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The Schwarz lemma in bicomplex analysis

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Summary

We investigate Schwarz lemma in the framework of bicomplex numbers, which are pairs of complex numbers making up a commutative ring with zero-divisors. The bicomplex is a generalization of complex which has closed relation with Fractal geometry, Minkowski Space-Time, Maxwell's equations, Schrödinger equation and Gaussian pulse wave. In this paper, we first construct a type of bicomplex Möbius transformation and obtain some results : the mapping properties on bicomplex spheres and bicomplex ball, preserving the inverse points with respect to the bicomplex sphere $\mathbb{B}(0,1)$. Then we obtain the Poisson integral formula in bicomplex setting, and by using the Poisson integral formula, we give the Schwarz lemma for bicomplex holomorphic functions in bicomplex setting. Finally, we shall give the Schwarz lemma and the Schwarz-Pick type lemma for holomorphic functions in bicomplex analysis. These results may give new energy for the development of quantum mechanics.

KEYWORDS:
Bicomplex numbers , Möbius transformation , Schwarz lemma

1 | INTRODUCTION

The bicomplex $\mathbb{BC}$ is a generalization of complex which has closed relation with Fractal geometry, Minkowski Space-Time, Maxwell's equations, Schrödinger equation and Gaussian pulse wave [1,2,3,4,5,6,7]. The theory of bicomplex numbers is a matter of active research in recent times and there are many further discussions in this direction. A. Banerjee studied Bicomplex Fourier transform [8], Bicomplex Laplace transform [9], Bicomplex indefinite inner product modules [10] and Bicomplex Harmonic and Isotonic Oscillators [11]. R. Agarwal discussed Bicomplex Maxwell's equations [7], Bicomplex Mittag-Leffler function [12] and Bicomplex polygamma function [13]. In addition, Bicomplex Riemann Zeta Function, finite and infinite dimensional Bicomplex Hilbert Spaces , Bicomplex Hardy Space and Bicomplex Quantum Mechanics were discussed in [14,15,16,17,18] .

Developing the corresponding theories in bicomplex analysis framework comparing with the theories in complex analysis is necessary. M. E. Luna-Elizarrar$\acute{a}$s has been generalized some classical complex analysis theories to bicomplex analysis, for example: Bicomplex Riemann mapping theorem [19, Theorem 8.6.2], Bicomplex Weierstrass' theorem [19, Theorem 10.2.3] , Bicomplex Abel's theorem [19, Theorem 10.4.1], Bicomplex Cauchy integral theorem [19, Theorem 11.1.1], Bicomplex Borel-Pompeiu formula [19, Theorem 11.2.2], Bicomplex Cauchy integral representation [19, Theorem 11.2.3], Bicomplex Laurent Theorem [20, Theorem 3.2] and Bicomplex Residue Theorem [21, Theorem 6]. See [19,20,21,22] etc. for more details.

Motivated by these developments, we will investigate Schwarz lemma in the framework of bicomplex numbers. The Schwarz lemmas play very important role in classical complex analysis and there are many further discussions. In [23,24,25,26,27], the Schwarz type lemmas were built in Euclidean space, linear space $C(V_n, 0)$, octonionic space and complex geometry. In this paper, we

[0]**Abbreviations:** Math Meth Appl Sci, Math Meth Appl Sci; Math Meth Appl Sci, Math Meth Appl Sci; Math Meth Appl Sci, Math Meth Appl Sci factor

obtain the Poisson integral formula in bicomplex setting, and by using the Poisson integral formula, we give the Schwarz lemma for bicomplex holomorphic functions in bicomplex setting. Finally, we shall give the Schwarz lemma and the Schwarz-Pick type lemma for holomorphic functions in bicomplex analysis.

The structure of the paper is as follows. Section 2 preliminaries present the basic and necessary concepts and results about the bicomplex numbers. In Section 3 , we first construct a type of bicomplex Möbius transformation and give detailed presentation of basic properties of bicomplex Möbius transformation. In Section 4 , we obtain the Poisson integral formula for the theory of bicomplex functions. In Section 5 , we shall give the Schwarz lemma and the Schwarz-Pick type lemma for holomorphic functions in bicomplex analysis.

2 | PRELIMINARIES

The commutative ring $\mathbb{BC}$ of bicomplex numbers is defined as

$$\mathbb{BC} := \left\{ Z = z_1 + \mathbf{j}z_2 | z_1, z_2 \in \mathbb{C}(\mathbf{i}) \right\},$$

where $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$ are different commutative imaginary units , that is

$$\mathbf{i} \neq \mathbf{j}, \ \ \mathbf{ij} = \mathbf{ji} = \mathbf{k}, \ \ \mathbf{i}^2 = \mathbf{j}^2 = -1.$$

and $\mathbb{C}(\mathbf{i})$ is the set of complex numbers with imaginary unit $\mathbf{i}$. If $z_1 = x_1$ is a real number and $z_2 = y_2\mathbf{i}$ is a purely imaginary complex number, then $Z = x_1 + y_2\mathbf{k}$ is an element of the ring of hyperbolic numbers $\mathbb{D}$.

