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RESEARCH ARTICLE

Relatively exact controllability of fractional neutral stochastic system with two incommensurate constant delays

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This paper is devoted to analysing a kind of fractional neutral stochastic system (FNSS). Firstly, by introducing the notion of newly defined two-parameter Mittag-Leffler matrix function, we derive the solution of the corresponding linear stochastic system. Subsequently, for the linear case, by virtue of the Grammian matrix, we give a suffcient and necessary condition to guarantee the relatively exact controllability for the addressed case. Furthermore, for the nonlinear one, the relatively exact controllability is obtained by fixed point and explore it via Banach contraction principle. Finally, two examples are provided to intensify our theoretical conclusions.

KEYWORDS:
fractional calculus, stochastic delay system, relatively exact controllability, two-parameter Mittag-Leffler matrix

MSC CLASSIFICATION
26A33; 93B05; 65C30; 60J65

1 | INTRODUCTION

The concept of fractional calculate originally motivated by a discussion between L'Hospital and Leibnitz. After the study of many outstanding mathematicians such as Riemann, Liouville, Euler and Hilfer, it has been developed into a successful tool in classical analysis. It is recognized as a powerful approach to apply the integral and differential operators of integer order into fractional even plural order, which is an useful tools in explaining real-life, particularly in stability theory [1 2 3], control theory [4 5 6] and stochastic analysis [7 8 9]. For have a more effective illustration, one can pay attention to refer the monographs [10 11 12 13] and previous studies [14 15 16 17 18 19 20 21].

Controllability issues with single delay have been addressed well. However, it is not many papers concerning the fractional system with two incommensurate delays. In fact, the relatively exact controllability [22] means when steer these delays systems to rest, it should not only require to control the value of the state at arbitrary final time but also exsist a solution that satisfies the initial function. Controllability plays a vital role in many application area including robotics, remote control, and so on.

The delay system can model real-world problems in a more accurate way. In [23] Khusainov et al. studied the existence of solutions about the first-order differential equation with a single delay. In [24] Li and Wang considered two parameter delayed matrix function of Mittag-Leffler and derived the solution of fractional delay equations. Furthermore, some scholars have begun to extend the case of single delay to two delays. In [25] Huseynov and Mahmudov analysed the following fractional neutral system

$$\begin{cases} \left({}^{C}\mathfrak{D}_{0^+}^{\alpha} x\right)(\rho) = \mathfrak{A}_0 x(\rho) + \mathfrak{A}_1 x\left(\rho-\tau_1\right) + \mathfrak{A}_2\left({}^{C}\mathfrak{D}_{0^+}^{\alpha} x\right)\left(\rho-\tau_2\right) + f(\rho, x(\rho), x(\rho-\tau_1), x(\rho-\tau_2)), \quad \rho \in [0,T], \\ x(\rho)=\varphi(\rho), \quad -\tau \leq \rho \leq 0, \quad \tau := \max\left\{\tau_1,\tau_2\right\}, \quad \tau_1,\tau_2>0. \end{cases} \tag{1}$$

We know that stochastic noise plays a significant role in fractional controllability problems. Because of our real life is full of stochastic disturbances, the deterministic systems should take this kind of disturbances into account. There are many experts discussed disparate disturbance. In[26] Wang et al. studied a kinds of stochastic oscillating delay systems driven by the Rosenblatt distribution. In[27] O'Regan et al. researched the controllability for stochastic systems with standard Brownian motion.

Inspired by the studies above, we will discuss the following neutral stochastic system with two different delays of the model

$$\begin{cases}({}^C\mathfrak{D}^\alpha_{0^+}x)(\rho) = & \mathfrak{A}_0 x(\rho) + \mathfrak{A}_1 x(\rho-\tau_1) + \mathfrak{A}_2({}^C\mathfrak{D}^\alpha_{0^+}x)(\rho-\tau_2) + \mathfrak{B}\mathfrak{u}(\rho) + F(\rho, x(\rho), x(\rho-\tau_1), x(\rho-\tau_2)) \\ & +\widetilde{\Delta}(\rho, x(\rho), x(\rho-\tau_1), x(\rho-\tau_2))\frac{dw}{d\rho}, \quad \rho \in [0,b], \\ x(\rho) = \varphi(\rho), & -\tau \le \rho \le 0, \quad \tau := \max\left\{\tau_1, \tau_2\right\}, \quad \tau_1, \tau_2 > 0, \end{cases} \tag{2}$$

where $\left({}^C\mathfrak{D}^\alpha_{0^+}x\right)(\cdot)$ is the Caputo fractional derivative, $\alpha \in (\frac{1}{2}, 1]$, $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathbb{R}^{n\times n}$. $\mathfrak{B} \in \mathbb{R}^{n\times m}$ denotes any real matrices and τ_1, τ_2 are the two different delays. Let $I = [-\tau, 0]$, $\vartheta = [0, b]$, $\varphi(\cdot)$: $I \to \mathbb{R}^n$ be an arbitrary vector function and $x(\cdot) \in \mathbb{R}^n$ is an analytical solution of the Cauchy problem (2). Here $\mathfrak{u}(\rho) \in \mathbb{R}^m$ is a control vector and the nonlinear functions F: $\vartheta \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\widetilde{\Delta}$: $\vartheta \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$ are continuous. $w(\cdot)$ is a standard d-dimensional Brownian motion.

Concerning relatively exact controllability of system (2), we would like to address the diffculties as follows

- Due to the complexity of the two parameter Mittag-Leffler type matrix funtion $\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\beta}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho\right)$ with two incommensurate delays, the estimation is much more diffcult.
- Different from the past-studied Grammian matrix[28], we introduce the generalized Grammian matrix and this matrix is given by newly defined delayed Mittag-Leffler type matrix function.
- With the help of Banach contraction principle and maximum weighted norm in Banach space, we give the sufficient and necessary condition to guarantee the fractional neutral stochastic system with two different constant delays, which is relatively exact controllability and it is essential new compared to some references[29].

