\alpha}u)]^{(i)}(0)=0, \quad i=0,1,2,...,n-2; \\ [{}_0^RD_t^{\beta}u(t)]_{t=1}=0, \ 0<\beta\le\alpha-1; \\ \Big[{}_0^RD_t^{\beta}\Big(\varphi_p\big({}_0^RD_t^{\alpha}u(t)\big)\Big)\Big]_{t=1}=0; \end{cases} \tag{1.6}$$

where $n-1<\alpha\le n$, ${}_0^RD_t^{\alpha}$ is standard Riemann-Liouville fractional derivative, φ_p is $p-$Laplacian operator. By employing functional-type cone expansion-compression fixed point theorem and Leggett-Williams fixed point theorem, we obtained some existence conclusions of multiple positive solutions for p-Laplacian differential systems(1.7).

Inspired by the above references, in this paper, we investigate the p-Laplacian fractional differential equation with integral boundary value conditions (1.1). So far, this kind of Riemann-Stieltjes integral boundary value problem involving p-Lalacian operator has seldom been researched. Comparing with the previous references, this article has the following characteristics, firstly, the tempered fractional derivative ${}_0^R\mathbb{D}_t^{\alpha,\lambda}$ is a more general definition than the standard Riemann-Liouvill fractional derivative ${}_0^RD_t^{\alpha}$, for instance, let $\lambda=0$, it is clear to see the operator ${}_0^R\mathbb{D}_t^{\alpha,\lambda}$ is equivalent to ${}_0^RD_t^{\alpha}$. Secondly, the Riemann-Stieltjes integral boundary conditions are more general cases, which covers the common integral boundary conditions as special cases. Finally, comparing with p-Laplacin differential systems (1.6) and (1.7), the integral operator in this paper need not be completely continuous or compact. Furthermore, we can not only obtain sufficient conditions for the existence of the unique positive solution, but also construct a Cauchy sequences to approximate the unique positive solution.

The organization of the article is as follow. In Section 2, we list some concepts, symbols, definitions and lemmas in the abstract Banach spaces, which need to be used in the subsequent proof process. In Section 3, by employing the sum-type mixed monotone operators fixed point theorem based on cone P_h, we show that the existence-uniqueness and monotone iteration of positive solutions for $p-$Laplacian differential equation two points boundary value problems (1.1). In Section 4, we present an example to demonstrate our main results.

2 Preliminaries

Definition 2.1 *([17]) $A: P \to P$ is said to be sub-homogeneous if it satisfies*

$$A(tx) \geq tAx, \quad \forall t \in (0,1), \quad x \in P.$$

Definition 2.2 *([18]) An operator $A: P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i (i=1,2) \in P$, $u_1 < u_2, v_1 > v_2$ imply $A(u_1,v_1) \leq A(u_2,v_2)$. Element $x \in P$ is called a fixed point of A if $A(x,x) = x$.*

Definition 2.3 *([10]) Let $p > 1$, the $p-$Laplacian operator is given by*

$$\varphi_p(x) = |x|^{p-2}x, \ \text{and} \ \varphi_p^{-1} = \varphi_q, \ \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Lemma 2.1 *([9]) Let $h(t) \in C[0,1] \bigcap L^1[0,1]$, $\alpha > 0$, then*

$${}_0I_t^{\alpha}\,{}_0^RD_t^{\alpha} h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in R$, $i = 1,2,3,...,n(n = [\alpha]+1)$.

Lemma 2.2 *([13])*
(1) *If $u \in L^1(0,1), \alpha > \beta > 0$, then*

$${}_0I_t^{\alpha}\,{}_0I_t^{\beta} u(t) = {}_0I_t^{\alpha+\beta} u(t), \ {}_0^RD_t^{\beta}\,{}_0I_t^{\alpha} u(t) = {}_0 I_t^{\alpha-\beta} u(t), \ {}_0^RD_t^{\beta}\,{}_0I_t^{\beta} u(t) = u(t);$$

(2) *If $\rho > 0, \mu > 0$, then*

$${}_0^RD_t^{\rho} t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\rho)} t^{\mu-\rho-1}.$$

Lemma 2.3 *Given $g \in C(0,1)$, then, the unique solution of*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda} u(t) + g(t) = 0, \ 2 < \alpha \leq 3, \ t \in (0,1), \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda} u(0) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda} u(1) = \int_0^{\eta} a(s) {}_0^R\mathbb{D}_t^{\beta_2,\lambda} u(s) dA(s), \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s) g(s) ds, \quad t \in [0,1], \tag{2.2}$$

where

$$G(t,s) = G_1(t,s) + \frac{t^{\alpha-1} e^{-\lambda t}}{\Delta \Gamma(\alpha - \beta_2)} \int_0^{\eta} a(t) G_2(t,s) dA(t), \tag{2.3}$$

in which

$$\Delta = \frac{e^{-\lambda}}{\Gamma(\alpha-\beta_1)} - \frac{\delta}{\Gamma(\alpha-\beta_2)}, \quad \delta = \int_0^\eta e^{-\lambda s} s^{\alpha-\beta_2-1} a(s) dA(s),$$

$$G_1(t,s) = \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-\beta_1-1} t^{\alpha-1} - (t-s)^{\alpha-1}, \ 0 \le s \le t \le 1, \\ (1-s)^{\alpha-\beta_1-1} t^{\alpha-1}, \ 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-\beta_1-1} t^{\alpha-\beta_2-1} - (t-s)^{\alpha-\beta_2-1}, \ 0 \le s \le t \le 1, \\ (1-s)^{\alpha-\beta_1-1} t^{\alpha-\beta_2-1}, \ 0 \le t \le s \le 1, \end{cases}$$

