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Lie group approach for constructing all reciprocal transformations. The two-dimensional stationary gas dynamics equations

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Recently, an infinitesimal approach for finding reciprocal transformations has been proposed. The method uses the group analysis approach and consists of similar steps as for finding an equivalence group of transformations. The new method provides a systematic tool for finding classes of reciprocal transformations (group of reciprocal transformations). Similar to the classical group analysis this approach can be also applied for finding all reciprocal transformations (not only composing a group) of the equations under study. The present paper provides this algorithm. As an illustration, the method is applied to the two-dimensional stationary gas dynamics equations. Equivalence group, group of reciprocal transformations and completeness of all discrete reciprocal transformations are presented in the paper. The results are stated in form of a theorem.

Keywords: Reciprocal transformations, equivalence group of transformations, Lie group of transformations

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1. Introduction

Nonlocal transformations play a significant role in continuum mechanics and mathematical physics. One such nonlocal transformation is the generalized Sundman transformation, which is defined by the formulae

$$U = F(x,u), \;\; dX = G(x,u)\,dx.$$

Using this transformation a function $u(x)$ is mapped into the function $U(X)$. This type of transformations was first proposed by the Finish astronomer K. F. Sundman [1], who used it to solve the 3-body problem. Later, Sundman's approach was used in solving various problems of continuum mechanics. The generalized Sundman transformation can be applied to equations with a single independent variable. In particular, the Sundman type transformation is effective for solving nonlinear ODEs. For example, it has been applied for linearization problems of ordinary differential equations [2, 3, 4, 5], and for finding families of analytical solutions [6, 7]†.

As for equations with two independent variables (x, y), one can use the following transformation. Let a system with the two independent variables (x, y) admit two conservation laws. Writing the conservation laws in the form of differentials

$$\Phi_1 dx + \Phi_2 dy = 0, \;\; \Psi_1 dx + \Psi_2 dy = 0,$$

one can construct a transformation

$$\begin{array}{c} U = F(x,y,u), \\ dX = \Phi_1 dx + \Phi_2 dy, \;\; dY = \Psi_1 dx + \Psi_2 dy. \end{array} \tag{1}$$

A single conservation law can also be considered here, assuming, for example, $\Psi_1 = 0$, and $\Psi_2 = 1$. Transformations (1) can be considered as the extension of the generalized Sundman transformations to two independent variables.

In 1938 Bateman [8] established the invariance of the two-dimensional isentropic irrotational gasdynamics equations under a multi-parameter class of relations of the form (1), which later became known as reciprocal transformations. These transformations leave invariant the governing equations, up to the equation of state. In nonlinear continuum mechanics, reciprocal transformations have likewise proved to have diverse physical applications [9, 10]‡. The preceding attests to the importance of reciprocal transformations in physical applications.

Just as for the generalized Sundman transformations, transformations (1) were applied for mapping equations to a simpler form or to well-studied equations. In these cases transformations (1) were also called reciprocal. The literature on this subject is very extensive. For example, the nonlinear heat equation

$$u_t = (u^{-2}u_x)_x$$

†See also references threin.

‡The literature on invariant reciprocal transformations is very extensive. A short review can be found in [11].

is mapped into the classical linear heat equation $u_t = u_{xx}$ by the transformation [12, 13]

$$U = u^{-1}, \;\; dX = u\,dx + u^{-2}u_x\,dt, \;\; dT = dt.$$

Applications of reciprocal transformations to the spectral problems of Korteveg de Vries (KdV) and modified Korteveg de Vries (mKdV) equations, and advantages of the reciprocal transformations can be found in [14].

The present paper deals with invariant reciprocal transformations, where a nondegenerate change (1) maps a system of differential equations of a given class into a system of the same class (only arbitrary elements can be changed). In the group analysis method such these types of transformations are called equivalence transformations. A link between a one-parameter subclass of infinitesimal reciprocal-type transformations in gasdynamics and the Lie group approach was established in [15], where it was shown that the transformations found in [16] can compose a Lie group of transformations. This idea was also applied in [11] for relativistic gas dynamics equations, where a connection between one-parameter subclasses found recently in [17, 18] and a Lie group procedure was shown. The results of [15, 11] led to the development of a method for finding reciprocal transformations by using the Lie group approach. In [19] this method was applied to the one-dimensional magnetogasdynamics equations of an ideal perfect gas with infinite electrical conductivity. It has been shown there that the magnetogasdynamics equations have reciprocal transformations different from the Bateman type.

In the present paper, the method used in [19] is combined with the automorphism-based algebraic method [20, 21][§]. This combination allows finding all reciprocal transformations (not only those composing a group). As for an illustration, the method is applied to the two-dimensional stationary gas dynamics equations. Equivalence group, group of reciprocal transformations, and all discrete reciprocal transformations are presented in the present paper. It is proven that the reciprocal transformations found in [25] compose a complete set of reciprocal transformations, up to equivalence transformations.

The paper is organized as follows.

The next Section discusses results obtained in [25, 15, 26]. Section 3 deals with the equivalence group of the two-dimensional stationary gas dynamics equations. Section 4 discusses applications of the first method for seeking reciprocal transformations using the infinitesimal approach. This method uses two conservation laws written in the form of differentials. Section 5 provides a generalization of the first method, where none of the assumptions about the differentials are necessary. Section 6 is devoted to the application of the results of the previous section for finding all reciprocal transformations for the two-dimensional stationary gas dynamics equations. Summary of the results is also given in this Section. The last Section gives the Conclusions. Some necessary formulas are given in the Appendix.

[§]Further development and extensions of the algebraic method can be found in [22, 23, 24].

2. The Bateman-type reciprocal transformations of 2D stationary gasdynamics

The two-dimensional gas dynamics equations considered in the present paper have the form (see, for example, [27])

$$\begin{gathered} F_1=(\rho u)_x+(\rho v)_y=0,\ \ F_2=\rho(uu_x+vu_y)+p_x=0,\\ F_3=\rho(uv_x+vv_y)+p_y=0,\ \ F_4=uS_x+vS_y=0, \end{gathered} \tag{2}$$

where $\boldsymbol{q}=(u,v)$ is the gas velocity, while p is the gas pressure, ρ is the gas density and S is the specific entropy. An appropriate state equation must be added to system (2)

$$p=G(\rho,S). \tag{3}$$

System (2) implies the pair of conservation laws

$$\begin{gathered} (\rho uv)_x+(p+\rho v^2)_y=0,\\ (p+\rho u^2)_x+(\rho uv)_y=0. \end{gathered} \tag{4}$$

Using these conservation laws, new independent variables can be introduced by the formulas

$$\begin{gathered} dx'=\beta_1^{-1}[(p+\beta_2+\rho v^2)dx-\rho uvdy],\\ dy'=\beta_1^{-1}[-\rho uvdx+(p+\beta_2+\rho u^2)dy] \end{gathered} \tag{5}$$

subject to the requirement that

$$p+\beta_2+\rho q^2\neq 0. \tag{6}$$

In [25] it was established that the gasdynamic system (2) is invariant under the 4-parameter class of the Bateman-type reciprocal transformations

$$\begin{gathered} u'=\frac{\beta_1 u}{p+\beta_2},\ \ v'=\frac{\beta_1 v}{p+\beta_2},\\ p'=\beta_4-\frac{\beta_1^2\beta_3}{p+\beta_2},\ \ \rho'=\frac{\beta_3\rho(p+\beta_2)}{p+\beta_2+\rho q^2},\ \ S'=F(S). \end{gathered} \tag{7}$$

This result has its roots in work of Bateman on lift and drag functions in planar irrotational gasdynamics [8, 28, 26]. Short review of the analysis and applications of these transformations can be found in [15].