$$\mathbb{D} := \{x + y\mathbf{k} \mid x, y \in \mathbb{R}\}$$

Both the ring $\mathbb{BC}$ and the ring $\mathbb{D}$ have zero-divisors. In the case of bicomplex numbers , the set of zero-divisors is

$$\mathfrak{S} := \{Z | Z \neq 0, z_1^2 + z_2^2 = 0\}$$

and by definition

$$\mathfrak{S}_0 := \mathfrak{S} \cup \{0\}.$$

There are two special zero-divisors $\mathbf{e} := \frac{1+\mathbf{k}}{2}$ and $\mathbf{e}^\dagger := \frac{1-\mathbf{k}}{2}$, which have the properties:

$$\mathbf{e} + \mathbf{e}^\dagger = 1; \quad \mathbf{e} - \mathbf{e}^\dagger = \mathbf{k}; \tag{1}$$

$$\mathbf{e}\mathbf{e}^\dagger = 0; \ \ \mathbf{e}\mathbf{e} = \mathbf{e}; \text{ and } \mathbf{e}^\dagger\mathbf{e}^\dagger = \mathbf{e}^\dagger.$$

Consider the sets:

$$\mathbb{BC}_{\mathbf{e}} = \left\{\beta_1\mathbf{e} | \beta_1 \in \mathbb{C}(\mathbf{i})\right\} \ \text{ and } \ \mathbb{BC}_{\mathbf{e}^\dagger} = \left\{\beta_2\mathbf{e}^\dagger | \beta_2 \in \mathbb{C}(\mathbf{i})\right\}.$$

Obviously, the set of zero-divisors in $\mathbb{BC}$ is given by

$$\mathfrak{S}_0 = \mathbb{BC}_{\mathbf{e}} \cup \mathbb{BC}_{\mathbf{e}^\dagger}.$$

Each bicomplex number $Z = z_1 + \mathbf{j}z_2$ can be written as

$$Z = \left(z_1 - z_2\mathbf{i}\right)\mathbf{e} + \left(z_1 + z_2\mathbf{i}\right)\mathbf{e}^\dagger = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \text{ where } \beta_1, \beta_2 \in \mathbb{C}(\mathbf{i}).$$

The following conjugations were introduced on bicomplex numbers [12,28,29,30]

(i) $\overline{Z} := \overline{z_1} + \overline{z_2}\mathbf{j}$ (the bar-conjugation);

(ii) $Z^\dagger := z_1 - z_2\mathbf{j}$ (the †-conjugation);

(iii) $Z^* := (\overline{Z})^\dagger = \overline{Z^\dagger} = \overline{z_1} - \overline{z_2}\mathbf{j}$ (the $*$-conjugation).

all of them are automorphisms on the ring $\mathbb{BC}$,

Besides, the above conjugations suggest to consider the three moduli for bicomplex numbers:

(i)$|Z|_{\mathbf{i}}^2 := Z \cdot Z^\dagger = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i})$;

(ii)$|Z|_{\mathbf{j}}^2 := Z \cdot \bar{Z} = \eta_1^2 + \eta_2^2 \in \mathbb{C}(\mathbf{j})$, where $Z = \eta_1 + \eta_2\mathbf{i} = \left(x_1 + x_2\mathbf{j}\right) + \left(y_1 + y_2\mathbf{j}\right)\mathbf{i}$;

(iii)$|Z|_{\mathbf{k}}^2 := Z \cdot Z^* = \left|\beta_1\right|^2\mathbf{e} + \left|\beta_2\right|^2\mathbf{e}^\dagger \in \mathbb{D}^+$, where the set $\mathbb{D}^+$can be described as $\mathbb{D}^+ = \left\{\upsilon\mathbf{e} + \mu\mathbf{e}^\dagger \mid \upsilon, \mu \geq 0\right\}$.

The following partial order on hyperbolic numbers, introduced by Luna-Elizarrarás ME in [19,21,31].

Given hyperbolic numbers $\mathfrak{z}$ and $\mathfrak{w}$, we say that $\mathfrak{z} \preceq \mathfrak{w}$ if $\mathfrak{w} - \mathfrak{z} \in \mathbb{D}^+$. When $\mathfrak{w} - \mathfrak{z} \in \mathbb{D}^+\backslash\{0\}$, we write $\boldsymbol{z} \prec \mathfrak{w}$.

The properties of $\mathbf{k}$-modulus are as follows:

1.$|Z|_{\mathbf{k}} = 0$ if and only if $Z = 0$.

2.it satisfies the multiplicative property:

$$|ZW|_{\mathbf{k}} = |Z|_{\mathbf{k}} \cdot |W|_{\mathbf{k}}$$

3.the **k**-modulus satisfies the triangle inequality:

$$|Z + W|_{\mathbf{k}} \preceq |Z|_{\mathbf{k}} + |W|_{\mathbf{k}}$$

Once we have this partial order, it is possible to define a bicomplex circumference $\mathbb{S}\left(Z_0, R\right)$ of hyperbolic radius $R = r_1\mathbf{e} + r_2\mathbf{e}^\dagger \in \mathbb{D}^+\backslash\{0\}$ and center Z_0:

$$\mathbb{S}\left(Z_0, R\right) := \left\{Z \in \mathbb{BC} \| Z - Z_0|_{\mathbf{k}} = R\right\}.$$

The extended set of bicomplex numbers was introduced in[20]. The extended set of bicomplex numbers is

$$\overline{\mathbb{BC}} := \overline{\mathbb{C}(\mathbf{i})}\mathbf{e} + \overline{\mathbb{C}(\mathbf{i})}\mathbf{e}^\dagger$$

where $\overline{\mathbb{C}(\mathbf{i})}$ is the well-known compactification of the $\mathbb{C}(\mathbf{i})$-complex plane.