Proof For the system (2.1), by means of Lemma 2.1, we have

$$e^{\lambda t} u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} g(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}. \tag{2.4}$$

From $u(0) = 0$, we get $c_3 = 0$, hence,

$$u(t) = -e^{-\lambda t} {}_0I_t^\alpha e^{\lambda t} g(t) + c_1 e^{-\lambda t} t^{\alpha-1} + c_2 e^{-\lambda t} t^{\alpha-2}. \tag{2.5}$$

By using the tempered fractional-order derivative operator ${}_0^R\mathbb{D}_t^{\gamma,\lambda}$ on both sides of (2.5), we obtain

$$\begin{aligned} {}_0^R\mathbb{D}_t^{\gamma,\lambda} u(t) &= {}_0^R\mathbb{D}_t^{\gamma,\lambda} \Big(- e^{-\lambda t} {}_0I_t^\alpha \big(e^{\lambda t} g(t)\big) + c_1 e^{-\lambda t} t^{\alpha-1} + c_2 e^{-\lambda t} t^{\alpha-2} \Big) \\ &= e^{-\lambda t} {}_0^R D_t^\gamma \Big(- {}_0I_t^\alpha \big(e^{\lambda t} g(t)\big) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \Big) \\ &= -e^{-\lambda t} {}_0I_t^{\alpha-\gamma} \big(e^{\lambda t} g(t)\big) + c_1 e^{-\lambda t} {}_0^R D_t^\gamma t^{\alpha-1} + c_2 e^{-\lambda t} {}_0^R D_t^\gamma t^{\alpha-2} \\ &= -\int_0^t \frac{(t-s)^{\alpha-\gamma-1} e^{\lambda(s-t)}}{\Gamma(\alpha-\gamma)} g(s) ds + c_1 \frac{\Gamma(\alpha) e^{-\lambda t}}{\Gamma(\alpha-\gamma)} t^{\alpha-1-\gamma} \\ &\quad + c_2 \frac{\Gamma(\alpha-1) e^{-\lambda t}}{\Gamma(\alpha-1-\gamma_1)} t^{\alpha-2-\gamma}. \end{aligned}$$

From ${}_0^R\mathbb{D}_t^{\gamma,\lambda} u(0) = 0$ and $1 < \alpha - \gamma \le 2$, we know that $c_2 = 0$. Hence, the equation (2.5) can be reduced as following

$$u(t) = -e^{-\lambda t} \int_0^t \frac{(t-s)^{\alpha-1} e^{\lambda s}}{\Gamma(\alpha)} g(s) ds + c_1 e^{-\lambda t} t^{\alpha-1}. \tag{2.6}$$

Once again, applying tempered fractional derivative operator ${}_0^R\mathbb{D}_t^{\beta_i,\lambda}$ on the both sides of (2.6), we have

$$\begin{aligned} {}_0^R\mathbb{D}_t^{\beta_i,\lambda} u(t) &= -{}_0^R\mathbb{D}_t^{\beta_i,\lambda} \Big(e^{-\lambda t} {}_0I_t^\alpha \big(e^{\lambda t} g(t)\big) \Big) + c_1 {}_0^R\mathbb{D}_t^{\beta_i,\lambda} (e^{-\lambda t} t^{\alpha-1}) \\ &= -e^{-\lambda t} {}_0^R D_t^{\beta_i} \Big({}_0I_t^\alpha \big(e^{\lambda t} g(t)\big) \Big) + c_1 e^{-\lambda t} {}_0^R D_t^{\beta_i} (t^{\alpha-1}) \\ &= -e^{-\lambda t} {}_0I_t^{\alpha-\beta_i} \big(e^{\lambda t} g(t)\big) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)} e^{-\lambda t} t^{\alpha-1-\beta_i} \\ &= -\int_0^t \frac{(t-s)^{\alpha-\beta_i-1} e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_i)} g(s) ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)} e^{-\lambda t} t^{\alpha-1-\beta_i}. \end{aligned} \tag{2.7}$$

From (2.7), it is clear to see that

$$\begin{cases} {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \frac{-1}{\Gamma(\alpha-\beta_1)}\int_0^1 (1-s)^{\alpha-\beta_1-1}e^{\lambda(s-1)}g(s)ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_1)}e^{-\lambda}, \\ {}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(t) = \frac{-1}{\Gamma(\alpha-\beta_2)}\int_0^t (t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}g(s)ds + c_1\frac{\Gamma(\alpha)e^{-\lambda t}}{\Gamma(\alpha-\beta_2)}t^{\alpha-1-\beta_2}. \end{cases} \tag{2.8}$$