This manuscript proceeds as follows. Section 2 is a preparatory part where we list some fundamental definitions and introductory results on fractional calculus. In Section 3, the relatively exact controllability issue of linear FNSS is analyzed by Grammian matrix and the relatively exact controllability of nonlinear case is obtained with the help of Banach contraction principle. The applications of two examples to intensify our results in section 4.

2 | PRELIMINARY

In order to carry out the following work, we will prepare the definitions. Moreover, we are going to give some fractional calculus formula and several necessary facts.

Let $H_2\left(\Omega, \mathfrak{F}_b, \mathbb{R}^n\right)$ be a Hilbert space of all $\mathfrak{F}_b$-measurable square integrable random variables with values in $\mathbb{R}^n$. $\mathbb{R}^n$ endowed with a norm $\|z\| = \sqrt{z_1^2 + \ldots + z_n^2}$ for any $z = \left(z_1, \ldots, z_n\right) \in \mathbb{R}^n$. $H_2^{\mathfrak{F}}([-\tau, b], \mathbb{R}^n)$ is the Hilbert space of all square integrable and $\mathfrak{F}_\rho$-measurable processes with values in $\mathbb{R}^n$. Let $\mathbb{J} = [0, T] \subset \mathbb{R}^n$, $\mathbb{C}(\mathbb{J}, \mathbb{R}^n)$ be the Banach space of all continuous functions mapping from $\mathbb{J} \to \mathbb{R}^n$ equipped with the norm $\|v\|_\infty = \max_{\rho\in\mathbb{J}} \|v(\rho)\|$. For any matrix $\mathfrak{A} = \{a_{ij}\} \in \mathbb{R}^{n\times d}$, the norm of the matrix $\mathfrak{A}$ is $\|\mathfrak{A}\| = \max_{1\le i\le n} \sum_{j=1}^{d} |a_{ij}|$.

Definition 1. (see Fečkan et al.[30]). If order $0 < \alpha \le 1$, for a function $v(\cdot) \in \mathbb{C}^1(\mathbb{J}, \mathbb{R}^n)$ the Caputo derivative is

$$\left({}^C\mathfrak{D}^\alpha_{0^+}v\right)(\rho) = \frac{1}{\Gamma(1-\alpha)} \int_0^\rho (\rho - s)^{-\alpha} \frac{d}{ds} v(s) \mathrm{d}s, \quad \rho > 0.$$

Definition 2. (see Luo et al.[22] and Li et al.[28]). The Mittag-Leffler matrix function $\mathcal{M}_\alpha(\mathfrak{A}\rho^\alpha)$ and $\mathcal{M}_{\alpha,\beta}(\mathfrak{A}\rho^\alpha)$ are defined by

$$\mathcal{M}_\alpha(\mathfrak{A}\rho^\alpha) = \sum_{k=0}^{\infty} \mathfrak{A}^k \frac{\rho^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \mathfrak{A} \in \mathbb{R}^{n\times n}, \quad \alpha > 0, \quad \rho \in \mathbb{R}.$$

$$\mathcal{M}_{\alpha,\beta}(\mathfrak{A}\rho^\alpha) = \sum_{k=0}^{\infty} \mathfrak{A}^k \frac{\rho^{k\alpha}}{\Gamma(k\alpha+\beta)}, \quad \mathfrak{A} \in \mathbb{R}^{n\times n}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad \rho \in \mathbb{R}.$$

Definition 3. (see Huseynov et al.[25]). If $\alpha > 0, \beta \in \mathbb{R}$, then the Mittag-Leffler matrix function of two parameter with two different delays $\tau_1, \tau_2 > 0$, $\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\beta}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \cdot\right) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathbb{R}^{n\times n}$ is

$$\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\beta}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho\right) := \begin{cases} \Theta, & -\tau \le \rho < 0, \\ I, & \rho = 0, \\ \sum\limits_{k=0}^{\infty}\sum\limits_{\omega_1=0}^{\infty}\sum\limits_{\omega_2=0}^{\infty} Q_{k+1}\left(\omega_1\tau_1, \omega_2\tau_2\right)\frac{(\rho-\omega_1\tau_1-\omega_2\tau_2)_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}, & \rho \in \mathbb{R}_+, \end{cases}$$

where

$$\left(\rho - \omega_1\tau_1 - \omega_2\tau_2\right)_+ = \begin{cases} \rho - \omega_1\tau_1 - \omega_2\tau_2, & \rho \ge \omega_1\tau_1 + \omega_2\tau_2, \\ 0, & \rho < \omega_1\tau_1 + \omega_2\tau_2. \end{cases}$$

Lemma 1. (see Huseynov et al.[25]). Let $\alpha > 0, \beta \in \mathbb{R}, \tau_1, \tau_2 > 0$, and $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathbb{R}^{n\times n}$. Then the following relation holds

$$\left\|\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\beta}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho\right)\right\| \le \mathscr{E}^{\tau_1,\tau_2}_{\alpha,\beta}\left(\|\mathfrak{A}_0\|, \|\mathfrak{A}_1\|, \|\mathfrak{A}_2\|; \rho\right) \le \rho^{\beta-1} M_{\alpha,\beta}\left(\|\mathfrak{A}_0\|, \|\mathfrak{A}_1\|, \|\mathfrak{A}_2\|; \rho\right), \quad \rho \in \mathbb{R}_+,$$

where $\rho^{\beta-1} M_{\alpha,\beta}\left(\|\mathfrak{A}_0\|, \|\mathfrak{A}_1\|, \|\mathfrak{A}_2\|; \rho\right)$ is the norm of it, such that

$$\rho^{\beta-1} M_{\alpha,\beta}\left(\|\mathfrak{A}_0\|, \|\mathfrak{A}_1\|, \|\mathfrak{A}_2\|; \rho\right) := \sum_{k=0}^{\infty}\sum_{\omega_1=0}^{\infty}\sum_{\omega_2=0}^{\infty}\left\|Q_{k+1}\left(\omega_1\tau_1, \omega_2\tau_2\right)\right\|\frac{\rho^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}.$$