In [15], a link between a one-parameter subclass of infinitesimal reciprocal-type transformations in gasdynamics and a Lie group approach was established. It was observed that if one sets

$$\beta_1=\beta_2=\beta_4=\epsilon^{-1},\ \ \beta_3=1, \tag{8}$$

then the one-parameter class of reciprocal transformations

$$\begin{gathered} u'=\frac{u}{1+\epsilon p},\ \ v'=\frac{v}{1+\epsilon p},\\ p'=\frac{p}{1+\epsilon p},\ \ \rho'=\frac{\rho(1+\epsilon p)}{1+\epsilon(p+\rho q^2)},\ \ S'=F(S) \end{gathered} \tag{9}$$

together with

$$dx' = \epsilon[(p+\rho v^2)dx - \rho uvdy] + dx,$$
$$dy' = \epsilon[-\rho uvdx + (p+\rho u^2)dy] + dy \tag{10}$$

composes transformations similar to the Lie group of transformations. This observation led to establishing a method [11, 29] for constructing reciprocal transformations by using procedures developed in the group analysis method. According to [29], transformations (9), (10) are called by the group of reciprocal transformations. Similar to the classical theory of Lie group of point transformations, it is convenient to present these transformations by the infinitesimal generators $X = F(S)\partial_S$ and

$$Y = \rho^2 q^2 \partial_\rho + pu\partial_u + pv\partial_v + p^2\partial_p + \big(-(p+\rho v^2)dx + \rho uvdy\big)\,\partial_{dx} + \big(\rho uvdx - (p+\rho u^2)dy\big)\,\partial_{dy},$$

where $F(S)$ is an arbitrary function.

3. Equivalence point transformations

Equivalence transformations preserve a structure of equations. The problem of finding an equivalence transformation consists of the construction a transformation of the variables (x, y, u, v, ρ, S, G) that preserves the equations changing only their representative $G = G(\rho, S)$. For this purpose there are several methods. One of these methods is the direct solution of the equations determining such transformations (see for example [30]). Despite its complexity this method gives a complete set of the equivalence point transformations [31]. The determining equations become simpler for the equivalence transformations composing a Lie group [32], which is called an equivalence group.

Consider an equivalence group preserving system (2). A generator of a one-parameter group of equivalence transformations is assumed to be in the form [32, 33]

$$X^{\mathrm{e}} = \xi^x\partial_x + \xi^y\partial_y + \zeta^\rho\partial_\rho + \zeta^u\partial_u + \zeta^v\partial_v + \zeta^S\partial_S + \zeta^p\partial_p,$$

where all coefficients of the generator depend on (x, y, ρ, u, v, S, p), and $p = G(\rho, S)$ is considered as an arbitrary element.

For finding the equivalence group of transformations the infinitesimal criterion [32] is used. For this purpose the generator X^e is prolonged, and using the prolonged generator, applied to equations (2), the determining equations of the equivalence transformations are derived. The solution of these determining equations gives the general form of the elements of the equivalence group. Calculations show that the basis elements of the corresponding Lie algebra are

$$X_1^e = \partial_x,\ \ X_2^e = \partial_y,\ \ X_3^e = -v\partial_u + u\partial_v - y\partial_x + x\partial_y,\ \ X_4^e = x\partial_x + y\partial_y,$$
$$X_5^e = \rho\partial_\rho + p\partial_p,\ \ X_6^e = \partial_p,\ \ X_h^e = h(S)(-2\rho\partial_\rho + u\partial_u + v\partial_v),\ \ X_F^e = F(S)\partial_S \tag{11}$$

where $h(S)$ and $F(S)$ are sufficiently differentiable arbitrary functions.

The transformations corresponding to the generators X_i^e, $(i = 1, 2, ..., 4)$ and X_F are well-known in the gas dynamics. The generators X_1^e, and X_2^e define the shifts with respect to x and y, respectively, X_3^e corresponds to the rotation transformation, X_4^e and X_5^e define scalings, X_6^e corresponds to the shift of p. The generator X_h^e defines the projective transformations

$$\rho' = \rho\psi^{-2}, \ \ u' = u\psi, \ \ v' = v\psi, \tag{12}$$

where $\psi = e^{\epsilon h}$, ϵ is the group-parameter, and only changeable variables are presented. This transformation corresponds to the Munk–Prim transformation [34], which is also well-known in the gas dynamics. The transformations related to the generator X_F^e allow one to change $S' = \Psi(S)$, where $\Psi(S)$ is a sufficiently differentiable arbitrary function.

There are also known two obvious involutions

$$E_1 : \ \ x' = -x, \ \ u' = -u,$$

$$E_2 : \ \ y' = -y, \ \ v' = -v.$$

For further presentation it is convenient to introduce the following generators

$$X_1 = -v\partial_u + u\partial_v - dy\partial_{dx} + dx\partial_{dy}, \ \ X_2 = dx\partial_{dx} + dy\partial_{dy},$$

$$\begin{aligned} X_3 = \rho^2 q^2 \partial_\rho + pu\partial_u + pv\partial_v + p^2\partial_p + \left(-(p+\rho v^2)dx + \rho uvdy\right)\partial_{dx} \\ + \left(\rho uvdx - (p+\rho u^2)dy\right)\partial_{dy}, \end{aligned} \tag{13}$$

$$X_4 = \tfrac{1}{2}\left(2p\partial_p + u\partial_u + v\partial_v - dx\partial_{dx} - dy\partial_{dy}\right), \ \ X_5 = \partial_p.$$

One notices that among these generators only the generator $X_3 = Y$ substantially defines reciprocal transformations: the transformations corresponding to other generators are equivalent to the transformations belonging to equivalence group.

4. The first infinitesimal approach

For constructing a group of reciprocal transformations one can apply the following two methods.

The first method consists of two steps. First, using invariance of the conservation laws, one finds the coefficients of the generator of the group of reciprocal transformations related with the differentials. Then, using this coefficients, one defines the prolongation formulas for the group of reciprocal transformations. Applying the prolonged generator to the studied differential equations, one derives determining equations for the coefficients of the infinitesimal generator related with the dependent and independent variables. The general solution of the determining equations gives the set of generators of one-parameter groups of reciprocal transformations. Solving the Lie equations corresponding to these generators, one finds the one-parameter reciprocal transformations. As in the classical group analysis method, the multi-parameter transformations are constructed by the composition of the one-parameter reciprocal transformations.

Equations (4) can be rewritten in the form of differentials

$$\begin{aligned} S_1 = q_{11}\left((p + q_{12} + \rho v^2)dx - (\rho uv + q_{13})\, dy\right) = 0, \\ S_2 = q_{21}\left(-(\rho uv + q_{23})\, dx + (p + q_{22} + \rho u^2)dy\right) = 0. \end{aligned} \tag{14}$$

where q_{ij}, $(i=1,2;\ j=1,2,3)$ are constant such that $q_{11}q_{21}\neq 0$.