Substituting (2.8) into ${}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)dA(s)$, we obtain

$$\begin{aligned} c_1 = [\Gamma(\alpha)\Delta]^{-1}\Big\{ &\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}e^{\lambda(s-1)}}{\Gamma(\alpha-\beta_1)}g(s)ds \\ &- \int_0^\eta a(t)dA(t)\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_2)}g(s)ds\Big\}. \end{aligned} \tag{2.9}$$

Finally, combining (2.9) with (2.6), we obtain

$$\begin{aligned} u(t) = &-e^{-\lambda t}\int_0^t \frac{(t-s)^{\alpha-1}e^{\lambda s}}{\Gamma(\alpha)}g(s)ds + \frac{e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)\Delta}\Big\{\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}e^{-\lambda}}{\Gamma(\alpha-\beta_1)}e^{\lambda s}g(s)ds \\ &- \int_0^\eta a(t)dA(t)\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_2)}g(s)ds\Big\} \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}e^{\lambda(s-t)}}{\Gamma(\alpha)}g(s)ds + \frac{e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta_1-1}e^{\lambda s}g(s)ds \\ &+ \frac{e^{-\lambda t}t^{\alpha-1}\delta}{\Gamma(\alpha)\Gamma(\alpha-\beta_2)\Delta}\int_0^1 (1-s)^{\alpha-\beta_1-1}e^{\lambda s}g(s)ds \\ &- \frac{e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha-\beta_2)\Delta}\int_0^\eta a(t)dA(t)\int_0^t (t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}g(s)ds \\ &= \int_0^1 G_1(t,s)g(s)ds + \frac{t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha-\beta_2)\Delta}\int_0^1 g(s)ds\int_0^\eta G_2(t,s)a(t)dA(t) \\ &= \int_0^1 G(t,s)g(s)ds, \end{aligned}$$

where $G(t,s)$ is Green function of systems (2.1). The proof is complete. □

Lemma 2.4 *Suppose that*

$$(H) \qquad \Gamma(\alpha-\beta_1)e^{\lambda}\delta < \Gamma(\alpha-\beta_2),$$

then, for all $(t,s)\in[0,1]\times[0,1]$, $G(t,s)$, $G_1(t,s)$ *and* $G_2(t,s)$ *satisfies*
(A_1) $G(t,s)$, $G_1(t,s)$ *and* $G_2(t,s)$ *are all continuous in* $(t,s)\in[0,1]\times[0,1]$*;*
(A_2) $G_i(t,s)\geq 0$ $(i=1,2)$*, and* $G(t,s)\geq 0$*;*
(A_3) $\frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-1} \leq G_1(t,s) \leq \frac{e^{\lambda s}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-1}$*;*
(A_4) $\frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-\beta_2-1}]}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-\beta_2-1} \leq G_2(t,s) \leq \frac{e^{\lambda s}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-\beta_2-1}$*;*
(A_5) $m(s)e^{-\lambda t}t^{\alpha-1} \leq G(t,s) \leq M(s)e^{-\lambda t}t^{\alpha-1}$*, where*

$$M(s) = [\frac{1}{\Gamma(\alpha)} + \frac{\delta}{\Delta\Gamma(\alpha)\Gamma(\alpha-\beta_2)}]e^{\lambda s}(1-s)^{\alpha-\beta_1-1}$$

and

$$m(s)=\frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}+\frac{\delta e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-\beta_2-1}]}{\Delta\Gamma(\alpha)\Gamma(\alpha-\beta_2)}.$$

Proof Firstly, for $(t,s)\in[0,1]\times[0,1]$, it is obvious that $G(t,s)$ and $G_i(t,s)(i=1,2)$ are continuous.

Secondly, for $G_i(t,s)(i=1,2)$ in (A_3) and (A_4), it is evident that the right sides of the inequalities hold, so we only need to prove the left sides of the inequalities. If $0\le s\le t\le 1$, it's easy to see $0\le t-s\le t-ts=(1-s)t$, thus $(t-s)^{\alpha-1}\le (1-s)^{\alpha-1}t^{\alpha-1}$. So we get

$$\begin{aligned}G_1(t,s)&\ge\frac{e^{\lambda(s-t)}}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}t^{\alpha-1}]\\&=\frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-1}.\end{aligned}$$

If $0\le t\le s\le 1$, evidently, $G_1(t,s)\ge\frac{e^{\lambda s}[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}]}{\Gamma(\alpha)}e^{-\lambda t}t^{\alpha-1}$ holds.

Furthermore, from $(1-s)^{\alpha-\beta_1-1}>(1-s)^{\alpha-1}$, we get $G_1(t,s)\ge 0$ for $\forall(t,s)\in[0,1]\times[0,1]$. In the same way, we can know that $G_2(t,s)\ge 0$ and the inequality (A_4) holds.