Lemma 2. (see Tian et al.[3]). For all $\gamma, \rho > 0$ and $\alpha \in \left(\frac{1}{2}, 1\right)$ the following inequality holds

$$\frac{\gamma}{\Gamma(2\alpha-1)}\int_0^{\rho}(\rho-s)^{2\alpha-2}\mathcal{M}_{2\alpha-1}\left(\gamma s^{2\alpha-1}\right)ds \le \mathcal{M}_{2\alpha-1}\left(\gamma\rho^{2\alpha-1}\right).$$

Lemma 3. (see Huseynov et al.[25]). The solution of the following system

$$\begin{cases} \left({}^C\mathfrak{D}^{\alpha}_{0^+}x\right)(\rho) = \mathfrak{A}_0 x(\rho) + \mathfrak{A}_1 x\left(\rho-\tau_1\right) + \mathfrak{A}_2\left({}^C\mathfrak{D}^{\alpha}_{0^+}x\right)\left(\rho-\tau_2\right) + f(\rho), & \rho \in [0, b], \\ x(\rho) = \varphi(\rho), \quad -\tau \le \rho \le 0, \quad \tau := \max\left\{\tau_1, \tau_2\right\}, \quad \tau_1, \tau_2 > 0 \end{cases}$$

can be represented as

$$\begin{aligned} x(\rho) =& \mathscr{E}^{\tau_1,\tau_2}_{\alpha,1}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho\right)\left(\varphi(0) - \mathfrak{A}_2\varphi\left(-\tau_2\right)\right) + \int_{-\tau_1}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds \\ &+ \int_{-\tau_2}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,0}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds \\ &+ \int_0^{\rho}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho-s\right)f(s)ds, \quad \rho \in [0, b], \quad \tau_1, \tau_2 > 0, \quad \tau := \max\left\{\tau_1, \tau_2\right\}. \end{aligned}$$

3 | MAIN RESULTS

3.1 | Linear case

We will consider the exact controllability of following linear stochastic system with two different delays

$$\begin{cases} \left({}^C\mathfrak{D}^{\alpha}_{0^+}x\right)(\rho) = \mathfrak{A}_0 x(\rho) + \mathfrak{A}_1 x\left(\rho-\tau_1\right) + \mathfrak{A}_2\left({}^C\mathfrak{D}^{\alpha}_{0^+}x\right)\left(\rho-\tau_2\right) + \mathfrak{B}\mathfrak{u}(\rho) + f(\rho) + \widetilde{\Delta}(\rho)\frac{dw(\rho)}{d\rho}, & \rho \in [0, b], \\ x(\rho) = \varphi(\rho), \quad -\tau \le \rho \le 0, \quad \tau := \max\left\{\tau_1, \tau_2\right\}, \quad \tau_1, \tau_2 > 0, \end{cases} \tag{3}$$

where $\widetilde{\Delta} : [0, b] \to \mathbb{R}^{n\times d}$ is continuous. We know that the corresponding linear deterministic control system is given as follows

$$\begin{cases} \left({}^C\mathfrak{D}^{\alpha}_{0^+}x\right)(\rho) = \mathfrak{A}_0 x(\rho) + \mathfrak{A}_1 x\left(\rho-\tau_1\right) + \mathfrak{A}_2\left({}^C\mathfrak{D}^{\alpha}_{0^+}x\right)\left(\rho-\tau_2\right) + \mathfrak{B}\mathfrak{u}(\rho) + f(\rho), & \rho \in [0, b], \quad \tau_1, \tau_2 > 0, \\ x(\rho) = \varphi(\rho), \quad -\tau \le \rho \le 0, \quad \tau := \max\left\{\tau_1, \tau_2\right\}, \quad \tau_1, \tau_2 > 0. \end{cases} \tag{4}$$

Using Lemma 3, the solution of (4) is

$$\begin{aligned} x(\rho) =& \mathscr{E}^{\tau_1,\tau_2}_{\alpha,1}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho\right)\left(\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right)+\int\limits_{-\tau_1}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds \\ &+\int\limits_{-\tau_2}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,0}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds+\int\limits_{0}^{\rho}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\mathfrak{B}\mathfrak{u}(s)ds \\ &+\int\limits_{0}^{\rho}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)f(s)ds, \quad \rho\in[0,b], \quad \tau_1,\tau_2>0. \end{aligned} \tag{5}$$

When substituting $\rho = b$ in (5), we have

$$\begin{aligned} x(b) =& \mathscr{E}^{\tau_1,\tau_2}_{\alpha,1}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b\right)\left(\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right)+\int\limits_{-\tau_1}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds \\ &+\int\limits_{-\tau_2}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,0}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds+\int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathfrak{B}\mathfrak{u}(s)ds \\ &+\int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)f(s)ds. \end{aligned} \tag{6}$$

The linear bounded operator $\mathcal{L}_b \in \mathbb{L}\left(H_2^{\mathfrak{F}}\left([0,b],\mathbb{R}^n\right), H_2\left(\Omega,\mathfrak{F}_\rho,\mathbb{R}^n\right)\right)$ can be written as

$$\mathcal{L}_b\mathfrak{u} = \int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathfrak{B}\mathfrak{u}(s)ds.$$