3 | MÖBIUS TRANSFORMATION AND PROPERTIES

It is well known that the Möbius transformation plays an important role in the study of complex analysis and there are many further discussions. John A. Emanuello[32] studied the Möbius transformation in $\mathbb{R}^{1,1}$ and discussed its conformality, transitivity, and fixed points. A. Golberg[33] defined the $\mathbb{D}-$Möbius transformation in $\mathbb{D}^n$ and came to the conclusion that $\mathbb{D}-$Möbius transformation can be expressed as a product of the six elementary Möbius transformations. C. Ghosh[34] defined the bicomplex Möbius transformation and discussed its fixed points . We will construct a type of bicomplex Möbius transformation and discuss some of its properties. The bicomplex Möbius transformation $F_A(Z)$ is denoted by:

$$F_A(Z) = (Z - A)(1 - A^*Z)^{-1} = (Z - A)\frac{1 - AZ^*}{|1 - A^*Z|^2_{\mathbf{k}}} \tag{2}$$

where $A = A_1\mathbf{e} + A_2\mathbf{e}^\dagger$, $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \mathbb{BC}$, $|A|_{\mathbf{k}} \neq 0, 1$ and $A^*Z \neq 1$.

Theorem 1. $F_A(Z)$ can be rewritten as when $|A|_{\mathbf{k}} \neq 0$.

$$F_A(Z) = -\frac{A}{|A|_{\mathbf{k}}} + \frac{(1 - |A|^2_{\mathbf{k}})A^2}{|A|^3_{\mathbf{k}}} \cdot \left(\frac{A}{|A|_{\mathbf{k}}} - |A|_{\mathbf{k}}Z\right)^{-1} \tag{3}$$

Proof. By (1), we have

$$\begin{aligned} F_A(Z) &= (Z - A)\frac{1 - AZ^*}{|1 - A^*Z|^2_{\mathbf{k}}} \\ &= (\beta_1 - A_1)\frac{1 - \overline{\beta}_1 A_1}{|1 - \overline{A}_1\beta_1|^2}\mathbf{e} + (\beta_2 - A_2)\frac{1 - \overline{\beta}_2 A_2}{|1 - \overline{A}_2\beta_2|^2}\mathbf{e}^\dagger \end{aligned} \tag{4}$$

Since

$$\begin{aligned} &(\beta_i - A_i)\frac{1 - \overline{\beta}_i A_i}{|1 - \overline{A}_i\beta_i|^2} \\ =&\frac{1}{|A_i|^2}\frac{(1 - |A_i|^2)(A_i - A_i^2\overline{\beta}_i) - |1 - \overline{A}_i\beta_i|^2}{|1 - \overline{A}_i\beta_i|^2} \\ =&-\frac{A_i}{|A_i|^2} + \frac{(1 - |A_i|^2)A_i^2}{|A_i|^3}\left(\frac{A_i}{|A_i|} - |A_i|\beta_i\right)^{-1}, i = 1, 2. \end{aligned} \tag{5}$$

Combining (4) with (5), (3) follows. □

Remark 1. By Theorem 3.1, Möbius transformation $F_A(Z)$ can be expressed as a product of the following three elementary Möbius transformations: (i) $F(Z) = Z + A, A \in \mathbb{BC}$. (ii) $F(Z) = \frac{1}{Z} = \frac{Z^*}{|Z|_k^2}, Z \in \mathbb{BC}, |Z|_k \neq 0$. (iii) $F(Z) = \zeta Z, \zeta \in \mathbb{BC}$.

Theorem 2. Möbius transformation $F_A^{-1}(Z)$ satisfies:

$$F_A^{-1}(Z) = (Z + A)\frac{1 + AZ^*}{|1 + A^*Z|_{\mathbf{k}}^2} = F_{-A}(Z)$$

Proof. Our approach is to verify directly $F_A(F_{-A}(Z)) = Z$.

$$\begin{aligned}
&F_A(F_{-A}(Z)) = F_A\left((Z + A)\frac{1 + AZ^*}{|1 + A^*Z|_{\mathbf{k}}^2}\right)\\
=&\left((Z + A)\frac{1 + AZ^*}{|1 + A^*Z|_{\mathbf{k}}^2} - A\right)\frac{1 - A(Z^* + A^*)\frac{1+A^*Z}{|1+A^*Z|_{\mathbf{k}}^2}}{\left|1 - A^*(Z + A)\frac{1+AZ^*}{|1+A^*Z|_{\mathbf{k}}^2}\right|_{\mathbf{k}}^2}\\
=&\frac{\left((\beta_1 + A_1)(1 + \overline{\beta}_1 A_1) - A_1|1 + \overline{A}_1\beta_1|^2\right)\left(|1 + \overline{A}_1\beta_1|^2 - A(1 + \overline{A}_1\beta_1)(\overline{A}_1 + \overline{\beta}_1)\right)}{\left||1 + \overline{A}_1\beta_1|^2 - (\beta_1 + A_1)(1 + \overline{\beta}_1 A_1)\right|^2}\mathbf{e}\\
+&\frac{\left((\beta_2 + A_2)(1 + \overline{\beta}_2 A_2) - A_2|1 + \overline{A}_2\beta_2|^2\right)\left(|1 + \overline{A}_2\beta_2|^2 - A(1 + \overline{A}_2\beta_2)(\overline{A}_2 + \overline{\beta}_2)\right)}{\left||1 + \overline{A}_2\beta_2|^2 - (\beta_2 + A_2)(1 + \overline{\beta}_2 A_2)\right|^2}\mathbf{e}^\dagger\\
=&\frac{|1 + \overline{A}_1\beta_1|^2\beta_1(1 - |A_1|^2)^2}{\left||1 + \overline{A}_1\beta_1|^2 - (\beta_1 + A_1)(1 + \overline{\beta}_1 A_1)\right|^2}\mathbf{e} + \frac{|1 + \overline{A}_2\beta_2|^2\beta_2(1 - |A_2|^2)^2}{\left||1 + \overline{A}_2\beta_2|^2 - (\beta_2 + A_2)(1 + \overline{\beta}_2 A_2)\right|^2}\mathbf{e}^\dagger\\
=&\frac{|1 + \overline{A}_1\beta_1|^2\beta_1(1 - |A_1|^2)^2}{|1 + \overline{A}_1\beta_1|^2(1 - |A_1|^2)^2}\mathbf{e} + \frac{|1 + \overline{A}_2\beta_2|^2\beta_2(1 - |A_2|^2)^2}{|1 + \overline{A}_2\beta_2|^2(1 - |A_2|^2)^2}\mathbf{e}^\dagger\\
=&\beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger = Z.
\end{aligned}$$