Finally, from (A_3) and (A_4), we can know that $m(s)e^{-\lambda t}t^{\alpha-1}\le G(t,s)\le M(s)e^{-\lambda t}t^{\alpha-1}$. In, addition, from the condition (H), we can deduce that $\Delta>0$. Combining $(1-s)^{\alpha-\beta_1-1}>(1-s)^{\alpha-1}$ with $\Delta>0$ together, we obtain $m(s)\ge 0$, that is, $G(s,t)\ge 0$ for $\forall(t,s)\in[0,1]\times[0,1]$. So, the proof is over. □

Lemma 2.5 *([18]) Let $\xi\in(0,1)$, $A:P\times P\to P$ is a mixed monotone operator and satisfies*

$$A(tx,t^{-1}y)\ge t^{\xi}A(x,y),\qquad\forall t\in(0,1),\quad x,y\in P. \tag{2.10}$$

$B:P\to P$ is increasing sub-homogeneous operator. Assume that
(I) there exists $h_0\in P_h$ such that $A(h_0,h_0)\in P_h$ and $Bh_0\in P_h$;
(II) there exists a constant $\delta_0>0$ such that $A(x,y)\ge\delta_0 Bx$, $\forall x,y\in P$;
Then,
(1) *$A:P_h\times P_h\to P_h$, $B:P_h\to P_h$;*
(2) *there exists $u_0,v_0\in P_h$ and $r\in(0,1)$ such that*

$$rv_0\le u_0<v_0,\quad u_0\le A(u_0,v_0)+Bu_0\le A(v_0,u_0)+B(v_0)\le v_0;$$

(3) *the operator equation $A(x,x)+Bx=x$ has a unique solution x^* in P_h;*
(4) *for any initial values $x_0,y_0\in P_h$, constructing successively the sequences*

$$x_n=A(x_{n-1},y_{n-1})+Bx_{n-1},\quad y_n=A(y_{n-1},x_{n-1})+By_{n-1},\quad n=1,2,\cdots,$$

we have $x_n\to x^*$ and $y_n\to x^*$ as $n\to\infty$.

3 Main results

Lemma 3.1 *If $\widetilde{g} \in C[0,1]$ is given, then the $p-$Laplacian tempered fractional differential system*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big)\Big) = \widetilde{g}(t),\ 2 < \alpha \le 3,\ 0 \le t \le 1; \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0, \\ \varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(0)\big) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(0)\big)\Big) = 0, \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)dA(s) \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(1)\big)\Big) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}\Big[\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(s)\big)\Big]dA(s) \end{cases} \tag{3.1}$$

has a unique integral formal solution

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\widetilde{g}(\tau)d\tau\right)ds, \tag{3.2}$$

where $G(t,s)$ is given in (2.3).

Proof Firstly, applying the fractional integral operator ${}_0I_t^\alpha$ on both sides of the first equation of $p-$Laplacian differential equation integral boundary value problems (3.1), we have

$$\begin{aligned} e^{\lambda t}\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big) &= {}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1t^{\alpha-1} + d_2t^{\alpha-2} + d_3t^{\alpha-3} \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\lambda s}\widetilde{g}(s)ds + d_1t^{\alpha-1} + d_2t^{\alpha-2} + d_3t^{\alpha-3}. \end{aligned}$$

From $\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u)(0) = 0$, we can deduce that $d_3 = 0$. So,

$$\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big) = e^{-\lambda t}{}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}t^{\alpha-1} + d_2e^{-\lambda t}t^{\alpha-2}. \tag{3.3}$$

Furthermore, applying the tempered fractional derivative operator ${}_0^R\mathbb{D}_t^{\gamma,\lambda}$ on both sides of (3.3), we have

$$\begin{aligned} {}_0^R\mathbb{D}_t^{\gamma,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big)\Big) &= {}_0^R\mathbb{D}_t^{\gamma,\lambda}\Big(e^{-\lambda t}{}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}t^{\alpha-1} + d_2e^{-\lambda t}t^{\alpha-2}\Big) \\ &= e^{-\lambda t}{}_0I_t^{\alpha-\gamma}\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}{}_0^RD_t^\gamma t^{\alpha-1} + d_2e^{-\lambda t}{}_0^RD_t^\gamma t^{\alpha-2} \\ &= \int_0^t \frac{(t-s)^{\alpha-\gamma-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\gamma)}\widetilde{g}(s)ds + d_1\frac{\Gamma(\alpha)e^{-\lambda t}}{\Gamma(\alpha-\gamma)}t^{\alpha-1-\gamma} \\ &\quad + d_2\frac{\Gamma(\alpha-1)e^{-\lambda t}}{\Gamma(\alpha-1-\gamma_1)}t^{\alpha-2-\gamma}. \end{aligned}$$

From ${}_0^R\mathbb{D}_t^{\gamma,\lambda}\big(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u)\big)(0) = 0$ and $1 < \alpha-\gamma < 2$, we deduce that $d_2 = 0$, that is,

$$\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big) = e^{-\lambda t}{}_0I_t^\alpha\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1e^{-\lambda t}t^{\alpha-1}. \tag{3.4}$$