Here the adjoint is expressed as

$$\mathcal{L}_b^* : H_2\left(\Omega,\mathfrak{F}_\rho,\mathbb{R}^n\right) \to H_2^{\mathfrak{F}}\left([0,b],\mathbb{R}^n\right),$$

and its defined as

$$\mathcal{L}_b^* x = \mathfrak{B}^*\left[\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\right]^*\mathbb{E}\left\{x \mid \mathfrak{F}_\rho\right\}.$$

Consider the operator $\Gamma_\tau^b \in \mathbb{L}\left(H_2\left(\Omega,\mathfrak{F}_\rho,\mathbb{R}^n\right), H_2\left(\Omega,\mathfrak{F}_\rho,\mathbb{R}^n\right)\right)$ about linear controllability

$$\begin{aligned} \Gamma_\tau^b\{\cdot\} &= \mathcal{L}_b\mathcal{L}_b^*\{\cdot\} \\ &= \int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathfrak{B}\mathfrak{B}^*\left[\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\right]^*\mathbb{E}\left\{\cdot \mid \mathfrak{F}_s\right\}ds, \end{aligned} \tag{7}$$

and the correspondingly deterministic Grammian matrix $G_\tau^b \in \mathbb{L}\left(\mathbb{R}^n,\mathbb{R}^n\right)$ has a form

$$G_\tau^b = \int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathfrak{B}\mathfrak{B}^*\left[\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\right]^*ds. \tag{8}$$

Definition 4. (see Wang et al.[29]). System (4) is called relatively controllable on $[0,b]$, if for an arbitrary initial vector function $\varphi(\rho)$ and the final state of the vector $x_1 \in \mathbb{R}^n$, there exists a control $\mathfrak{u}(\rho) \in \mathbb{R}^n$ such that the system (4) has a solution x that satisfies the initial condition $x(\rho) = \varphi(\rho)$ and $x(b) = x_1$.

Definition 5. (see Luo et al.[4]). System (3) is called relatively controllable if

$$\mathscr{R}_b\left(\mathscr{U}_{ac}\right) = H_2\left(\Omega,\mathfrak{F}_b,\mathbb{R}^n\right),$$

where $\mathscr{R}_b\left(\mathscr{U}_{ac}\right) = \left\{x(b,\mathfrak{u}) \in H_2\left(\Omega,\mathfrak{F}_b,\mathbb{R}^n\right) : \mathfrak{u}(\cdot) \in \mathscr{U}_{ac}\right\}$ and $\mathscr{U}_{ac} = H_2^{\mathfrak{F}}\left([0,b],\mathbb{R}^n\right)$ denotes set of all admissible controls.

Lemma 4. System (4) is relatively controllable if and only if the Grammian matrix (8) is nonsingular.

Proof. **Suffciency**: Assume that G_τ^b is nonsingular, so there exists its well defined inverse $\left[G_\tau^b\right]^{-1}$. The function $\mathfrak{u}(s) \in \mathbb{R}^n$ is expressed by

$$\mathfrak{u}(s) = \mathfrak{B}^* \left[\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\right]^* \left[G_\tau^b\right]^{-1} \beta, \tag{9}$$

where

$$\begin{aligned}\beta =& x_1 - \mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b\right)\left(\varphi(0) - \mathfrak{A}_2\varphi\left(-\tau_2\right)\right) - \int_{-\tau_1}^{0} \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds \\ &- \int_{-\tau_2}^{0} \mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds - \int_{0}^{b} \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right) f(s)ds,\end{aligned}$$

with the chosen arbitrarily vector $x_1 \in \mathbb{R}^n$.

Inserting (9) in (6), one can derive

$$\begin{aligned}x(b) =& \mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b\right)\left(\varphi(0) - \mathfrak{A}_2\varphi\left(-\tau_2\right)\right) + \int_{-\tau_1}^{0} \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds \\ &+ \int_{-\tau_2}^{0} \mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds + \int_{0}^{b} \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right) f(s)ds \\ &+ \int_{0}^{b} \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\mathfrak{B}\mathfrak{B}^*\left[\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\right]^*\left[G_\tau^b\right]^{-1}\beta ds \\ =& x_1.\end{aligned}$$

The boundary condition $x(\rho) = \varphi(\rho), -\tau \leq \rho \leq 0, \tau := \max\left\{\tau_1, \tau_2\right\}, \tau_1, \tau_2 > 0$ holds by Lemma 3. Thus the system (4) is relatively controllable according to Definition 4.

Necessity: Under the assumption that G_τ^b satisfy singular, then there remains at least one nonzero state $\tilde{x}_1 \in \mathbb{R}^n$. Therefore, we have

$$\begin{aligned}0 &= \tilde{x}_1^* G_\tau^b \tilde{x}_1 \\ &= \tilde{x}_1^* \int_0^b \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\mathfrak{B}\mathfrak{B}^*\left[\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\right]^* ds\tilde{x}_1 \\ &= \int_0^b \left[\tilde{x}_1^*\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\mathfrak{B}\right]\left[\tilde{x}_1^*\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\mathfrak{B}\right]^* ds \\ &= \int_0^b \left\|\tilde{x}_1^*\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\mathfrak{B}\right\|^2 ds,\end{aligned}$$

which can derive that

$$\tilde{x}_1^*\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; b-s\right)\mathfrak{B} = \mathbf{0}, \forall s \in [0, b], \tag{10}$$

where $\mathbf{0}$ represents n dimensional zero vector. Because system (4) is relatively controllable, according to the Definition 4, a control function $\mathfrak{u}_0(\rho)$ exist and enable the initial state to zero at time b, namely

$$\begin{aligned} x(b) =&\mathscr{E}^{\tau_1,\tau_2}_{\alpha,1}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b\right)\left(\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right)+\int\limits_{-\tau_1}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds \\ &+\int\limits_{-\tau_2}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,0}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds+\int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathfrak{B}\mathfrak{u}_0(s)ds \\ &+\int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)f(s)ds \\ =&\mathbf{0}. \end{aligned} \tag{11}$$