The eauality $F_A(F_{-A}(Z)) = Z$ is verified. □

In [19], the bicomplex spheres were introduced. Let $Z_0 = \beta_{1,0}\mathbf{e} + \beta_{2,0}\mathbf{e}^\dagger \in \mathbb{BC}$ and $R = r_1\mathbf{e} + r_2\mathbf{e}^\dagger \in \overline{\mathbb{D}}_1^+$. If $r_1 > 0, r_2 > 0, r_1 \neq +\infty \neq r_2$,

$$\begin{aligned}
\mathbb{B}(Z_0, R) &= \left\{Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \mathbb{BC} | 0 \leq |\beta_1 - \beta_{1,0}| < r_1, 0 \leq |\beta_2 - \beta_{2,0}| < r_2\right\}\\
\overline{\mathbb{B}(Z_0, R)}^C &= \left\{Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \mathbb{BC} | |\beta_1 - \beta_{1,0}| > r_1, |\beta_2 - \beta_{2,0}| > r_2\right\}\\
&\cup \left\{Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \mathbb{BC} | 0 \leq |\beta_1 - \beta_{1,0}| \leq r_1, |\beta_2 - \beta_{2,0}| > r_2\right\}\\
&\cup \left\{Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \mathbb{BC} | |\beta_1 - \beta_{1,0}| > r_1, 0 \leq |\beta_2 - \beta_{2,0}| \leq r_2\right\}.\\
&:= \overline{\mathbb{B}_1(Z_0, R)}^C \cup \overline{\mathbb{B}_2(Z_0, R)}^C \cup \overline{\mathbb{B}_3(Z_0, R)}^C.
\end{aligned}$$

Theorem 3. Möbius transformation $F_A(Z)$ satisfies the following mapping properties:

1.$F_A(Z)$ maps $\mathbb{S}(0, 1)$ one-to-one onto itself.

2.If $|A|_{\mathbf{k}} \prec 1$, then

(i) $F_A(Z)$ maps $\mathbb{B}(0, 1)$ one-to-one onto itself.

(ii) $F_A(Z)$ maps $\overline{\mathbb{B}(0, 1)}^C$ one-to-one onto itself.

3.If $|A|_{\mathbf{k}} \succ 1$, then

(i) $F_A(Z)$ maps $\mathbb{B}(0, 1)$ one-to-one onto $\overline{\mathbb{B}_1(0, 1)}^C$.

(ii) $F_A(Z)$ maps $\overline{\mathbb{B}_1(0, 1)}^C$ one-to-one onto $\mathbb{B}(0, 1)$.

(iii) $F_A(Z)$ maps $\overline{\mathbb{B}_2(0, 1)}^C$ one-to-one onto $\overline{\mathbb{B}_3(0, 1)}^C$.

(iv) $F_A(Z)$ maps $\overline{\mathbb{B}_3(0, 1)}^C$ one-to-one onto $\overline{\mathbb{B}_2(0, 1)}^C$.

Proof. For item 1,

$$|F_A(Z)|_{\mathbf{k}} = \frac{|Z - A|_{\mathbf{k}}}{|1 - A^*Z|_{\mathbf{k}}} = \frac{|Z - A|_{\mathbf{k}}}{|ZZ^* - A^*Z|_{\mathbf{k}}} = \frac{1}{|Z|_{\mathbf{k}}} = 1$$

For item 2 ,

$$\begin{aligned}1-|F_A(Z)|_{\mathbf{k}}^2 &= \frac{|1-A^*Z|_{\mathbf{k}}^2-|Z-A|_{\mathbf{k}}^2}{|1-A^*Z|_{\mathbf{k}}^2}\\ &= \frac{\left((|\beta_1|^2-1)(1-|A_1|^2)\right)}{|1-\overline{A}_1\beta_1|^2}\mathbf{e}+\frac{\left((|\beta_2|^2-1)(1-|A_2|^2)\right)}{|1-\overline{A}_2\beta_2|^2}\mathbf{e}^\dagger\end{aligned}$$

If $|A|_{\mathbf{k}} \prec 1$, we have

$$\begin{cases}|F_A(Z)|_{\mathbf{k}} \prec 1, |Z|_{\mathbf{k}} \prec 1\\ |F_A(Z)|_{\mathbf{k}} \succ 1, |Z|_{\mathbf{k}} \succ 1\end{cases}$$

Similar discussions as above will obtain item 3. □

In[33], a reflection with respect to $\mathbb{S}(Z_0, R)$ is defined as

$$\begin{aligned}F_{\mathbb{S}(Z_0,R)}(Z) &= Z_0+\frac{R^2(Z-Z_0)}{|Z-Z_0|_{\mathbf{k}}^2}\\ &= \left(\beta_{1,0}+\frac{r_1^2(\beta_1-\beta_{1,0})}{|\beta_1-\beta_{1,0}|^2}\right)\mathbf{e}+\left(\beta_{2,0}+\frac{r_2^2(\beta_2-\beta_{2,0})}{|\beta_2-\beta_{2,0}|^2}\right)\mathbf{e}^\dagger\\ F_{\mathbb{S}(Z_0,R)}(Z_0) &= \infty_{\mathbb{BC}} \qquad F_{\mathbb{S}(Z_0,R)}(\infty_{\mathbb{BC}}) = Z_0,\end{aligned}$$

where $\infty_{\mathbb{BC}} = \infty\mathbf{e}+\infty\mathbf{e}^\dagger$.