Consider the transformations defined by the generator

$$X=\zeta^{\rho}\partial_{\rho}+\zeta^{u}\partial_{u}+\zeta^{v}\partial_{v}+\zeta^{p}\partial_{p}+\zeta^{S}\partial_{S}+\zeta^{dx}\partial_{dx}+\zeta^{dy}\partial_{dy}, \tag{15}$$

where the coefficients ζ^{ρ}, ζ^{u}, ζ^{v}, ζ^{p} and ζ^{S} depend on the variables (x,y,ρ,u,v,p,S), ζ^{dx} and ζ^{dy} are linear forms with respect to the differentials dx and dy with the coefficients ${}^{x}\zeta^{dx}$, ${}^{y}\zeta^{dx}$, ${}^{x}\zeta^{dy}$ and ${}^{y}\zeta^{dy}$ also depending on the variables (x,y,ρ,u,v,p,S):

$$\zeta^{dx}={}^{x}\zeta^{dx}\,dx+{}^{y}\zeta^{dx}\,dy,\ \ \zeta^{dy}={}^{x}\zeta^{dy}\,dx+{}^{y}\zeta^{dy}\,dy.$$

Requiring that equations (14) are invariants of the generator X:

$$XS_i=0,$$

one finds that

$$\begin{gathered}\zeta^{dy}=\Delta^{-1}((\zeta^{u}\rho v(p+q_{12}+\rho v^2)+\zeta^{v}\rho(pu+q_{12}u-2q_{23}v-\rho uv^2)\\ +\zeta^{\rho}v(pu+q_{12}u-q_{23}v)-\zeta^{p}(q_{23}+\rho uv))dx\\ +(\zeta^{u}\rho(-2pu-2q_{12}u+q_{23}v-\rho uv^2)+\zeta^{v}\rho u(q_{23}+\rho uv)\\ +\zeta^{\rho}u(-pu-q_{12}u+q_{23}v)-\zeta^{p}(p+q_{12}+\rho v^2))dy)\end{gathered}$$

$$\begin{gathered}\zeta^{dx}=\Delta^{-1}((\zeta^{u}\rho v(q_{13}+\rho uv)+\zeta^{v}\rho(-2pv+q_{13}u-2q_{22}v-\rho u^2v)\\ +\zeta^{\rho}v(-pv+q_{13}u-q_{22}v)-\zeta^{p}(p+q_{22}+\rho u^2))dx\\ +(\zeta^{u}\rho(pv-2q_{13}u+q_{22}v-\rho u^2v)+\zeta^{v}\rho u(p+q_{22}+\rho u^2)\\ +\zeta^{\rho}u(pv-q_{13}u+q_{22}v)-\zeta^{p}(q_{13}+\rho uv))dy),\end{gathered}$$

where

$$\Delta=p^2+(q_{12}+q_{22})p+p\rho(u^2+v^2)+q_{12}q_{22}-q_{13}q_{23}+\rho(q_{12}u^2+q_{22}v^2)-(q_{13}+q_{23})\rho uv.$$

The next step consists of finding the coefficients ζ^{ρ}, ζ^{u}, ζ^{v}, ζ^{p} and ζ^{S}, satisfying the determining equations

$$(XF_i)_{|(2)}=0,\ \ (i=1,2,3,4). \tag{16}$$

Here X is the prolongation of the generator (15) with the prolongation formulas

$$\zeta^{f_x}=D_x\zeta^{f}-f_x\frac{\partial\zeta^{dx}}{\partial(dx)}-f_y\frac{\partial\zeta^{dy}}{\partial(dx)},\ \ \zeta^{f_y}=D_y\zeta^{f}-f_x\frac{\partial\zeta^{dx}}{\partial(dy)}-f_y\frac{\partial\zeta^{dy}}{\partial(dy)}, \tag{17}$$

where $f=\rho,\ u,\ \ v,\ \ p,\ \ S$, D_x and D_y are operators of the total derivatives with respect to x and y

$$D_x=\partial_x+\partial_{\rho}\rho_x+\partial_u u_x+\partial_v v_x+\partial_p p_x+\partial_S S_x,$$

$$D_y=\partial_y+\partial_{\rho}\rho_y+\partial_u u_y+\partial_v v_y+\partial_p p_y+\partial_S S_y,$$

the derivatives $\dfrac{\partial\zeta^{dx}}{\partial(dx)}$, $\dfrac{\partial\zeta^{dx}}{\partial(dy)}$, $\dfrac{\partial\zeta^{dy}}{\partial(dx)}$ and $\dfrac{\partial\zeta^{dy}}{\partial(dy)}$ mean the coefficients of the 1-forms ζ^{dx} and ζ^{dt}. The prolongation formulas (17) are derived by using the invariance of the differentials *df* during the transformations. Using this invariance, one finds formulas for transformation of derivatives, differentiating them with respect to the group-parameter, and setting them to zero, one obtains the prolongation formulas (17).

Calculations show that for solving the determining equations, it is necessary to consider three cases (a) $\zeta^p = 0$, (b) $\zeta^p q_{13} \neq 0$, and (c) $\zeta^p \neq 0$ and $q_{13} = 0$.

The case $\zeta^p = 0$ gives that $X = X_h^e + X_F^e$, which means that there are no reciprocal transformations for this case.

4.1. *Case* $\zeta^p q_{13} \neq 0$

In this case one obtains that $q_{23} = -q_{13}$, $q_{22} = q_{12}$, and the general solution of the determining equations (16) gives that

$$X = k(X_3 + 2q_{12}X_4 + q_{13}X_1 + (q_{12}^2 + q_{13}^2)X_5) + X_h^e + X_F^e,$$

where k is an arbitrary constant, $h(S)$ and $F(S)$ are arbitrary functions. For finding reciprocal transformations the generators X_h^e and X_F^e can be excluded. One first notices that $\zeta^{dx} = -k(q_{11}^{-1}S_1 + 2q_{13}dy)$ and $\zeta^{dy} = -k(q_{21}^{-1}S_2 - 2q_{13}dx)$. As S_1 and S_2 are invariants, then $dx' = -\epsilon k(q_{11}^{-1}S_1 + 2q_{13}dy) + dx$ and $dy' = -\epsilon k(q_{21}^{-1}S_2 - 2q_{13}\,dx) + dy$ or

$$\begin{aligned} dx' &= -\epsilon k\left((p + q_{12} + \rho v^2)dx - (\rho uv - q_{13})\,dy\right) + dx, \\ dy' &= \epsilon k\left((\rho uv + q_{13})\,dx - (p + q_{12} + \rho u^2)dy\right) + dy. \end{aligned} \tag{18}$$

It is more convenient to obtain transformations of the dependent variables in polar coordinates for the velocity defined by the change

$$u = R\sin(\theta), \quad v = R\cos(\theta).$$

The solution of the Lie equations becomes

$$\begin{gathered} R' = \frac{q_{13}R\sqrt{1+\lambda^2}}{q_{13} - \lambda(p + q_{12})}, \quad \theta' = \theta - \epsilon q_{13}, \quad \rho' = \rho\frac{\lambda(p + q_{12}) - q_{13}}{\lambda(p + q_{12} + \rho R^2) - q_{13}}, \\ p' = \frac{q_{13}p + \lambda(pq_{12} + q_{12}^2 + q_{13}^2)}{q_{13} - \lambda(p + q_{12})}, \quad S' = S, \end{gathered} \tag{19}$$

where $\lambda = \tan(\epsilon q_{13})$.

4.2. *Case* $\zeta^p \neq 0$ *and* $q_{13} = 0$

One has that $q_{23} = 0$, $q_{22} = q_{12}$, and the general solution of the determining equations (16) gives that

$$X = k_2(X_3 + 2q_{12}X_4 + q_{12}^2X_5) + k_1(2X_4 + 2q_{12}X_5 - X_2) + X_h^e + X_F^e,$$

where k_1 and k_2 are arbitrary constants, $h(S)$ and $F(S)$ are arbitrary functions. One obtains that $\zeta^{dx} = -k_2 q_{11}^{-1} S_1 - 2k_1\, dx$ and $\zeta^{dy} = -k_2 q_{21}^{-1} S_2 - 2k_1\, dy$. As S_1 and S_2 are invariants, then

$$\begin{aligned} dx' &= \epsilon\left(k_2\left(-(p+q_{12}+\rho v^2)dx + \rho uv dy\right) - 2k_1 dx\right) + dx, \\ dy' &= \epsilon\left(k_2\left(\rho uv dx - (p+q_{12}+\rho u^2)dy\right) - 2k_1 dy\right) + dy. \end{aligned} \tag{20}$$

The formulas for changing the dependent variables depend on k_1.