Moreover, by Definition 4, there also exists a control $\mathfrak{u}_1(\rho)$ that transfers the complete state to the state $\tilde{x}_1$ at b, namely

$$\begin{aligned} x(b) =&\mathscr{E}^{\tau_1,\tau_2}_{\alpha,1}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b\right)\left(\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right)+\int\limits_{-\tau_1}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds \\ &+\int\limits_{-\tau_2}^{0}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,0}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds+\int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathfrak{B}\mathfrak{u}_1(s)ds \\ &+\int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)f(s)ds \\ =&\tilde{x}_1. \end{aligned} \tag{12}$$

Linking the formula (11) and (12), then

$$\tilde{x}_1=\int\limits_{0}^{b}\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathfrak{B}\left(\mathfrak{u}_1(s)-\mathfrak{u}_0(s)\right)ds,$$

multiplying both sides of the above equation by $\tilde{x}_1^*$, and we have

$$\tilde{x}_1^*\tilde{x}_1=\int\limits_{0}^{b}\tilde{x}_1^*\mathscr{E}^{\tau_1,\tau_2}_{\alpha,\alpha}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathfrak{B}\left(\mathfrak{u}_1(s)-\mathfrak{u}_0(s)\right)\mathrm{d}s.$$

We acquire $\tilde{x}_1=\mathbf{0}$, which is contradicted with $\tilde{x}_1$ being nonzero. Thus, G^b_τ is nonsingular. □

Lemma 5. (see Klamka[31]). The following conditions are equivalent
(i) System (4) is relatively controllable on [0, b],
(ii) System (3) is relatively exactly controllable on [0, b].

3.2 | Nonlinear case

Before starting this part, we assume that the following assumptions hold

- (H_1) The functions $\mathcal{H}\in\mathbb{C}(\vartheta\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $\widetilde{\Delta}\in\mathbb{C}(\vartheta\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^{n\times d})$, then there exist two positive constants $\mathcal{L}_1,\mathcal{L}_2$, such that
  - (i)
$$\begin{aligned}&\left\|\mathcal{H}\left(\rho,\omega_1,\kappa_1,\varsigma_1\right)-\mathcal{H}\left(\rho,\omega_2,\kappa_2,\varsigma_2\right)\right\|^2\\ &\le\mathcal{L}_1\left(\left\|\omega_1-\omega_2\right\|^2+\left\|\kappa_1-\kappa_2\right\|^2+\left\|\varsigma_1-\varsigma_2\right\|^2\right),\end{aligned}$$

- (ii)

$$\left\|\widetilde{\Delta}\left(\rho,\omega_1,\kappa_1,\varsigma_1\right)-\widetilde{\Delta}\left(\rho,\omega_2,\kappa_2,\varsigma_2\right)\right\|^2$$
$$\leq \mathcal{L}_2\left(\left\|\omega_1-\omega_2\right\|^2+\left\|\kappa_1-\kappa_2\right\|^2+\left\|\varsigma_1-\varsigma_2\right\|^2\right),$$
$$\rho\in[0,b],\omega_1,\omega_2,\kappa_1,\kappa_2,\varsigma_1,\varsigma_2\in\mathbb{R}^n.$$

- (H_2) Set

$$\begin{aligned}
N &:= \max_{0\leq\rho\leq b}\mathbb{E}\|\mathcal{H}(\rho,0,0,0)\|^2,\\
M &:= \max_{0\leq\rho\leq b}\mathbb{E}\|\Delta(\rho,0,0,0)\|^2,\\
M_1 &:= \max_{-\tau\leq\rho\leq 0}\|\varphi(\rho)\|,\\
K_0 &:= \max_{0\leq\rho\leq b}\mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho\right)^2,\\
K_1 &:= \max_{0\leq\rho\leq b}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\left\|\mathfrak{A}_0\right\|,\left\|\mathfrak{A}_1\right\|,\left\|\mathfrak{A}_2\right\|;\rho\right)^2,\\
K_2 &:= \max_{0\leq\rho\leq b}\mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\left\|\mathfrak{A}_0\right\|,\left\|\mathfrak{A}_1\right\|,\left\|\mathfrak{A}_2\right\|;\rho\right)^2,\\
K_3 &:= \max_{0\leq s\leq\rho\leq b}M_{\alpha,\alpha}\left(\left\|\mathfrak{A}_0\right\|,\left\|\mathfrak{A}_1\right\|,\left\|\mathfrak{A}_2\right\|;\rho-s\right)^2.
\end{aligned}$$

- (H_3) Set $K_4 :=\| G_\tau^b \|^2$, $K_5 :=\| [(\Gamma_\tau)_0^b]^{-1} \|^2$, and

$$K := \frac{3\Gamma(2\alpha-1)K_3}{\lambda}\left(1+2K_4K_5\right)\left(b\mathcal{L}_1+\mathcal{L}_2\right)<1.$$

Now we give the solution of (2) with this form

$$\begin{aligned}
x(\rho) =&\mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho\right)\left(\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right)+\int_{-\tau_1}^{0}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds\\
&+\int_{-\tau_2}^{0}\mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds+\int_{0}^{\rho}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\mathfrak{B}\mathfrak{u}_x(s)ds\\
&+\int_{0}^{\rho}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)ds\\
&+\int_{0}^{\rho}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\widetilde{\Delta}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)dw(s).
\end{aligned}\tag{13}$$

Furthermore, the admissible control function

$$\mathfrak{u}_x(\rho)=\mathfrak{B}^*\left[\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\right]^*\mathbb{E}\left\{\left[(\Gamma_\tau)_0^b\right]^{-1}\eta\mid\mathfrak{F}_\rho\right\}\tag{14}$$

defined for $\rho\in[0,b]$, where

$$\begin{aligned}
\eta =&x_1-\mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b\right)\left(\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right)-\int_{-\tau_1}^{0}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds\\
&-\int_{-\tau_2}^{0}\mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds-\int_{0}^{b}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\mathcal{H}\left(s,x(s),x(s-\tau_1),x(s-\tau_2)\right)ds\\
&-\int_{0}^{\rho}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-s\right)\widetilde{\Delta}\left(s,x(s),x(s-\tau_1),x(s-\tau_2)\right)dw(s),
\end{aligned}$$

and $x_1\in\mathbb{R}^n$ is arbitrarily.