Remark 3.2. $F_{\mathbb{S}(Z_0,R)}(Z)$ is also called the inverse point of Z with respect to $\mathbb{S}(Z_0, R)$.

Theorem 4. Let Möbius transformation $F_A(Z)$ be as above in (2). Then for any $Z \in \mathbb{BC}$,

$$F_A(F_{\mathbb{S}(0,1)}(Z)) = F_{\mathbb{S}(0,1)}(F_A(Z))$$

Proof. According to the expression for Möbius transformation, we have

$$\begin{aligned}F_A(F_{\mathbb{S}(0,1)}(Z)) = F_A\left(\frac{Z}{|Z|_{\mathbf{k}}^2}\right) &= \frac{(1-AZ^*)(Z-A)}{|Z-A|_{\mathbf{k}}^2}\\ &= \frac{(1-A^*Z)^*}{(Z-A)^*} = F_{\mathbb{S}(0,1)}(F_A(Z)).\end{aligned}$$

□

Let we write the bicomplex function as $F = f_{11}+\mathbf{i}f_{12}+\mathbf{j}f_{21}+\mathbf{k}f_{22}$ in terms of its real components, which are all real functions of a bicomplex variable. For Möbiusbius transformation $F_A(Z)$, donote

$$J(F_A(Z)) = \frac{\partial(f_{11}, f_{12}, f_{21}, f_{22},)}{\partial(x_1, y_1, x_2, y_2,)}, \tag{6}$$

where $F_A(Z) = f_{11}+\mathbf{i}f_{12}+\mathbf{j}f_{21}+\mathbf{k}f_{22}, Z = x_1+\mathbf{i}y_1+\mathbf{j}x_2+\mathbf{k}y_{22}$

Lemma 1. Suppose that $F(Z) = \frac{1}{Z}$, then

$$J(F(Z)) = \frac{1}{\left||Z|_{\mathbf{i}}^2\right|^4}.$$

Proof. Since

$$F(Z) = \frac{1}{Z} = \frac{1}{\beta_1\mathbf{e}+\beta_2\mathbf{e}^\dagger} = \frac{1}{\beta_1}\mathbf{e}+\frac{1}{\beta}\mathbf{e}^\dagger$$

we have

$$\begin{aligned}f_{11} &= \frac{1}{2}\left(\frac{x_1+y_2}{|\beta_1|^2}+\frac{x_1-y_2}{|\beta_2|^2}\right) \quad f_{12} = \frac{1}{2}\left(\frac{x_2+y_1}{|\beta_1|^2}-\frac{x_2-y_1}{|\beta_2|^2}\right)\\ f_{21} &= \frac{1}{2}\left(\frac{x_2-y_1}{|\beta_1|^2}+\frac{x_2+y_1}{|\beta_2|^2}\right) \quad f_{22} = \frac{1}{2}\left(\frac{x_1+y_2}{|\beta_1|^2}-\frac{x_1-y_2}{|\beta_2|^2}\right).\end{aligned} \tag{7}$$

Combining (6) with (7),

$$
\begin{aligned}
J(F(Z)) &= \begin{vmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{vmatrix} \\
&= \left((b+c)^2 + (a-d)^2\right) \cdot \left((b-c)^2 + (a+d)^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \frac{1}{2}\left(\frac{(y_1-x_2)^2-(x_1+y_2)^2}{|\beta_1|^4} - \frac{(x_1-y_2)^2+(x_2-y_1)^2}{|\beta_2|^4}\right) \\
b &= \frac{(y_1-x_2)(x_1+y_2)}{|\beta_1|^4} + \frac{(x_1-y_2)(x_2+y_1)}{|\beta_2|^4} \\
c &= -\frac{(y_1-x_2)(x_1+y_2)}{|\beta_1|^4} + \frac{(x_1-y_2)(x_2+y_1)}{|\beta_2|^4} \\
d &= \frac{1}{2}\left(\frac{(y_1-x_2)^2-(x_1+y_2)^2}{|\beta_1|^4} + \frac{(x_1-y_2)^2+(x_2-y_1)^2}{|\beta_2|^4}\right)
\end{aligned}
$$

This leads to the conclusion easily.

□

Theorem 5. Suppose that $F_A(Z) = (Z-A)(1-A^*Z)^{-1}$, then

$$
J(F_A(Z)) = \frac{\left|\left|1-|A|_{\mathbf{k}}\right|_{\mathbf{i}}^2\right|^2}{\left|\left|1-A^*Z\right|_{\mathbf{i}}^2\right|^4}.
$$

where $A, Z \in \mathbb{BC}$, $|A|_{\mathbf{k}} \neq 0, 1$ and $A^*Z \neq 1$.