Secondly, applying tempered fractional derivative operator ${}_0^R\mathbb{D}_t^{\beta_i,\lambda}(i=1,2)$ on both sides of (3.4), we get

$$\begin{aligned}
&{}_0^R\mathbb{D}_t^{\beta_i,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big)\Big) = {}_0^R\mathbb{D}_t^{\beta_i,\lambda}\Big(e^{-\lambda t}{}_0I_t^{\alpha}\big(e^{\lambda t}\widetilde{g}(t)\big)\Big) + d_1{}_0^R\mathbb{D}_t^{\beta_i,\lambda}(e^{-\lambda t}t^{\alpha-1})\\
&= e^{-\lambda t}{}_0I_t^{\alpha-\beta_i}\big(e^{\lambda t}\widetilde{g}(t)\big) + d_1 e^{-\lambda t}{}_0^R D_t^{\beta_i}(t^{\alpha-1})\\
&= \int_0^t \frac{(t-s)^{\alpha-\beta_i-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_i)}\widetilde{g}(s)ds + d_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)}e^{-\lambda t}t^{\alpha-1-\beta_i}.
\end{aligned} \tag{3.5}$$

From (3.5), it is clear to see that

$$\begin{cases}
{}_0^R\mathbb{D}_t^{\beta_1,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(1)\big)\Big) = \int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}e^{\lambda(s-1)}}{\Gamma(\alpha-\beta_1)}\widetilde{g}(s)ds + d_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_1)}e^{-\lambda},\\
{}_0^R\mathbb{D}_t^{\beta_2,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big)\Big) = \int_0^t \frac{(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_2)}\widetilde{g}(s)ds + d_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_2)}e^{-\lambda t}t^{\alpha-1-\beta_2}.
\end{cases} \tag{3.6}$$

Combining (3.6) with the Riemann-Stieltjes integral boundary value condition ${}_0^R\mathbb{D}_t^{\beta_1,\lambda}\big(\varphi_p({}_0^R\mathbb{D}_t^{\alpha,\lambda}u)\big)(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}\Big[\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(s)\big)\Big]dA(s)$, we obtain

$$\begin{aligned}
d_1 = \frac{-1}{\Gamma(\alpha)\Delta}\Big\{&\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}e^{\lambda(s-1)}}{\Gamma(\alpha-\beta_1)}\widetilde{g}(s)ds\\
&-\int_0^\eta a(t)dA(t)\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha-\beta_2)}\widetilde{g}(s)ds\Big\}.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.4), we obtain

$$\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big) = -\int_0^1 G(t,s)\widetilde{g}(s)ds. \tag{3.8}$$

Furthermore, applying the $p-$Laplacian operator φ_q on both sides of (3.8), we get

$${}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) + \varphi_q\left(\int_0^1 G(t,s)\widetilde{g}(s)ds\right) = 0. \tag{3.9}$$

Finally, letting $g(t) :\triangleq \varphi_q\big(\int_0^1 G(t,s)\widetilde{g}(s)ds\big)$. It's easy to see that the $p-$Laplacin fractional differential equation (3.1) is equivalent to the fraction differential system as following:

$$\begin{cases}
{}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) + g(t) = 0,\ t\in(0,1),\ 2<\alpha\le 3;\\
u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0,\\
{}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)dA(s).
\end{cases} \tag{3.10}$$

By means of Lemma 2.3, we know that the $p-$Lalacian tempered fractional differential systems (3.10) has a unique integral solution

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)g(s)ds\\
&= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\widetilde{g}(\tau)d\tau\right)ds.
\end{aligned}$$

This constitutes the complete proof. □

From Lemma 3.1, we can deduce that the systems (1.1) is equivalent to integral formulation given by

$$u(t)=\int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)\Big[f\big(\tau,u(\tau),u(\tau)\big)+g\big(\tau,u(\tau)\big)\Big]d\tau\Big)ds. \quad (3.11)$$

For the convenience of further research, we define the operator T by

$$T(u,v)(t)=\int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)\Big[f\big(\tau,u(\tau),v(\tau)\big)+g\big(\tau,u(\tau)\big)\Big]d\tau\Big)ds. \quad (3.12)$$

It's obvious that u^* is a solution of (1.1) if and only if $T(u^*,u^*)=u^*$.

Theorem 3.1 *Suppose that the condition (H) in lemma 2.4 is true, $a(t):[0,1]\to R^+$, $f(t,u,v):[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ and $g(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ are all continuous functions with $g(t,u)\not\equiv 0$, and the following conditions are satisfied:*

(H_1) *for fixed $t\in[0,1]$, $f(t,u,v)$ is increasing in $u\in[0,+\infty)$ and decreasing in $v\in[0,+\infty)$. In addition, for $\forall\gamma\in(0,1)$, $u,v\in[0,+\infty)$, there exists a constant $\xi\in(0,1)$ such that*

$$f(t,\gamma u,\gamma^{-1}v)\ge\varphi_p^{\xi}(\gamma)f(t,u,v); \quad (3.13)$$

(H_2) *for fixed $t\in[0,1]$, $g(t,u)$ is increasing in $u\in[0,+\infty)$. In addition, for $\forall t\in[0,1]$, $\gamma\in(0,1)$, $u\in[0,+\infty)$,*

$$g(t,\gamma u)\ge\varphi_p(\gamma)g(t,u); \quad (3.14)$$

(H_3) *for $\forall u,v\in[0,+\infty)$, there exists a constant $\delta_0>0$ such that*

$$f(t,u,v)\ge\varphi_p(\delta_0)g(t,u),\quad t\in[0,1]. \quad (3.15)$$

Then, we have:

(I) *the $p-$Laplacian tempered fractional differential equation Riemann-Stieltjes integral boundary value problem (1.1) has a unique positive solution $u^*\in P_h$, where $h(t)=e^{-\lambda t}t^{\alpha-1},\ t\in[0,1]$;*

(II) *for $\forall t\in[0,1]$, there exist $u_0,v_0\in P_h$ and $r\in(0,1)$ such that $rv_0\le u_0<v_0$ and*

$$u_0(t)\le\int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)\Big[f\big(\tau,u_0(\tau),v_0(\tau)\big)+g\big(\tau,u_0(\tau)\big)\Big]d\tau\Big)ds,$$
$$v_0(t)\ge\int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)\Big[f\big(\tau,v_0(\tau),u_0(\tau)\big)+g\big(\tau,v_0(\tau)\big)\Big]d\tau\Big)ds;$$

(III) for any initial values $x_0, y_0 \in P_h$, making successively the sequences

$$\begin{aligned}
&x_n = \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)\Big[f\big(\tau, x_{n-1}(\tau), y_{n-1}(\tau)\big) + g\big(\tau, x_{n-1}(\tau)\big)\Big]d\tau\Big)ds,\\
&y_n = \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)\Big[f\big(\tau, y_{n-1}(\tau), x_{n-1}(\tau)\big) + g\big(\tau, y_{n-1}(\tau)\big)\Big]d\tau\Big)ds,\\
&n = 0,1,2,...,
\end{aligned}$$

we obtain $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$.

Proof To begin with, we define two operators $A : P \times P \to E$ and $B : P \to E$ by

$$A(u,v)(t) = \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau, u(\tau), v(\tau)\big)d\tau\Big)ds, \tag{3.16}$$

$$B(u)(t) = \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)g\big(\tau, u(\tau)\big)d\tau\Big)ds. \tag{3.17}$$

So, we have $T(u,v) = A(u,v) + B(u)$, it's evident that u^* is the solution of the systems (1.1) if and only if $A(u^*,u^*) + B(u^*) = u^*$. From Lemma 2.4, we get $A : P \times P \to P$ and $B : P \to P$. Furthermore, it follows from (H_1) and (H_2) that A is a mixed monotone operator and B is an increasing operator. For $\forall\gamma \in (0,1)$ and $u, v \in P$, from (3.13), we obtain

$$\begin{aligned}
A(\gamma u, \gamma^{-1}v)(t) &= \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau, \gamma u(\tau), \gamma v(\tau)\big)d\tau\Big)ds\\
&\geq \int_0^1 G(t,s)\varphi_q\Big(\varphi_p^{\xi}(\gamma)\int_0^1 G(s,\tau)f\big(\tau, u(\tau), v(\tau)\big)d\tau\Big)ds\\
&= \gamma^{\xi}A(u,v)(t).
\end{aligned} \tag{3.18}$$

Hence, the mixed monotone operator A satisfies the condition (2.10) in Lemma 2.5. In addition, for $\forall\gamma \in (0,1)$ and $u \in P$, from (3.14) we have

$$\begin{aligned}
B(\gamma u)(t) &= \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)g\big(\tau, \gamma u(\tau)\big)d\tau\Big)ds\\
&\geq \varphi_q\big(\varphi_p(\gamma)\big)\int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)g\big(\tau, u(\tau)\big)d\tau\Big)ds\\
&= \gamma B(u)(t).
\end{aligned} \tag{3.19}$$

So, the operator B is a sub-homogeneous operator.

Next, we show that $A(h,h) \in P_h$ and $Bh \in P_h$. From Lemma 2.4, we have

$$\begin{aligned}
A(h,h)(t) &= \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau,h(\tau),h(\tau)\big)d\tau\Big)ds \\
&\leq \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 M(\tau)e^{-\lambda s}s^{\alpha-1}f\big(\tau,h(\tau),h(\tau)\big)d\tau\Big)ds \\
&\leq \int_0^1 M(s)e^{-\lambda t}t^{\alpha-1}\varphi_q\Big(\int_0^1 M(\tau)e^{-\lambda s}s^{\alpha-1}f\big(\tau,h(\tau),h(\tau)\big)d\tau\Big)ds \\
&\leq \Big(\int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\Big(\int_0^1 M(\tau)f\big(\tau,h_{max},0\big)d\tau\Big)ds\Big)\, e^{-\lambda t}t^{\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
A(h,h)(t) &= \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau,h(\tau),h(\tau)\big)d\tau\Big)ds \\
&\geq \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 m(\tau)e^{-\lambda s}s^{\alpha-1}f\big(\tau,h(\tau),h(\tau)\big)d\tau\Big)ds \\
&\geq \int_0^1 m(s)e^{-\lambda t}t^{\alpha-1}\varphi_q\Big(\int_0^1 m(\tau)e^{-\lambda s}s^{\alpha-1}f\big(\tau,h(\tau),h(\tau)\big)d\tau\Big)ds \\
&\geq \Big(\int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\Big(\int_0^1 m(\tau)f\big(\tau,0,h_{max}\big)d\tau\Big)ds\Big)\, e^{-\lambda t}t^{\alpha-1},
\end{aligned}$$