Inserting (14) in (13), it is easy to check that the control $\mathfrak{u}_x(\rho)$ steers x_0 to x_1 at time b. In order to establish sufficient conditions, we let

$$\mathbb{C}_\varphi := \{x(\cdot) \in \mathbb{C}([-\tau, b], \mathbb{R}^n) : x(\rho) = \varphi(\rho), -\tau \le \rho \le 0\}$$

be a Banach space with norm $\|\cdot\|_\lambda$ as follows

$$\|x\|_\lambda^2 = \max_{0\le\rho\le b} \left\{ \frac{\mathbb{E}\|x^*(\rho)\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)} \right\},$$

where $\lambda > 0$, and $\|x^*(\rho)\|^2 = \max\limits_{-\tau\le\xi\le\rho} \|x(\xi)\|^2$, $\tau = \max\{\tau_1, \tau_2\}$, where $\tau_1, \tau_2 > 0$. Since two norms $\|\cdot\|_\infty$ and $\|\cdot\|_\lambda$ are equivalent, $(C_\varphi, \|\cdot\|)$ is also a Banach space. We will use the following fact that $\max\limits_{-\tau\le s\le\rho} \|\hat{x}(s)\| = \hat{x}^*(\rho)$ and $\max\limits_{-\tau\le s\le\rho} \|\hat{x}(s) - \hat{y}(s)\| = \hat{x}^*(\rho) - \hat{y}^*(\rho)$. In addition, we denote $\hat{x}(\rho) := \max\limits_{-\tau\le h\le 0} x(\rho + h)$. Let

$$\Theta_k := \left\{ x \in \mathbb{C}_\varphi : \|x\|_\lambda = \max_{0\le\rho\le b} \left\{ \frac{\mathbb{E} \parallel x^*(\rho) \parallel^2}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)} \right\} \le k \right\},$$

then $\Theta_k \subset \mathbb{C}_\varphi$ is defined as

$$k = \frac{36K_4K_5\lambda\mathbb{E}\|x\|_1^2 + \left(6 + 36K_4K_5\right)\lambda\left[K_0\left\|\varphi(0) - \mathfrak{A}_2\varphi\left(-\tau_2\right)\right\|^2 + \|\mathfrak{A}_1\|^2\tau^2K_1M_1^2 + \|\mathfrak{A}_2\|^2\tau^2K_2M_1^2 + \frac{2b^{2\alpha}}{2\alpha-1}K_3N + \frac{2b^{2\alpha-1}}{2\alpha-1}K_3M\right]}{\lambda - 6K_3\Gamma(2\alpha-1)(b\mathcal{L}_1 + \mathcal{L}_2)}. \tag{15}$$

The operator $\boldsymbol{\phi} : \Theta_k \to \Theta_k$ is described as

$$\begin{aligned}(\boldsymbol{\phi}x)(\rho) =& \mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho\right)\left(\varphi(0) - \mathfrak{A}_2\varphi\left(-\tau_2\right)\right) + \int_{-\tau_1}^{0} \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho - \tau_1 - s\right)\mathfrak{A}_1\varphi(s)ds \\ &+ \int_{-\tau_2}^{0} \mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho - \tau_2 - s\right)\mathfrak{A}_2\varphi(s)ds + \int_0^\rho \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho - s\right)\mathfrak{B}\mathfrak{u}_x(s))ds \\ &+ \int_0^\rho \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho - s\right)\mathcal{H}\left(s, x(s), x(s-\tau_1), x(s-\tau_2)\right)ds \\ &+ \int_0^\rho \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho - s\right)\widetilde{\Delta}\left(s, x(s), x(s-\tau_1), x(s-\tau_2)\right)dw(s).\end{aligned} \tag{16}$$

Under the condition of operator $\boldsymbol{\phi}$ has a fixed point, therefore system (2) has a solution $x(\rho)$ for $\mathfrak{u}_x(\cdot) \in \mathscr{U}_{ac}$, which satisfy $(\boldsymbol{\phi}x)(b) = x(b) = x_1, x(\rho) = \varphi(\rho), \rho \in [-\tau, 0], \tau := \max\left\{\tau_1, \tau_2\right\}$. In other words, system (2) is relatively exact controllable.

Theorem 1. Suppose that hypothesis (H_1)-(H_3) set up and system (3) is relatively exactly controllable. Then system (2) is relatively exactly controllable on $[0, b]$.

Proof. In order to make the following process clear we divide it into the following steps.

Step 1: We prove that $\boldsymbol{\phi}$ maps Θ_k into itself.