Proof. It is easy to check that

$$
J(F(Z)) = \begin{cases} 1 & F(Z) = Z + A,\ A \in \mathbb{BC} \\ 2|\xi_1\xi_2|^2 & F(Z) = \xi Z,\ \xi = \xi_1\mathbf{e} + \xi_2\mathbf{e}^\dagger \in \mathbb{BC} \end{cases} \tag{8}
$$

According to Theorem 1, lemma 1 and (8), we have

$$
\begin{aligned}
J(F_A(Z)) &= J\left(-\frac{A}{|A|_{\mathbf{k}}} + \frac{(1-|A|_{\mathbf{k}}^2)A^2}{|A|_{\mathbf{k}}^3} \cdot \left(\frac{A}{|A|_{\mathbf{k}}} - |A|_{\mathbf{k}} Z\right)^{-1}\right) \\
&= J\left(\frac{1-|A|_{\mathbf{k}}^2}{|A|_{\mathbf{k}}} \cdot \frac{A^2}{|A|_{\mathbf{k}}^2} \cdot \left(\frac{A}{|A|_{\mathbf{k}}} - |A|_{\mathbf{k}} Z\right)^{-1}\right) \\
&= \frac{(1-|A_1|)^2(1-|A_2|)^2}{|A_1A_2|^2} J\left(\left(\frac{A}{|A|_{\mathbf{k}}} - |A|_{\mathbf{k}} Z\right)^{-1}\right) \\
&= \frac{(1-|A_1|)^2(1-|A_2|)^2}{|A_1A_2|^2} \cdot \frac{1}{\left|\left|\frac{A}{|A|_{\mathbf{k}}} - |A|_{\mathbf{k}} Z\right|_{\mathbf{i}}^2\right|^4} J\left(\frac{A}{|A|_{\mathbf{k}}} - |A|_{\mathbf{k}} Z\right) \\
&= \frac{\left|\left|1-|A|_{\mathbf{k}}\right|_{\mathbf{i}}^2\right|^2}{\left|\left|1-A^*Z\right|_{\mathbf{i}}^2\right|^4}.
\end{aligned}
$$

□

4 | THE BICOMPLEX POISSON INTEGRAL FORMULA

Theorem 6. Let f be a function $\mathbb{BC}$-holomorphic in $\mathbb{B}(0,1)\cup\mathbb{S}(0,1)$. Then

$$f(Z)=\frac{1}{2\pi}\int\limits_{\Gamma_1}\frac{(R^2-|Z-Z_0|_{\mathbf{k}}^2)f(Y)d\theta}{|Y-Z|_{\mathbf{k}}^2}\mathbf{e}+\frac{1}{2\pi}\int\limits_{\Gamma_2}\frac{(R^2-|Z-Z_0|_{\mathbf{k}}^2)f(Y)d\theta}{|Y-Z|_{\mathbf{k}}^2}\mathbf{e}^{\dagger},$$

where

$$\Gamma_1:=\{Y\mathbf{e}\in\mathbb{BC}_{\mathbf{e}}|\left|(Y-Z_0)\mathbf{e}\right|_{\mathbf{k}}=R\mathbf{e}\}$$

and

$$\Gamma_2:=\{Y\mathbf{e}^{\dagger}\in\mathbb{BC}_{\mathbf{e}^{\dagger}}|\left|(Y-Z_0)\mathbf{e}^{\dagger}\right|_{\mathbf{k}}=R\mathbf{e}^{\dagger}\}$$

$\theta=\theta_1\mathbf{e}+\theta_2\mathbf{e}^{\dagger}, Y=\beta_{Y,1}\mathbf{e}+\beta_{Y,2}\mathbf{e}^{\dagger}, \theta_1=\arg\beta_{Y,1}, \theta_2=\arg\beta_{Y,2}$.

Proof. Since f is a $\mathbb{BC}$-holomorphic function, by[30, Theorem 7.3], we have

$$f(Z)=f(\beta_1\mathbf{e}+\beta_2\mathbf{e}^{\dagger})=f_1(\beta_1)\mathbf{e}+f_2(\beta_2)\mathbf{e}^{\dagger}$$

where

$$f_1:D_1\to\mathbb{C}(\mathbf{i})\ \text{ and }\ f_2:D_2\to\mathbb{C}(\mathbf{i})$$

are $\mathbb{C}(\mathbf{i})$-holomorphic functions, with

$$D_1:=\{\beta_{Y,1}\in\mathbb{C}(\mathbf{i})|\left|\beta_{Y,1}-\beta_{1,0}\right|\le r_1\}\ \text{ and }\ D_2:=\{\beta_{Y,2}\in\mathbb{C}(\mathbf{i})|\left|\beta_{Y,2}-\beta_{2,0}\right|\le r_2\}.$$

Now we can apply the complex Poisson integral formula to the functions f_1 and f_2, then

$$f_1(\beta_1)=\frac{1}{2\pi}\int\limits_{\partial D_1}\frac{(r_1^2-|\beta_1-\beta_{1,0}|^2)f_1(\beta_{Y,1})d\theta_1}{\left|\beta_1-\beta_{Y,1}\right|^2}$$

and

$$f_2(\beta_2)=\frac{1}{2\pi}\int\limits_{\partial D_2}\frac{(r_2^2-|\beta_2-\beta_{2,0}|^2)f_1(\beta_{Y,2})d\theta_2}{\left|\beta_2-\beta_{Y,2}\right|^2}.$$