For $k_1 \neq 0$, one finds

$$\begin{gathered} u' = -\frac{2k_1 u\lambda}{k_2(\lambda^2-1)(p+q_{12}) - 2k_1}, \quad v' = -\frac{2k_1 v\lambda}{k_2(\lambda^2-1)(p+q_{12}) - 2k_1}, \\ \rho' = \rho\frac{k_2(\lambda^2-1)(p+q_{12}) - 2k_1}{k_2(\lambda^2-1)(p+q_{12}+\rho(u^2+v^2)) - 2k_1}, \\ p' = \frac{k_2 q_{12}(p+q_{12})(1-\lambda^2) + 2k_1(q_{12}(1-\lambda^2) - \lambda^2 p)}{k_2(p+q_{12})(\lambda^2-1) - 2k_1}, \end{gathered} \tag{21}$$

where $\lambda = e^{k_1\epsilon}$.

For $k_1 = 0$, one has

$$\begin{gathered} u' = \frac{u}{1-a(p+q_{12})}, \quad v' = \frac{v}{1-a(p+q_{12})}, \\ \rho' = \rho\frac{1-a(p+q_{12})}{1-a(p+q_{12}+\rho(u^2+v^2))}, \quad p' = \frac{p - a q_{12}(p+q_{12})}{1-a(p+q_{12})}, \end{gathered} \tag{22}$$

where $a = k_2\epsilon$.

5. Application of the second method

In the previous sections the differential forms S_1 and S_2 are used for constructing reciprocal transformations, while this section shows that they are not needed. In the second method there is no separation in the two steps like in the first method, as the second method does not use the conservation laws. The prolongation formulas of the generator of the group of reciprocal transformations are constructed by using the general form of the coefficients related with the differentials of the generator of the group of reciprocal transformations. This method is a little bit more complicated, but it can also be applied to systems without knowing their conservation laws.

5.1. Group of reciprocal transformations of (2)

Consider the generator

$$X = \zeta^\rho \partial_\rho + \zeta^v \partial_v + \zeta^p \partial_p + \zeta^S \partial_S + \zeta^{dt}\partial_{dt} + \zeta^{dx}\partial_{dx}, \tag{23}$$

where the coefficients ζ^ρ, ζ^u, ζ^v, ζ^p and ζ^S depend on the variables (x, y, ρ, u, v, p, S), ζ^{dx} and ζ^{dy} are linear 1-forms with respect to dx and dy with the coefficients also depending on the variables (x, y, ρ, u, v, p, S):

$$\zeta^{dx} = {}^x\zeta^{dx}\, dx + {}^y\zeta^{dx}\, dy, \quad \zeta^{dy} = {}^x\zeta^{dy}\, dx + {}^y\zeta^{dy}\, dy.$$

Requiring that equations (2) compose an invariant manifold of the generator X, one derives the determining equations

$$
\begin{gathered}
(u\zeta^{\rho_x}+v\zeta^{\rho_y}+\rho_x\zeta^u+\rho_y\zeta^v+\rho(\zeta^{u_x}+\zeta^{v_y})+\zeta^\rho(u_x+v_y))_{|(2)}=0,\\
(\rho u\zeta^{u_x}+\rho v\zeta^{u_y}+\rho u_x\zeta^u+\rho u_y\zeta^v+\zeta^\rho(uu_x+vu_y)+\zeta^{p_x})_{|(2)}=0,\\
(\rho u\zeta^{v_x}+\rho v\zeta^{v_y}+\rho v_x\zeta^u+\rho v_y\zeta^v+\zeta^\rho(uv_x+vv_y)+\zeta^{p_y})_{|(2)}=0,\\
\left(S_x\zeta^u+S_y\zeta^v+u\zeta^S_x+v\zeta^S_y\right)_{|(2)}=0,
\end{gathered}
\tag{24}
$$

where the sign $_{|(2)}$ has the usual meaning of considering the relations in parenthesis on the manifold defined by equations (2), where for ζ^{f_x} and ζ^{f_y}, $(f=\rho,\, u,\, v, p\,,\, S)$ one has to apply the prolongation formulas (17).

Beside equations (24) one also has to satisfy the equations $D_x\left({}^y\zeta^{dx}\right)-D_y\left({}^x\zeta^{dx}\right)=0$ and $D_x\left({}^y\zeta^{dy}\right)-D_y\left({}^x\zeta^{dy}\right)=0$, which are

$$
\begin{gathered}
-{}^x\zeta^{dx}_S S_y-{}^x\zeta^{dx}_p p_y-{}^x\zeta^{dx}_\rho \rho_y-{}^x\zeta^{dx}_u u_y-{}^x\zeta^{dx}_v v_y-{}^x\zeta^{dx}_y\\
+{}^y\zeta^{dx}_S S_x+{}^y\zeta^{dx}_p p_x+{}^y\zeta^{dx}_\rho \rho_x+{}^y\zeta^{dx}_u u_x+{}^y\zeta^{dx}_v v_x+{}^y\zeta^{dx}_x=0,\\
-{}^x\zeta^{dy}_S S_y-{}^x\zeta^{dy}_p p_y-{}^x\zeta^{dy}_\rho \rho_y-{}^x\zeta^{dy}_u u_y-{}^x\zeta^{dy}_v v_y-{}^x\zeta^{dy}_y\\
+{}^y\zeta^{dy}_S S_x+{}^y\zeta^{dy}_p p_x+{}^y\zeta^{dy}_\rho \rho_x+{}^y\zeta^{dy}_u u_x+{}^y\zeta^{dy}_v v_x+{}^y\zeta^{dy}_x=0.
\end{gathered}
\tag{25}
$$

The method of solving the determining equations (24), (25) is similar as in the classical group analysis method [32]. Calculations show that, solving the determining equations (24), (25), one obtains that

$$
X=\sum_{i=2}^{5}k_iX_i+X^e_h+X^e_F,
$$

where $h(S)$ and $F(S)$ are arbitrary functions, k_i, $(i=1,2,...,5)$ are arbitrary constants, and the generators X_i, $(i=1,2,...,5)$ are defined by formulas (13):

$$
\begin{gathered}
X_1=-v\partial_u+u\partial_v-dy\partial_{dx}+dx\partial_{dy},\;\; X_2=dx\partial_{dx}+dy\partial_{dy},\\
X_3=\rho^2q^2\partial_\rho+pu\partial_u+pv\partial_v+p^2\partial_p+\left(-(p+\rho v^2)dx+\rho uvdy\right)\partial_{dx}\\
+\left(\rho uvdx-(p+\rho u^2)dy\right)\partial_{dy},\\
X_4=\tfrac{1}{2}\left(2p\partial_p+u\partial_u+v\partial_v-dx\partial_{dx}-dy\partial_{dy}\right),\;\; X_5=\partial_p.
\end{gathered}
\tag{26}
$$

5.2. Algebraic properties of the Lie algebra $L_{rt} = \{X_1, X_2, X_3, X_4, X_5, X_h, X_F\}$

The commutator table of L_{rt} is

	X_1	X_2	X_3	X_4	X_5	X_h	X_F
X_1	0	0	0	0	0	0	0
X_2	0	0	0	0	0	0	0
X_3	0	0	0	$-X_3$	$-X_4$	0	0
X_4	0	0	X_3	0	$-X_5$	0	0
X_5	0	0	X_4	X_5	0	0	0
X_h	0	0	0	0	0	0	$-X_{h_1}$
X_F	0	0	0	0	0	X_{h_1}	0

where $h_1 = h'F$. From the commutator table, one concludes that the generators X_1 and X_2 compose the center, the generators X_3, X_4 and X_5 compose an ideal, the derivatives of L_{rt} are

$$L'_{rt} = [L_{rt}, L_{rt}] = \{X_3, X_4, X_5, X_h\}, \;\; L''_{rt} = [L'_{rt}, L'_{rt}] = \{X_3, X_4, X_5\}.$$

6. Complete set of reciprocal transformations

For finding discrete reciprocal transformations, it is adapted the method, proposed in [20] for finding discrete symmetries, and extended in [35] for equivalence transformations. The main idea of the method consists of using invariant sets of a group of reciprocal transformations.