By using $\left(H_2\right)$ and Jensen inequality, we can acquire

$$\begin{aligned}&\frac{\mathbb{E}\|(\boldsymbol{\phi}x)(\rho)\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)} \\ \le& \frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho\right)\left(\varphi(0) - \mathfrak{A}_2\varphi\left(-\tau_2\right)\right)\right\|^2 \\ &+ \frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\int_{-\tau_1}^{0} \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho - \tau_1 - s\right)\mathfrak{A}_1\varphi(s)ds\right\|^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\int_{-\tau_2}^{0}\mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds\right\|^2\\
&+\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\int_{0}^{\rho}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)ds\right\|^2\\
&+\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\int_{0}^{\rho}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\widetilde{\Delta}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)dw(s)\right\|^2\\
&+\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\int_{0}^{\rho}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\mathfrak{B}\mathfrak{u}(s)ds\right\|^2\\
&:=I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

With the aid of (H_2), we have

$$
\begin{aligned}
I_1=&\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho\right)\left(\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right)\right\|^2\\
\le&6K_0\left\|\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right\|^2.
\end{aligned}
$$

Motived by Hölder inequality and (H_2), we have

$$
\begin{aligned}
I_2=&\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\int_{-\tau_1}^{0}\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds\right\|^2\\
\le&6\left\|\mathfrak{A}_1\right\|^2\tau^2K_1M_1^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3=&\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}6\mathbb{E}\left\|\int_{-\tau_2}^{0}\mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds\right\|^2\\
\le&6\left\|\mathfrak{A}_2\right\|^2\tau^2K_2M_1^2.
\end{aligned}
$$

By employing Hölder inequality, (H_1), (H_2), Lemma 1, and Lemma 2, we have the following

$$
\begin{aligned}
I_4\le&6b\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\mathbb{E}\left\|\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)\right\|^2ds\\
\le&12b\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\mathbb{E}\left\|\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\mathcal{H}(s,0,0,0)\right\|^2ds\\
&+12b\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\mathbb{E}\left\|\mathcal{H}(s,0,0,0)\right\|^2ds\\
\le&12b\mathcal{L}_1\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\frac{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}\mathbb{E}\left\|x(s)\right\|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&+12b\mathcal{L}_1\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\frac{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}\mathbb{E}\left\|x(s-\tau_1)\right\|^2ds\\
&+12b\mathcal{L}_1\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\frac{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}\mathbb{E}\left\|x(s-\tau_2)\right\|^2ds\\
&+12b\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\mathbb{E}\left\|\mathcal{H}(s,0,0,0)\right\|^2ds\\
\leq&36b\mathcal{L}_1\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)ds\max_{0\leq\rho\leq b}\frac{\mathbb{E}\left\|\hat{x}^*(\rho)\right\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\\
&+12bK_3N\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho(\rho-s)^{2\alpha-2}ds\\
\leq&36bK_3\mathcal{L}_1\|\hat{x}\|_\lambda^2\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho(\rho-s)^{2\alpha-2}\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)ds+\frac{12b^{2\alpha}}{2\alpha-1}K_3N\\
\leq&\frac{36bK_3\mathcal{L}_1\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{12b^{2\alpha}}{2\alpha-1}K_3N.
\end{aligned}
$$

Similarly like above algorithm and by Itô's isometry, we get

$$
\begin{aligned}
I_5\leq&12\mathcal{L}_2\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\frac{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}\mathbb{E}\left\|x(s)\right\|^2ds\\
&+12\mathcal{L}_2\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\frac{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}\mathbb{E}\left\|x(s-\tau_1)\right\|^2ds\\
&+12\mathcal{L}_2\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\frac{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)}\mathbb{E}\left\|x(s-\tau_2)\right\|^2ds\\
&+12K_3M\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho(\rho-s)^{2\alpha-2}ds\\
\leq&36\mathcal{L}_2\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)ds\max_{0\leq\rho\leq b}\frac{\mathbb{E}\left\|\hat{x}^*(\rho)\right\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}+12K_3M\frac{1}{2\alpha-1}\rho^{2\alpha-1}\\
\leq&36K_3\mathcal{L}_2\|\hat{x}\|_\lambda^2\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho(\rho-s)^{2\alpha-2}\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)ds+\frac{12b^{2\alpha-1}}{2\alpha-1}K_3M\\
\leq&\frac{36K_3\mathcal{L}_2\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{12b^{2\alpha-1}}{2\alpha-1}K_3M.
\end{aligned}
$$

Motived by Jensen inequality and (H_3) we have

$$
\begin{aligned}
I_6\leq&36\left\|G_\tau^b\right\|^2\left\|[(\Gamma_\tau)_0^b]^{-1}\right\|^2\frac{1}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\Bigg[\mathbb{E}\left\|x_1\right\|^2+\mathbb{E}\left\|\mathscr{E}_{\alpha,1}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho\right)\left(\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right)\right\|^2\\
&+\mathbb{E}\left\|\int_{-\tau_1}^0\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_1-s\right)\mathfrak{A}_1\varphi(s)ds\right\|^2+\mathbb{E}\left\|\int_{-\tau_2}^0\mathscr{E}_{\alpha,0}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;b-\tau_2-s\right)\mathfrak{A}_2\varphi(s)ds\right\|^2\\
&+\mathbb{E}\left\|\int_0^\rho\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)ds\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\left\|\int_0^\rho \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\widetilde{\Delta}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)dw(s)\right\|^2\Bigg]\\
\leq& 36K_4K_5\Big[\mathbb{E}\|x\|_1^2+K_0\left\|\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right\|^2+\left\|\mathfrak{A}_1\right\|^2\tau^2K_1M_1^2+\left\|\mathfrak{A}_2\right\|^2\tau^2K_2M_1^2\\
&+\frac{6bK_3\mathcal{L}_1\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{2b^{2\alpha}}{2\alpha-1}K_3N+\frac{6K_3\mathcal{L}_2\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{2b^{2\alpha-1}}{2\alpha-1}K_3M\Big].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\mathbb{E}\|(\boldsymbol{\phi}x)(\rho)\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^\alpha\right)}\\
\leq& I_1+I_2+I_3+I_4+I_5+I_6\\
\leq& 6K_0\left\|\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right\|^2+6\left\|\mathfrak{A}_1\right\|^2\tau^2K_1M_1^2+6\left\|\mathfrak{A}_2\right\|^2\tau^2K_2M_1^2\\
&+\frac{36bK_3\mathcal{L}_1\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{12b^{2\alpha}}{2\alpha-1}K_3N+\frac{36K_3\mathcal{L}_2\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{12b^{2\alpha-1}}{2\alpha-1}K_3M\\
&+36K_4K_5\Big[\mathbb{E}\|x\|_1^2+K_0\left\|\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right\|^2+\left\|\mathfrak{A}_1\right\|^2\tau^2K_1M_1^2+\left\|\mathfrak{A}_2\right\|^2\tau^2K_2M_1^2\\
&+\frac{6bK_3\mathcal{L}_1\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{2b^{2\alpha}}{2\alpha-1}K_3N+\frac{6K_3\mathcal{L}_2\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{2b^{2\alpha-1}}{2\alpha-1}K_3M\Big]\\
\leq& 36K_4K_5\mathbb{E}\|x\|_1^2+(6+36K_4K_5)\Big[K_0\left\|\varphi(0)-\mathfrak{A}_2\varphi\left(-\tau_2\right)\right\|^2+\left\|\mathfrak{A}_1\right\|^2\tau^2K_1M_1^2+\left\|\mathfrak{A}_2\right\|^2\tau^2K_2M_1^2\\
&+\frac{6bK_3\mathcal{L}_1\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{2b^{2\alpha}}{2\alpha-1}K_3N+\frac{6K_3\mathcal{L}_2\Gamma(2\alpha-1)}{\lambda}\|\hat{x}\|_\lambda^2+\frac{2b^{2\alpha-1}}{2\alpha-1}K_3M\Big].
\end{aligned}
$$