Thus for any $Z\in\mathbb{B}(Z_0,R)$ there holds that

$$\begin{aligned}
f(Z)&=f_1(\beta_1)\mathbf{e}+f_2(\beta_2)\mathbf{e}^{\dagger}\\
&=\frac{1}{2\pi}\int\limits_{\partial D_1}\frac{(r_1^2-|\beta_1-\beta_{1,0}|^2)f_1(\beta_{Y,1})d\theta_1}{\left|\beta_1-\beta_{Y,1}\right|^2}\mathbf{e}\\
&+\frac{1}{2\pi}\int\limits_{\partial D_2}\frac{(r_2^2-|\beta_2-\beta_{2,0}|^2)f_2(\beta_{Y,2})d\theta_2}{\left|\beta_2-\beta_{Y,2}\right|^2}\mathbf{e}^{\dagger}\\
&=\frac{1}{2\pi}\int\limits_{\Gamma_1}\frac{(r_1^2-|\beta_1-\beta_{1,0}|^2)f_1(\beta_{Y,1})d\theta_1}{\left|\beta_1-\beta_{Y,1}\right|^2}\mathbf{e}\\
&+\frac{1}{2\pi}\int\limits_{\Gamma_2}\frac{(r_2^2-|\beta_2-\beta_{2,0}|^2)f_2(\beta_{Y,2})d\theta_2}{\left|\beta_2-\beta_{Y,2}\right|^2}\mathbf{e}^{\dagger}\\
&=\frac{1}{2\pi}\int\limits_{\Gamma_1}\frac{(R^2-|Z-Z_0|_{\mathbf{k}}^2)f(Y)d\theta}{|Y-Z|_{\mathbf{k}}^2}\mathbf{e}\\
&+\frac{1}{2\pi}\int\limits_{\Gamma_2}\frac{(R^2-|Z-Z_0|_{\mathbf{k}}^2)f(Y)d\theta}{|Y-Z|_{\mathbf{k}}^2}\mathbf{e}^{\dagger}.
\end{aligned}$$

□

5 | THE BICOMPLEX SCHWARZ LEMMA

Theorem 7. Let f be a function $\mathbb{BC}$-holomorphic in $\mathbb{B}(0,1)\cup\mathbb{S}(0,1)$ and $f(0)=0, |f(Z)|_{\mathbf{k}}\preceq 1, Z\in\mathbb{B}(0,1)$. Then for any $Z\in\mathbb{B}(0,1)$

$$|f(Z)|_{\mathbf{k}}\preceq\frac{1}{\sqrt{2}-1}|Z|_{\mathbf{k}}.$$

Proof. By Theorem 6,

$$f(Z)=\frac{1}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}}\frac{(R^2-|Z|_{\mathbf{k}}^2)f(Y)d\theta}{|Y-Z|_{\mathbf{k}}^2}\mathbf{e}+\frac{1}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}^{\dagger}}\frac{(R^2-|Z|_{\mathbf{k}}^2)f(Y)d\theta}{|Y-Z|_{\mathbf{k}}^2}\mathbf{e}^{\dagger}$$

Since $f(0)=0$, we have

$$f(0)=\frac{1}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}}f(Y)d\theta\ \mathbf{e}+\frac{1}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}^{\dagger}}f(Y)d\theta\ \mathbf{e}^{\dagger}$$

Then

$$\begin{aligned}f(Z)=&\frac{R^2-|Z|_{\mathbf{k}}^2}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}}\left(\frac{1}{|Y-Z|_{\mathbf{k}}^2}-\frac{1}{|Y|_{\mathbf{k}}^2}\right)f(Y)d\theta\ \mathbf{e}\\&+\frac{R^2-|Z|_{\mathbf{k}}^2}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}^{\dagger}}\left(\frac{1}{|Y-Z|_{\mathbf{k}}^2}-\frac{1}{|Y|_{\mathbf{k}}^2}\right)f(Y)d\theta\ \mathbf{e}^{\dagger}\end{aligned}$$

According to the triangle inequality and $|f(Y)|_{\mathbf{k}}^2\preceq 1, Y\in\mathbb{B}(0,1)$,

$$\begin{aligned}|f(Z)|_{\mathbf{k}}\preceq&\frac{R^2-|Z|_{\mathbf{k}}^2}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}}\left|\frac{1}{|Y-Z|_{\mathbf{k}}^2}-\frac{1}{|Y|_{\mathbf{k}}^2}\right|_{\mathbf{k}}|f(Y)|_{\mathbf{k}}d\theta\ \mathbf{e}\\&+\frac{R^2-|Z|_{\mathbf{k}}^2}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}^{\dagger}}\left|\frac{1}{|Y-Z|_{\mathbf{k}}^2}-\frac{1}{|Y|_{\mathbf{k}}^2}\right|_{\mathbf{k}}|f(Y)|_{\mathbf{k}}d\theta\ \mathbf{e}^{\dagger}\\\preceq&\frac{R^2-|Z|_{\mathbf{k}}^2}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}}\frac{|Z|_{\mathbf{k}}}{|Y-Z|_{\mathbf{k}}^2|Y|_{\mathbf{k}}}\left(1+\frac{|Y|_{\mathbf{k}}+|Z|_{\mathbf{k}}}{|Y|_{\mathbf{k}}^2}\right)d\theta\ \mathbf{e}\\&+\frac{R^2-|Z|_{\mathbf{k}}^2}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}^{\dagger}}\frac{|Z|_{\mathbf{k}}}{|Y-Z|_{\mathbf{k}}^2|Y|_{\mathbf{k}}}\left(1+\frac{|Y|_{\mathbf{k}}+|Z|_{\mathbf{k}}}{|Y|_{\mathbf{k}}^2}\right)d\theta\ \mathbf{e}^{\dagger}\end{aligned}\tag{9}$$

and

$$1=\frac{1}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}}\frac{(R^2-|Z|_{\mathbf{k}}^2)d\theta}{|Y-Z|_{\mathbf{k}}^2}\mathbf{e}+\frac{1}{2\pi}\int\limits_{\mathbb{S}(0,R)\mathbf{e}^{\dagger}}\frac{(R^2-|Z|_{\mathbf{k}}^2)d\theta}{|Y-Z|_{\mathbf{k}}^2}\mathbf{e}^{\dagger}\tag{10}$$