Let be given a system of differential equations (S), and L be its (maximal) Lie algebra of reciprocal transformations, where $Aut(L)$ is the automorphism group. Any reciprocal transformation T provides an automorphism $T_* \in Aut(L)$ of the Lie algebra L. The transformation T_* is defined by the change of the coefficients of a generator under the transformation T. Notice that the change of the coefficients of a generator of a group of reciprocal transformations under a reciprocal transformation follows the same formulas as the change of the coefficients of a generator of a group of point transformations under a point transformation [32]. The property that $T_* \in Aut(L)$ follows from commutativeness of the operation of commutation of generators and the operation of the change of the variables.

For further study we use the following notations. Details and other properties one can see in [35] and references therein.

A megaideal τ of a Lie algebra L is a vector subspace of L, which is invariant under any mapping from the automorphism group $Aut(L)$. That is, for any megaideal τ of L, and any transformation A from $Aut(L)$ one has that $A\tau = \tau$.

Some megaideals of L can be computed without knowing $Aut(L)$. In particular, the center and the derivative of a Lie algebra are its megaideals [35]. Hence, $L' = [L, L]$ is a megaideal. We also use the property that a megaideal τ_2 of a megaideal τ_1 of L is also a megaideal of L. In particular, $L'' = (L')'$ is also a megaideal [35].

Thus, one obtains the megaideals of L_{rt} :

$$L'_{rt} = \{X_3, X_4, X_5, X_h\},\ \ L''_{rt} = \{X_3, X_4, X_5\},\ \ C_1 = \{X_1\},\ \ C_2 = \{X_2\}.$$

Notice also that the nonzero structure constants for L''_{rt} are

$$c^3_{34} = -1,\ \ c^4_{35} = -1,\ \ c^5_{45} = -1.$$

Consider an invertible transformation T defined by the relations

$$\begin{gathered}
\rho' = R(\rho, u, v, p, S),\ \ u' = U(\rho, u, v, p, S),\ \ v' = V(\rho, u, v, p, S),\\
p' = P(\rho, u, v, p, S),\ \ S' = H(\rho, u, v, p, S),\\
dx' = {}^{x}f^{dx}(\rho, u, v, p, S)\,dx + {}^{y}f^{dx}(\rho, u, v, p, S)\,dy,\\
dy' = {}^{x}f^{dy}(\rho, u, v, p, S)\,dx + {}^{y}f^{dy}(\rho, u, v, p, S)\,dy.
\end{gathered} \tag{27}$$

The transformation T is a reciprocal transformation if

$$\begin{gathered}
-RUdy' + RVdx' = 0,\ \ (P + RV^2)dx' - RUVdy' = 0,\\
RUVdx' - (RU^2 + P)dy' = 0,\ \ RHUdy' - RVHdx' = 0.
\end{gathered} \tag{28}$$

The transformation T_* acts on a generator $X \in L_{rt}$

$$X = \zeta^{\rho}\partial_{\rho} + \zeta^{v}\partial_{v} + \zeta^{p}\partial_{p} + \zeta^{S}\partial_{S} + \zeta^{dt}\partial_{dt} + \zeta^{dx}\partial_{dx}, \tag{29}$$

as follows

$$T_*X = \zeta^{\rho'}\partial_{\rho'} + \zeta^{v'}\partial_{v'} + \zeta^{p'}\partial_{p'} + \zeta^{S'}\partial_{S'} + \zeta^{dt'}\partial_{dt'} + \zeta^{dx'}\partial_{dx'}, \tag{30}$$

where

$$\zeta^{\rho'} = X\rho',\ \ \zeta^{v'} = Xv',\ \ \zeta^{p'} = Xp',\ \ \zeta^{S'} = XS',\ \ \zeta^{dt'} = X(dt'),\ \ \zeta^{dx'} = X(dx').$$

One of the sets of equations for the invertible transformation T to be a reciprocal transformation is defined by the conditions that dx' and dy' are differentials:

$$D_y({}^{x}f^{dx}) = D_x({}^{y}f^{dx}),\ \ D_y({}^{x}f^{dy}) = D_x({}^{y}f^{dy}). \tag{31}$$

Analysis of these equations is similar to the analysis of determining equations. As these equations have to be satisfied for any solution of equations (2), then substituting the main derivatives of equations (2) into (31), and splitting them with respect to parametrical derivatives, one derives the set of equations, which we also call determining equations.

6.1. Use of the megaideal L''_{rt}

Let X belongs to the megaideal $L''_{rt} = \{X_3, X_4, X_5\}$. As $T_* \in Aut(L''_{rt})$, then $T_*X \in L''_{rt}$, which means that there exist constants a_3, a_4, a_5 such that

$$T_*X = a_3X'_3 + a_4X'_4 + a_5X'_5,$$

where prime in X'_i means that it is the generator X_i with the primed variables:

$$X'_3 = {\rho'}^2{q'}^2\partial_{\rho'} + p'u'\partial_{u'} + p'v'\partial_{v'} + {p'}^2\partial_{p'}$$

$$+\left(-(p' + \rho'{v'}^2)dx' + \rho'u'v'dy'\right)\partial_{dx'} + \left(\rho'u'v'dx' - (p' + \rho'{u'}^2)dy'\right)\partial_{dy'},$$

$$X'_4 = \frac{1}{2}\left(2p'\partial_{p'} + u'\partial_{u'} + v'\partial_{v'} - dx'\partial_{dx'} - dy'\partial_{dy'}\right), \ \ X'_5 = \partial_{p'}.$$

In particular,

$$T_*X_i = a_{3i}X'_3 + a_{4i}X'_4 + a_{5i}X'_5, \ \ (i = 3, 4, 5), \tag{32}$$

where the matrix

$$A = \begin{pmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{pmatrix}$$

is nonsingular.

The constants a_{ij}, $(i, j = 3, 4, 5)$ have to satisfy the following relations. As $T_* \in Aut(L''_{rt})$, then

$$[T_*X_i, T_*X_j] = T_*[X_i, X_j] = \sum_{k=3}^{5} c^k_{ij}T_*X_k, \ \ (i, j = 3, 4, 5),$$

which lead to the conditions

$$\sum_{k=3}^{5}\sum_{s=3}^{5}\sum_{n=3}^{5} a_{ik}a_{js}c^n_{ks}X'_n = \sum_{k=3}^{5}\sum_{n=3}^{5} c^k_{ij}a_{nk}X'_n, \ \ (i, j = 3, 4, 5)$$

or after splitting

$$\sum_{k=3}^{5}\sum_{s=3}^{5} a_{ik}a_{js}c^n_{ks} = \sum_{k=3}^{5} c^k_{ij}a_{nk}, \ \ (i, j, n = 3, 4, 5) \tag{33}$$

Calculations show that equations (32) define the derivatives with respect ρ, u and p of all unknown functions (27). The solution of equations (32) and the representation of equations (33) are given in the Appendix. Analysis of equations (33) leads to the study of two cases (a) $a_{35} = 0$, and (b) $a_{35} \neq 0$.

All calculations were performed using symbolic manipulation system Reduce [36].

