Appendix A. Start-up of the Scheme

To find a desired accuracy for $\widetilde{y}_1$ and $\widetilde{y}_2$, we find the approximate solutions at points $t_{\frac{1}{4}}$ and $t_{\frac{1}{2}}$ using the constant, linear and quadratic interpolation. Let $I^0(f_a)$ be the constant interpolation of f at point $t = a$ that is $I^0(f_a) = f(a)$. Let $I^1(f_a, f_b)$ and $I^2(f_a, f_b, f_c)$ be linear and quadratic interpolation of f with grids (a, b) and (a, b, c) respectively. In the algorithm below we describe how to find the approximate solution of $y(t)$ at points $t_{\frac{1}{4}}$, $t_{\frac{1}{2}}$, t_1, and t_2 (i.e. $\widetilde{y}_{\frac{1}{4}}$, $\widetilde{y}_{\frac{1}{2}}$, $\widetilde{y}_1$, and $\widetilde{y}_2$) using a predictor-corrector scheme:

- Approximate solution of $y(t)$ at point $t_{\frac{1}{4}}$:

$$\begin{aligned}\widetilde{y}^P_{\frac{1}{4}} &= y_0 + AB_f \widetilde{f}(t_{\frac{1}{4}}, \widetilde{y}_0) + AB_y \int_0^{\frac{1}{4}} (t_{\frac{1}{4}} - s)^{\alpha-1} I^0(\widetilde{f}_0) ds, \\ \widetilde{y}_{\frac{1}{4}} &= y_0 + AB_f \widetilde{f}^P_{\frac{1}{4}} + AB_g \int_0^{t_{\frac{1}{4}}} I^1(\widetilde{f}_0, \widetilde{f}^P_{\frac{1}{4}})(t_{\frac{1}{4}} - s)^{\alpha-1} ds.\end{aligned}$$

- Approximate solution of $y(t)$ at point $t_{\frac{1}{2}}$:

$$\begin{aligned}\widetilde{y}^{P_1}_{\frac{1}{2}} &= y_0 + AB_f (2\widetilde{f}_{\frac{1}{4}} - \widetilde{f}_0) + AB_g \int_0^{t_{\frac{1}{2}}} I^0(\widetilde{f}_{\frac{1}{4}})(t_{\frac{1}{2}} - s)^{\alpha-1} ds, \\ \widetilde{y}^{P_2}_{\frac{1}{2}} &= y_0 + AB_f \widetilde{f}^{P_1}_{\frac{1}{2}} + AB_g \int_0^{t_{\frac{1}{2}}} I^1(\widetilde{f}_{\frac{1}{4}}, \widetilde{f}^{P_1}_{\frac{1}{2}})(t_{\frac{1}{2}} - s)^{\alpha-1} ds, \\ \widetilde{y}_{\frac{1}{2}} &= y_0 + AB_f \widetilde{f}^{P_2}_{\frac{1}{2}} + AB_g \int_0^{t_{\frac{1}{2}}} I^2(\widetilde{f}_0, \widetilde{f}_{\frac{1}{4}}, \widetilde{f}^{P_2}_{\frac{1}{2}})(t_{\frac{1}{2}} - s)^{\alpha-1} ds.\end{aligned}$$

- Approximate solution of $y(t)$ at point t_1:

$$\begin{aligned}\widetilde{y}^{P_1}_1 &= y_0 + AB_f (6\widetilde{f}_{\frac{1}{2}} - 8\widetilde{f}_{\frac{1}{4}} - 3\widetilde{f}_0) + AB_g \int_0^{t_n} I^0(\widetilde{f}_{\frac{1}{2}})(t_n - s)^{\alpha-1} ds, \\ \widetilde{y}^{P_2}_1 &= y_0 + AB_f \widetilde{f}^{P_1}_1 + AB_g \int_0^{t_1} I^1(\widetilde{f}_{\frac{1}{2}}, \widetilde{f}^{P_1}_1)(t_1 - s)^{\alpha-1} ds, \\ \widetilde{y}_1 &= y_0 + AB_f \widetilde{f}^{P_2}_1 + AB_g \int_0^{t_1} I^2(\widetilde{f}_0, \widetilde{f}_{\frac{1}{2}}, \widetilde{f}^{P_2}_1)(t_1 - s)^{\alpha-1} ds.\end{aligned}$$

- Approximate solution of $y(t)$ at point t_2:

$$\begin{aligned}\widetilde{y}^{P_1}_2 &= y_0 + AB_f (6\widetilde{f}_1 - 8\widetilde{f}_{\frac{1}{2}} - 3\widetilde{f}_0) + \int_0^{t_2} I^0(\widetilde{f}_1)(t_2 - s)^{\alpha-1} ds, \\ \widetilde{y}^{P_2}_2 &= y_0 + AB_f \widetilde{f}^{P_1}_2 + \int_0^{t_2} I^1(\widetilde{f}_1, \widetilde{f}^{P_1}_2)(t_2 - s)^{\alpha-1} ds, \\ \widetilde{y}_2 &= y_0 + AB_f \widetilde{f}^{P_2}_2 + \int_0^{t_2} I^2(\widetilde{f}_0, \widetilde{f}_1, \widetilde{f}^{P_2}_2)(t_2 - s)^{\alpha-1} ds.\end{aligned}$$

where

$$\left\{ AB_f = \frac{1-\alpha}{AB(\alpha)} \quad and \quad AB_g = \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \right\}.$$