where $h_{\max} = \max\{h(t) : t \in [0,1]\}$. Letting

$$\begin{aligned}
L_1 &\triangleq \int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\Big(\int_0^1 M(\tau)f\big(\tau,h_{max},0\big)d\tau\Big)ds, \\
l_1 &\triangleq \int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\Big(\int_0^1 m(\tau)f\big(\tau,0,h_{max}\big)d\tau\Big)ds.
\end{aligned}$$

It is clear to see that $L_1 > l_1 > 0$. Hence, $l_1 h(t) \leq A(h,h) \leq L_1 h(t)$, that is, $A(h,h) \in P_h$. Similarly, for the sub-homogenous operator B, from Lemma 2.4, we get

$$\begin{aligned}
B(h)(t) &= \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)g\big(\tau,h(\tau)\big)d\tau\Big)ds \\
&\leq \Big(\int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\Big(\int_0^1 M(\tau)g\big(\tau,h_{max}\big)d\tau\Big)ds\Big)\, e^{-\lambda t}t^{\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
B(h)(t) &= \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)g\big(\tau,h(\tau)\big)d\tau\Big)ds \\
&\geq \Big(\int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\Big(\int_0^1 m(\tau)g\big(\tau,0\big)d\tau\Big)ds\Big)\, e^{-\lambda t}t^{\alpha-1}.
\end{aligned}$$

Letting

$$L_2 \triangleq \int_0^1 \frac{M(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}} \varphi_q\Big(\int_0^1 M(\tau)g\big(\tau, h_{max}\big)d\tau\Big)ds,$$
$$l_2 \triangleq \int_0^1 \frac{m(s)s^{(\alpha-1)(q-1)}}{e^{\lambda s(q-1)}} \varphi_q\Big(\int_0^1 m(\tau)g\big(\tau, 0\big)d\tau\Big)ds.$$

From $L_2 > l_2 > 0$ and $l_2h(t) \leq B(h) \leq L_2h(t)$, we get $Bh \in P_h$. Since $h \in P_h$, letting $h_0 = h$, then the condition (I_1) of Lemma 2.5 is satisfied.

At last, we show that the condition (I_2) of Lemma 2.5 is also satisfied. For $\forall u, v \in P$, from (3.15), we obtain

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau, u(\tau), v(\tau)\big)d\tau\Big)ds \\ &\geq \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 \varphi_p(\delta_0)G(s,\tau)g\big(\tau, u(\tau)\big)d\tau\Big)ds \\ &= \delta_0 B(u)(t)t \end{aligned} \tag{3.20}$$

that is, $A(u,v) \geq \delta_0 Bu$. Now, all the conditions of Lemma 2.5 are satisfied. Hence, the conclusions of Theorem 3.1 follows from Lemma2.5. □

Corollary 3.1 *Assume that the condition (H) holds and*

(H_1') $a(t) : [0,1] \to R^+$ *and* $f(t,u,v) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *are all continuous functions.*

(H_2') *for fixed* $t \in [0,1]$, $f(t,u,v)$ *is increasing in* $u \in [0,+\infty)$ *and decreasing in* $v \in [0,+\infty)$, *respectively;*

(H_3') *for* $\forall u, v \in [0,+\infty)$, $\gamma \in (0,1)$, *there exists a constant* $\xi \in (0,1)$ *such that*

$$f(t, \gamma u, \gamma^{-1}v) \geq \varphi_p^{\xi}(\gamma)f(t,u,v), \quad t \in [0,1]. \tag{3.21}$$

Then we have:

(I) *the p-Laplacian differential equation Riemann-Stieltjes integral boundary value problem*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t)\big)\Big) = f\big(t, u(t), u(t)\big), \ 0 \leq t \leq 1; \\ u(0) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}u(0) = 0; \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}u(s)dA(s); \\ \varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(0)\big) = {}_0^R\mathbb{D}_t^{\gamma,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(0)\big)\Big) = 0; \\ {}_0^R\mathbb{D}_t^{\beta_1,\lambda}\Big(\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(1)\big)\Big) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\beta_2,\lambda}\Big[\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha,\lambda}u(s)\big)\Big]dA(s) \end{cases}$$

has a unique positive solution $u^* \in P_h$, *where* $h(t) = e^{-\lambda t}t^{\alpha-1}$.

(II) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \leq u_0 < v_0$ *and*

$$u_0(t) \leq \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau, u_0(\tau), v_0(\tau)\big)d\tau\Big)ds,$$
$$v_0(t) \geq \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau, v_0(\tau), u_0(\tau)\big)d\tau\Big)ds;$$

(III) for any initial values $x_0, y_0 \in P_h$, making successively the sequences

$$x_n = \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau, x_{n-1}(\tau), y_{n-1}(\tau)\big)d\tau\Big)ds,$$
$$y_n = \int_0^1 G(t,s)\varphi_q\Big(\int_0^1 G(s,\tau)f\big(\tau, y_{n-1}(\tau), x_{n-1}(\tau)\big)d\tau\Big)ds,$$
$$n = 0, 1, 2, ...,$$

we obtain $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$.