6.2. Case $a_{35}=0$

In this case, by virtue of that $\det A\neq 0$, one derives from equations (33) that $a_{33}\neq 0$ and

$$a_{34}=0,\ \ a_{43}=a_{54}a_{33},\ \ a_{44}=1,\ \ a_{45}=0,\ \ a_{53}=\frac{1}{2}a_{54}^2a_{33},\ \ a_{55}=a_{33}^{-1}. \tag{34}$$

6.2.1. Using the megaideal $\{X_1\}$ The generator X_1 is a center of the Lie algebra L. As any center is a megaideal, then there exists constant a_{11} such that

$$T_*X_1=a_{11}X_1'. \tag{35}$$

Calculations give that the latter equation defines derivatives with respect to v of all unknown functions:

$$\begin{gathered}R_v=0,\ U_v=\frac{Uv-Va_{11}u}{u^2+v^2},\ V_v=\frac{Ua_{11}u+Vv}{u^2+v^2},\ P_v=\frac{2Pa_{33}v+2v(a_{54}a_{33}-p)}{a_{33}(u^2+v^2)},\ H_v=0,\\ {}^x f^{dx}_v=-u\frac{{}^y f^{dx}+{}^x f^{dy}a_{11}}{u^2+v^2},\ \ {}^y f^{dx}_v=u\frac{{}^x f^{dx}-{}^y f^{dy}a_{11}}{u^2+v^2},\\ {}^x f^{dy}_v=u\frac{{}^x f^{dx}a_{11}-{}^y f^{dy}}{u^2+v^2},\ \ {}^y f^{dy}_v=u\frac{{}^y f^{dx}a_{11}+{}^x f^{dy}}{u^2+v^2}.\end{gathered} \tag{36}$$

The remaining equations are equations (31).

6.2.2. Analysis of equations (31) Splitting these equations with respect to parametric derivatives of equations (2), one obtains the determining equations

$$a_{11}^2=1,\ \ {}^y f^{dx}=-{}^x f^{dy}a_{11},\ \ {}^x f^{dx}={}^y f^{dy}a_{11}, \tag{37}$$

$$\begin{aligned}&{}^x f^{dy}a_{11}(p+\rho u^2-a_{33}(P+RV^2+a_{54}))+{}^y f^{dy}(a_{11}\rho uv-RUVa_{33})=0,\\ &{}^x f^{dy}(a_{11}\rho uv+RUVa_{33})+{}^y f^{dy}a_{11}(p+\rho v^2-a_{33}(P+RV^2+a_{54}))=0,\\ &-{}^x f^{dy}a_{11}(RUVa_{33}+a_{11}\rho uv)+{}^y f^{dy}(p+\rho u^2-a_{33}(P+RU^2+a_{54}))=0,\\ &-{}^x f^{dy}a_{11}(p+\rho v^2-a_{33}(P+RU^2+a_{54}))+{}^y f^{dy}(a_{11}\rho uv-RUVa_{33})=0,\end{aligned} \tag{38}$$

Considering equations (38) as a system of linear algebraic equations with respect to ${}^x f^{dy}$ and ${}^y f^{dy}$, and because of $({}^x f^{dy})^2+({}^y f^{dy})^2\neq 0$, one derives that

$$P=\frac{1}{2}\left(\frac{2p+\rho(u^2+v^2)}{a_{33}}-R(U^2+V^2+2a_{54})\right),\ \ R=\mu\rho\frac{u^2+v^2}{a_{33}(U^2+V^2)}, \tag{39}$$

where $\mu^2 = 1$, and equations (38) are reduced to the equations

$$\begin{gathered}
\mu\left({}^x f^{dy}(V^2-U^2)+2\,{}^y f^{dy}UVa_{11}\right)(u^2+v^2)\\
+\left(-2\,{}^y f^{dy}uv+{}^x f^{dy}(v^2-u^2)\right)(V^2+U^2)=0,\\
\left(-2^x f^{dy}uv+{}^y f^{dy}(u^2-v^2)\right)(U^2+V^2)\\
+\mu\left({}^y f^{dy}(V^2-U^2)-2^x f^{dy}UVa_{11}\right)(u^2+v^2)=0.
\end{gathered} \tag{40}$$

Integrating equations (32), which are presented in detail in the Appendix, and equations (36) for U and V, one derives that

$$U=\rho^{(1-\mu)/2}\left(\varphi_1 u+\varphi_2 v\right),\;\; V=a_{11}\rho^{(1-\mu)/2}(\varphi_1 v-\varphi_2 u), \tag{41}$$

where $\varphi_1(S)$ and $\varphi_2(S)$ are arbitrary functions such that $\varphi_1^2+\varphi_2^2\neq 0$. Equations (39) become

$$P=\frac{p}{a_{33}}-a_{54}+\frac{(1-\mu)}{2a_{33}}\rho(u^2+v^2),\;\; R=\frac{\mu\rho^{\mu}}{a_{33}(\varphi_1^2+\varphi_2^2)}. \tag{42}$$

Equations (37) give

$$dx'=a_{11}({}^y f^{dy}dx-{}^x f^{dy}dy),\;\; dy'={}^x f^{dy}dx+{}^y f^{dy}dy. \tag{43}$$

Further analysis of finding the coefficients ${}^x f^{dy}$ and ${}^y f^{dy}$depends on μ.

If $\mu=1$, then equations (40) provide

$${}^x f^{dy}\varphi_1+{}^y f^{dy}\varphi_2=0.$$

In a symmetric form one can represent a solution of the latter equation as

$${}^y f^{dy}=\varphi_1\psi^{-1},\;\; {}^x f^{dy}=-\varphi_2\psi^{-1},$$

where $\psi(S)$ is an arbitrary function. Equations (31) require that

$$\varphi_1=\alpha\psi,\;\; \varphi_2=\beta\psi,$$

where α and β are constant such that $\alpha^2+\beta^2\neq 0$.

Thus, one obtains that the transformation can be written in the form

$$\begin{gathered}
P=\frac{p}{a_{33}}-a_{54},\;\; R=\frac{\rho}{a_{33}\psi^2(\alpha^2+\beta^2)},\;\; U=\psi(\alpha u+\beta v),\;\; V=a_{11}\psi(\alpha v-\beta u),\;\; H=F(S),\\
dx'=a_{11}(\alpha dx+\beta dy),\;\; dy'=-\beta dx+\alpha dy.
\end{gathered} \tag{44}$$

As the coefficients ${}^x f^{dx}$, ${}^y f^{dx}$, ${}^x f^{dy}$ and ${}^y f^{dy}$ are constant, then the transformation (44) is equivalent to an equivalence transformation, in particular, composition of (12) with the rotation.

If $\mu=-1$, then (41), (42) become

$$P=\frac{p}{a_{33}}-a_{54}+\frac{1}{a_{33}}\rho(u^2+v^2),\ \ R=-\frac{1}{a_{33}\rho(\varphi_1^2+\varphi_2^2)},\tag{45}$$

$$U=\rho\left(\varphi_1 u+\varphi_2 v\right),\ \ V=a_{11}\rho(\varphi_1 v-\varphi_2 u).\tag{46}$$

Equations (40) give that

$${}^{x}f^{dy}\varphi_2-{}^{y}f^{dy}\varphi_1=0,$$

In a symmetric form one can represent a solution of the latter equation as ${}^{y}f^{dy}=\varphi_2\psi^{-1},\ \ {}^{x}f^{dy}=\varphi_1\psi^{-1}$. Equations (31) require that $\varphi_1=\alpha\psi,\ \ \varphi_2=\beta\psi$. Equations (28) in this case are only satisfied if the original solution is isentropic and irrotational. Hence, the case $\mu=-1$ also does not provide a reciprocal transformation.