Combining (9) and (10), we have

$$|f(Z)|_{\mathbf{k}}\preceq\frac{|Z|_{\mathbf{k}}}{R}\left(1+\frac{R+|Z|_{\mathbf{k}}}{R}\right).$$

Taking $R\to 1$,

$$|f(Z)|_{\mathbf{k}}\preceq|Z|_{\mathbf{k}}(2+|Z|_{\mathbf{k}}).$$

For $0\prec|Z|_{\mathbf{k}}\prec 1$,

$$|f(Z)|_{\mathbf{k}}\preceq\frac{1}{|Z|_{\mathbf{k}}}|Z|_{\mathbf{k}}.$$

It is easy to check that

$$\begin{aligned}|f(Z)|_{\mathbf{k}} &\leq |Z|_{\mathbf{k}}\left\{\sup_{0\leq|\beta_1|<1}\min\left\{2+|\beta_1|,\frac{1}{|\beta_1|}\right\}\mathbf{e}+\sup_{0\leq|\beta_2|<1}\min\left\{2+|\beta_2|,\frac{1}{|\beta_2|}\right\}\mathbf{e}^{\dagger}\right\}\\ &\leq |Z|_{\mathbf{k}}\left(\frac{1}{\sqrt{2}-1}\mathbf{e}+\frac{1}{\sqrt{2}-1}\mathbf{e}^{\dagger}\right)\\ &=\frac{1}{\sqrt{2}-1}|Z|_{\mathbf{k}}.\end{aligned}$$

□

Theorem 8. (Schwarz-Pick lemma). Let f be a function $\mathbb{BC}$-holomorphic in $\mathbb{B}(0,1)\cup\mathbb{S}(0,1)$ and $f(A)=0$, $|A|_{\mathbf{k}}\prec 1$, $|f(Z)|_{\mathbf{k}}\preceq 1$, $Z\in\mathbb{B}(0,1)$. Then for any $Z\in\mathbb{B}(0,1)$

$$|f(Z)|_{\mathbf{k}}\preceq\frac{1+|A|_{\mathbf{k}}}{\sqrt{2}-1}\frac{|Z-A|_{\mathbf{k}}}{|1-A^*Z|^2_{\mathbf{k}}}.$$

Proof. For $|A|_{\mathbf{k}}\prec 1$, taking Möbius transformation

$$Y=F_A(Z)=(Z-A)\frac{1-AZ^*}{|1-A^*Z|^2_{\mathbf{k}}}$$

$F_A(Z)$ maps $\mathbb{B}(0,1)$ one-to-one onto itself. For any $Y\in\mathbb{B}(0,1)$, denote

$$F(Y)=(1-|A|_{\mathbf{k}})\frac{1+A^*Y}{|1+A^*Y|^2_{\mathbf{k}}}f(F_A^{-1}(Y))\tag{11}$$

It is easy to check that $F(0)=0$ since $f(A)=0$, $|A|_{\mathbf{k}}\prec 1$. For any $Y\in\mathbb{B}(0,1)$,

$$|F(Y)|_{\mathbf{k}}\preceq\frac{\left|1-|A|_{\mathbf{k}}\right|_{\mathbf{k}}}{|1+A^*Y|_{\mathbf{k}}}\prec\frac{1-|A^*Y|_{\mathbf{k}}}{|1+A^*Y|_{\mathbf{k}}}\prec 1.$$

According to [19, Theorem 7.2.6], we have $F(Y)$ is also $\mathbb{BC}$-holomorphic in $\mathbb{B}(0,1)\cup\mathbb{S}(0,1)$. Using Theorem 7, we obtain for any $Y\in\mathbb{B}(0,1)$

$$|F(Y)|_{\mathbf{k}}\preceq\frac{1}{\sqrt{2}-1}|Y|_{\mathbf{k}}\tag{12}$$

Combining (11) with (12), we have

$$\left|(1-|A|_{\mathbf{k}})\frac{1+A^*Y}{|1+A^*Y|^2_{\mathbf{k}}}f(F_A^{-1}(Y))\right|_{\mathbf{k}}\preceq\frac{1}{\sqrt{2}-1}|Y|_{\mathbf{k}}$$

Then

$$\frac{\left|1-|A|_{\mathbf{k}}\right|_{\mathbf{k}}}{|1+A^*Y|_{\mathbf{k}}}\left|f(Z)\right|_{\mathbf{k}}\preceq\frac{1}{\sqrt{2}-1}\left|\frac{Z-A}{1-A^*Z}\right|_{\mathbf{k}}$$

It follows that

$$\begin{aligned}\left|f(Z)\right|_{\mathbf{k}}&\preceq\frac{1}{\sqrt{2}-1}\frac{|Z-A|_{\mathbf{k}}}{|1-A^*Z|_{\mathbf{k}}}\frac{|1+A^*Y|_{\mathbf{k}}}{1-|A|_{\mathbf{k}}}\\ &=\frac{1}{\sqrt{2}-1}\frac{|Z-A|_{\mathbf{k}}}{|1-A^*Z|_{\mathbf{k}}}\frac{1}{1-|A|_{\mathbf{k}}}\frac{1-|A|^2_{\mathbf{k}}}{|1-A^*Z|}\\ &=\frac{1+|A|_{\mathbf{k}}}{\sqrt{2}-1}\frac{|Z-A|_{\mathbf{k}}}{|1-A^*Z|^2_{\mathbf{k}}}.\end{aligned}$$

□

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Conflict of interest

This work does not have any conflicts of interest.

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