Proof Setting $g(t,u(t)) \equiv 0$, by means of Theorem 3.1, we get the conclusions. □

4 Applications

Example 4.1 we consider the following tempered fractional differential systems involving p-Laplacian operator:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\frac{5}{2},1}\Big(\varphi_3\big({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(t)\big)\Big) = F(t,u(t)),\ 0 \le t \le 1; \\ u(0) = {}_0^R\mathbb{D}_t^{\frac{3}{4},1}u(0) = 0; \\ {}_0^R\mathbb{D}_t^{1,1}u(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\frac{5}{8},1}u(s)dA(s); \\ \varphi_3\big({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(0)\big) = {}_0^R\mathbb{D}_t^{\frac{3}{4},1}\Big(\varphi_3\big({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(0)\big)\Big) = 0; \\ {}_0^R\mathbb{D}_t^{1,1}\Big(\varphi_3\big({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(1)\big)\Big) = \int_0^1 {}_0^R\mathbb{D}_t^{\frac{5}{8},1}\Big[\varphi_3\big({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(s)\big)\Big]dA(s); \end{cases} \tag{4.1}$$

where $F(t,u(t)) = f\big(t,u(t),u(t)\big)+g\big(t,u(t)\big)$, and $f(t,u,v) = (1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}u^{\frac{1}{3}}+v^{-\frac{1}{5}}$, $g(t,u) = (1-t)^{-\frac{1}{8}}t^{-\frac{1}{6}}u^{\frac{1}{3}}$, $p = 3,\ \lambda = 1 > 0,\ \eta = 1$ and $A(t) = \frac{t}{2}$. For any $t \in (0,1)$, $u > 0$ and $v > 0$ and we see that $\alpha = \frac{5}{2},\ \beta_1 = 1,\ \beta_2 = \frac{5}{8},\ \gamma = \frac{3}{4},\ a(t) \equiv 1$ in the systems (4.1).

Let us check that all the required conditions of Theorem 3.1 are satisfied.

(1) From $\delta = \int_0^\eta e^{-\lambda s}s^{\alpha-\beta_2-1}a(s)dA(s) = 0.1432$, we can know $\Gamma(\alpha-\beta_1)e^\lambda\delta = 0.345 < 0.9534 = \Gamma(\alpha-\beta_2)$, clearly, the condition (H) is satisfied.

(2) From the expressions of f and g, it is evidently that $f(t,u,v) : (0,1)\times R^+\times R^+ \to R^+$ and $g(t,u) : (0,1)\times R^+ \to R^+$ are continuous. Furthermore, $f(t,u,v)$ is increasing in u for fixed $t \in (0,1)$ and $v \in R^+$, decreasing in v for fixed $t \in (0,1)$ and $u \in R^+$; in addition, for fixed $t \in (0,1)$, $g(t,u)$ is increasing in u.

(3) For any $\gamma \in (0,1)$, $t \in (0,1)$, $u, v > 0$, taking $\xi = \frac{1}{2} \in (0,1)$, we have

$$\begin{aligned} f(t,\gamma u,\gamma^{-1}v) &= (1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}(\gamma u)^{\frac{1}{3}} + (\gamma^{-1}v)^{-\frac{1}{5}} \\ &\ge \gamma^{\frac{1}{2}}[(1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}u^{\frac{1}{3}} + v^{-\frac{1}{5}}] \\ &\ge \gamma[(1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}u^{\frac{1}{3}} + v^{-\frac{1}{5}}] \\ &= \varphi_p^\xi(\gamma)f(t,u,v) \end{aligned}$$

and

$$\begin{aligned} g(t,\gamma u) &= (1-t)^{-\frac{1}{8}}t^{-\frac{1}{6}}(\gamma u)^{\frac{1}{3}} \\ &\ge \gamma^2[(1-t)^{-\frac{1}{8}}t^{-\frac{1}{6}}u^{\frac{1}{3}}] \\ &= \varphi_p(\gamma)g(t,u). \end{aligned}$$

(4) Taking $\delta_0 = \frac{1}{2}$, for $\forall t \in (0,1)$ and $u, v \in [0, +\infty)$, we have

$$\begin{aligned} f(t,u,v) &= (1-t)^{-\frac{1}{3}} t^{-\frac{2}{3}} u^{\frac{1}{3}} + v^{-\frac{1}{5}} \\ &\geq \frac{1}{4}[(1-t)^{-\frac{1}{8}} t^{-\frac{1}{6}} u^{\frac{1}{3}}] \\ &= \varphi_p(\delta_0) g(t,u). \end{aligned}$$

From the above conclusions, obviously, Theorem 3.1 implies that the tempered fractional differential equation integral boundary value problem (4.1) has a unique positive solution $u^* \in P_h$, where $h(t) = e^{-t} t^{\frac{3}{2}}$.

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