6.3. *Case* $a_{35}\neq 0$

For this case

$$a_{33}=\frac{a_{34}^2}{2a_{35}},\ \ a_{43}=\frac{a_{34}(a_{45}a_{34}-2a_{35})}{2a_{35}^2},\ \ a_{44}=\frac{a_{45}a_{34}}{a_{35}}-1,$$
$$a_{53}=\frac{a_{45}^2a_{34}^2-4a_{45}a_{35}a_{34}+4a_{35}^2}{4a_{35}^3},\ \ a_{54}=\frac{a_{45}(a_{45}a_{34}-2a_{35})}{2a_{35}^2},\ \ a_{55}=\frac{a_{45}^2}{2a_{35}},$$

and the analysis of the equations defining the reciprocal transformations is similar to the case $a_{35}=0$, but more cumbersome. Because of their cumbersomeness, we only describe the main steps of the finding the reciprocal transformations.

Using the megaideal $\{X_1\}$, one finds the derivatives R_v, U_v, V_v, P_v, ${}^{x}f^{dx}_v$, ${}^{y}f^{dx}_v$, ${}^{x}f^{dy}_v$, and ${}^{y}f^{dy}_v$. After that equations (31) give the relation $u\,{}^{x}f^{dx}_S+v\,{}^{y}f^{dx}_S=0$, and an algebraic system of ten homogeneous linear equations with respect to ${}^{x}f^{dx}$, ${}^{y}f^{dx}$, ${}^{x}f^{dy}$, and ${}^{y}f^{dy}$. As $({}^{x}f^{dx})({}^{y}f^{dy})-({}^{y}f^{dx})({}^{x}f^{dy})\neq 0$, then the rank r of the matrix with respect to these variables satisfies the inequality $r\leq 3$. From the analysis of the minors of the latter system of linear homogeneous equations one finds P and R. Integrating the overdetermined system of equations for the functions U and V, one finds them. Substituting all the expressions of R, U, V, and P into a linear system for ${}^{x}f^{dx}$, ${}^{y}f^{dx}$, ${}^{x}f^{dy}$, and ${}^{y}f^{dy}$, one finds the reciprocal transformations

$$\begin{gathered}R=\frac{2\rho(p-g)}{\psi^2a_{35}(\alpha^2+\beta^2)(p+\rho q^2-g)},\ \ P=-\frac{a_{45}}{a_{35}}-\frac{2}{a_{35}(p-g)},\\ U=\frac{\psi(\alpha v+\beta u)}{p-g},\ \ V=a_{11}\frac{\psi(-\alpha u+\beta v)}{p-g},\ \ H=F,\\ dx'=k\left((\alpha\rho uv-\beta(p+\rho v^2-g))\,dx+(-\alpha(p+\rho u^2-g)+\beta\rho uv)\,dy\right),\\ dy'=ka_{11}\left((\alpha(p+\rho v^2-g)+\beta\rho uv)\,dx-(\alpha\rho uv+\beta(p+\rho u^2-g))\,dy\right),\end{gathered}\tag{47}$$

where $g=a_{34}a_{35}^{-1}$, $a_{11}^2=1$, α, β, and k are constant, $\psi(S)$ and $F(S)$ are arbitrary functions. Recall that a_{34}, and a_{45} are arbitrary constants, and $a_{35}\neq 0$.

Notice that because of the equivalence transformation corresponding to the involution E_2 and the rotation, one can assume that $a_{11} = 1$ and $\alpha = 0$. As $\alpha^2 + \beta^2 \neq 0$, one has that $\beta \neq 0$. By virtue of the equivalence transformation (12), one can reduce ψ from formulas (47). Introducing the constants $\beta_i,\ (i = 1, 2, 3, 4)$:

$$\beta = \frac{2}{\beta_1\beta_3},\ \ a_{34} = -\frac{2\beta_2}{\beta_1^2\beta_3},\ \ a_{35} = \frac{2}{\beta_1^2\beta_3},\ \ a_{45} = -\frac{2\beta_4}{\beta_1^2\beta_3},\ \ k = -\frac{\beta_3}{2},$$

formulas (47) coincide with (5), (7).

From the above study one can conclude that for finding all reciprocal transformations it was sufficient to use the megaideal $L''_{tr} = \{X_3, X_4, X_5\}$ and the center $\{X_1\}$. The final results obtained can be formulated as follows.

Theorem 6.1 *The complete set of reciprocal transformations of the two-dimensional stationary gas dynamics equations, considered up to the equivalence transformations corresponding to (11) and the involution E_2, consists of the transformations (5), (7):*

$$\begin{gathered} u' = \frac{\beta_1 u}{p + \beta_2},\ \ v' = \frac{\beta_1 v}{p + \beta_2}, \\ p' = \beta_4 - \frac{\beta_1^2\beta_3}{p + \beta_2},\ \ \rho' = \frac{\beta_3\rho(p + \beta_2)}{p + \beta_2 + \rho q^2},\ \ S' = F(S), \end{gathered} \tag{48}$$

$$\begin{aligned} dx' &= \beta_1^{-1}[(p + \beta_2 + \rho v^2)dx - \rho uv dy], \\ dy' &= \beta_1^{-1}[-\rho uv dx + (p + \beta_2 + \rho u^2)dy]. \end{aligned} \tag{49}$$

Remark 6.2 *The transformations (49), (48) can be further simplified by the equivalence transformations corresponding to (11). In particular, using the transformation corresponding to X_6^e, one can assume that $\beta_2 = 0$. Because of the transformation corresponding to X_5^e, one can assume that $\beta_1 = 1$. The reciprocal transformations (49), (48) become*

$$\begin{gathered} u' = \frac{u}{p},\ \ v' = \frac{v}{p},\ \ p' = \beta_4 - \frac{\tilde{\beta}_3}{p},\ \ \rho' = \frac{\tilde{\beta}_3\rho p}{p + \rho q^2},\ \ S' = F(S), \\ dx' = (p + \rho v^2)dx - \rho uv dy,\ \ dy' = -\rho uv dx + (p + \rho u^2)dy. \end{gathered} \tag{50}$$

7. Conclusions

Two methods for constructing a group of reciprocal transformations are presented in the paper. These methods are demonstrated by the two-dimensional stationary gas dynamics equations. Both methods use the infinitesimal approach. The first method requires two properties to be satisfied. The first property is that two conservation laws written in the form of differentials are required to be invariant under these transformations. This property gives the representation of the coefficients of the generator corresponding to the differentials. Using these coefficients, the prolongation of the generator is obtained. The second method provides a generalization of the first one, where none of the assumptions about the differentials are required. This method can also be applied to systems without knowing their conservation laws.

Another advantage of the second method is that the use of this method in the combination with the automorphism-based algebraic method can also be applied for finding all reciprocal transformations (not only constituting a group) of the equations under study. The present paper provides this algorithm and demonstrates it by the two-dimensional stationary gas dynamics equations, for which it is shown that, up to the equivalence transformations corresponding to (11) and the involution E_2, the complete set of reciprocal transformations consists of the transformations (50).

The proposed in the paper methods provide systematic tools for finding reciprocal transformations. The developed approach can be also extended to equations with more than two independent variables, while, as far as the authors know, there is only one example of reciprocal transformations of a system with three independent variables [37]. Further work is needed to explore the applications of the methods proposed in this paper.

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Appendix

Equations (32) can be written in the forms

$$R_\rho = (R^2U^2(a_{35}p^2 - 2a_{34}p + 2a_{33}) + R^2V^2(a_{35}p^2 - 2a_{34}p + 2a_{33}))/(2\rho^2(u^2 + v^2)),$$

$$R_u = (-R_v v + R^2U^2(-a_{35}p + a_{34}) + R^2V^2(-a_{35}p + a_{34}))/u,$$

$$R_p = R^2a_{35}(U^2 + V^2)/2,$$

$$U_\rho = (PU(a_{35}p^2 - 2a_{34}p + 2a_{33}) + U(a_{45}p^2 - 2a_{44}p + 2a_{43}))/(2\rho^2(u^2 + v^2)),$$

$$U_u = (-U_v v + PU(-a_{35}p + a_{34}) + U(-a_{45}p + a_{44}))/u,$$

$$U_p = (PUa_{35} + Ua_{45})/2,$$

$$V_\rho = (PV(a_{35}p^2 - 2a_{34}p + 2a_{33}) + V(a_{45}p^2 - 2a_{44}p + 2a_{43}))/(2\rho^2(u^2 + v^2)),$$

$$V_u = (-V_v v + PV(-a_{35}p + a_{34}) + V(-a_{45}p + a_{44}))/u,$$

$$V_p = (PVa_{35} + Va_{45})/2,$$

$$P_\rho = (P^2(a_{35}p^2 - 2a_{34}p + 2a_{33}) + 2P(a_{45}p^2 - 2a_{44}p + 2a_{43}) + 2(a_{55}p^2 - 2a_{54}p + 2a_{53}))/(2\rho^2(u^2 + v^2)),$$

$$P_u = (-P_v v + P^2(-a_{35}p + a_{34}) + 2P(-a_{45}p + a_{44}) + 2(-a_{55}p + a_{54}))/u,$$

$$P_p = (P^2a_{35} + 2Pa_{45} + 2a_{55})/2,$$

$$H_\rho = 0,\ \ H_u = -H_v v/u,\ \ H_p = 0,$$

$$\begin{aligned}{}^x f^{dx}_\rho = ({}^x f^{dx} P(-a_{35}p^2 + 2a_{34}p - 2a_{33}) + {}^x f^{dx} RV^{\,2}(-a_{35}p^2 + 2a_{34}p - 2a_{33})\\ +{}^x f^{dx}(-a_{45}p^2 + 2a_{44}p - 2a_{43} + 2\rho v^2) - 2{}^y f^{dx}\rho uv\\ +{}^x f^{dy} RUV(a_{35}p^2 - 2a_{34}p + 2a_{33}))/(2\rho^2(u^2+v^2)),\end{aligned}$$

$$\begin{aligned}{}^x f^{dx}_u = (-{}^x f^{dx}_v v + {}^x f^{dx} P(a_{35}p - a_{34}) + {}^x f^{dx} RV^{\,2}(a_{35}p - a_{34}) + {}^x f^{dx}(a_{45}p - a_{44} + 1)\\ +{}^x f^{dy} RUV(-a_{35}p + a_{34}))/u,\end{aligned}$$

$${}^x f^{dx}_p = (-{}^x f^{dx} Pa_{35} - {}^x f^{dx} RV^{\,2} a_{35} - {}^x f^{dx} a_{45} + {}^x f^{dy} RUV a_{35})/2,$$

$$\begin{aligned}{}^y f^{dx}_\rho = (-2{}^x f^{dx}\rho uv + {}^y f^{dx} P(-a_{35}p^2 + 2a_{34}p - 2a_{33}) + {}^y f^{dx} RV^{\,2}(-a_{35}p^2 + 2a_{34}p - 2a_{33})\\ +{}^y f^{dx}(-a_{45}p^2 + 2a_{44}p - 2a_{43} + 2\rho u^2) + {}^y f^{dy} RUV(a_{35}p^2 - 2a_{34}p + 2a_{33}))/(2\rho^2(u^2+v^2)),\end{aligned}$$

$$\begin{aligned}{}^y f^{dx}_u = (-{}^y f^{dx}_v)v + {}^y f^{dx} P(a_{35}p - a_{34}) + {}^y f^{dx} RV^{\,2}(a_{35}p - a_{34}) + {}^y f^{dx}(a_{45}p - a_{44} + 1)\\ +{}^y f^{dy} RUV(-a_{35}p + a_{34}))/u,\end{aligned}$$

$${}^y f^{dx}_p = (-{}^y f^{dx} Pa_{35} - {}^y f^{dx} RV^{\,2} a_{35} - {}^y f^{dx} a_{45} + {}^y f^{dy} RUV a_{35})/2,$$

$$\begin{aligned}{}^x f^{dy}_\rho = ({}^x f^{dx} RUV(a_{35}p^2 - 2a_{34}p + 2a_{33}) + {}^x f^{dy} P(-a_{35}p^2 + 2a_{34}p - 2a_{33})\\ +{}^x f^{dy} RU^{\,2}(-a_{35}p^2 + 2a_{34}p - 2a_{33}) + {}^x f^{dy}(-a_{45}p^2 + 2a_{44}p - 2a_{43} + 2\rho v^2)\\ -2{}^y f^{dy}\rho uv)/(2\rho^2(u^2+v^2)),\end{aligned}$$

$$\begin{aligned}{}^x f^{dy}_u = (-{}^x f^{dy}_v v + {}^x f^{dx} RUV(-a_{35}p + a_{34}) + {}^x f^{dy} P(a_{35}p - a_{34}) + {}^x f^{dy} RU^{\,2}(a_{35}p - a_{34})\\ +{}^x f^{dy}(a_{45}p - a_{44} + 1))/u,\end{aligned}$$

$${}^x f^{dy}_p = {}^x f^{dy}_p = ({}^x f^{dx} RUV a_{35} - {}^x f^{dy} Pa_{35} - {}^x f^{dy} RU^{\,2} a_{35} - {}^x f^{dy} a_{45})/2,$$

$$\begin{aligned}{}^y f^{dy}_\rho = ({}^y f^{dx} RUV(a_{35}p^2 - 2a_{34}p + 2a_{33}) - 2{}^x f^{dy}\rho uv + {}^y f^{dy} P(-a_{35}p^2 + 2a_{34}p - 2a_{33}))\\ +{}^y f^{dy} RU^{\,2}(-a_{35}p^2 + 2a_{34}p - 2a_{33}) + {}^y f^{dy}(-a_{45}p^2 + 2a_{44}p - 2a_{43} + 2\rho u^2))/(2\rho^2(u^2+v^2)),\end{aligned}$$

$$\begin{aligned}{}^y f^{dy}_u = (-{}^y f^{dy}_v v + {}^y f^{dx} RUV(-a_{35}p + a_{34}) + {}^y f^{dy} P(a_{35}p - a_{34}) + {}^y f^{dy} RU^{\,2}(a_{35}p - a_{34})\\ +{}^y f^{dy}(a_{45}p - a_{44} + 1))/u,\end{aligned}$$

$${}^y f^{dy}_p = ({}^y f^{dx} RUV a_{35} - {}^y f^{dy} Pa_{35} - {}^y f^{dy} RU^{\,2} a_{35} - {}^y f^{dy} a_{45})/2,$$

Equations (33) are

$$a_{44}a_{33} - a_{43}a_{34} - a_{33} = 0,\ \ a_{54}a_{33} - a_{53}a_{34} - a_{43} = 0,\ \ a_{54}a_{43} - a_{53}a_{44} - a_{53} = 0,$$

$$a_{45}a_{33} - a_{43}a_{35} - a_{34} = 0,\ \ a_{55}a_{33} - a_{53}a_{35} - a_{44} = 0,\ \ a_{55}a_{43} - a_{54} - a_{53}a_{45} = 0,$$

$$a_{45}a_{34} - a_{44}a_{35} - a_{35} = 0,\ \ a_{55}a_{34} - a_{54}a_{35} - a_{45} = 0,\ \ a_{55}a_{44} - a_{55} - a_{54}a_{45} = 0.$$

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