From the above, one can concludes that there exists a constant $C>0$, such that

$$
\mathbb{E}\|(\boldsymbol{\phi}x)(\rho)\|^2\leq C(1+\|\hat{x}\|_\lambda^2).
$$

Hence, $\boldsymbol{\phi}$ maps Θ_k into itself.

Step 2: We claim that $\boldsymbol{\phi}$ is a contraction mapping. In fact, for any $x,z\in\Theta_k$, by applying Jensen inequality, we derive that

$$
\begin{aligned}
&\mathbb{E}\frac{\left\|\boldsymbol{\phi}_1(x)(\rho)-(\boldsymbol{\phi}_1z)(\rho)\right\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\\
\leq&\frac{3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\mathbb{E}\left\|\int_0^\rho\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\left[\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\mathcal{H}\left(s,z(s),z\left(s-\tau_1\right),z\left(s-\tau_2\right)\right)\right]ds\right\|^2\\
&+\frac{3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\mathbb{E}\left\|\int_0^\rho\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\left[\widetilde{\Delta}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\widetilde{\Delta}\left(s,z(s),z\left(s-\tau_1\right),z\left(s-\tau_2\right)\right)\right]dw(s)\right\|^2\\
&+\frac{3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\mathbb{E}\left\|\int_0^\rho\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\mathfrak{B}(\mathfrak{u}_x(s)-\mathfrak{u}_z(s))ds\right\|^2\\
:=&J_1+J_2+J_3.
\end{aligned}
$$

Applying Hölder inequality, (H_1), Jensen inequality, Lemma 1, (H_2), and Lemma 2, one can get

$$
\begin{aligned}
J_1=&\frac{3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\mathbb{E}\left\|\int_0^\rho\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\left[\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\mathcal{H}\left(s,z(s),z\left(s-\tau_1\right),z\left(s-\tau_2\right)\right)\right]ds\right\|^2\\
\leq&\frac{3b}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho\left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2\mathbb{E}\left\|\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\mathcal{H}\left(s,z(s),z\left(s-\tau_1\right),z\left(s-\tau_2\right)\right)\right\|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{3b\mathcal{L}_1 K_3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)} \int_0^\rho (\rho-s)^{2\alpha-2}\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right) ds \max_{0\leq s\leq b} \frac{\mathbb{E}\left\|\hat{x}^*(s)-\hat{z}^*(s)\right\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)} \\
&\leq \frac{3b\Gamma(2\alpha-1)}{\lambda}\mathcal{L}_1 K_3 \left\|\hat{x}-\hat{z}\right\|_\lambda^2 .
\end{aligned}
$$

Similarly like above algorithm and by Itô's isometry, we get

$$
\begin{aligned}
J_2 =& \frac{3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\mathbb{E}\left\|\int_0^\rho \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\left[\widetilde{\Delta}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\widetilde{\Delta}\left(s,z(s),z\left(s-\tau_1\right),z\left(s-\tau_2\right)\right)\right]dw(s)\right\|^2 \\
\leq& \frac{3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho \left\|\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\right\|^2 \mathbb{E}\left\|\widetilde{\Delta}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\widetilde{\Delta}\left(s,z(s),z\left(s-\tau_1\right),z\left(s-\tau_2\right)\right)\right\|^2 ds \\
\leq& \frac{3\mathcal{L}_2 K_3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\int_0^\rho (\rho-s)^{2\alpha-2}\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right) ds \max_{0\leq s\leq b}\frac{\mathbb{E}\left\|\hat{x}^*(s)-\hat{z}^*(s)\right\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda s^{2\alpha-1}\right)} \\
\leq& \frac{3\Gamma(2\alpha-1)}{\lambda}\mathcal{L}_2 K_3 \left\|\hat{x}-\hat{z}\right\|_\lambda^2 .
\end{aligned}
$$

One can apply Jensen inequality, Itô's isometry, and (H_3) to derive that

$$
\begin{aligned}
J_3 =& \frac{3}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\mathbb{E}\left\|\int_0^\rho \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\mathfrak{B}(\mathfrak{u}_x(s)-\mathfrak{u}_z(s))ds\right\|^2 \\
\leq& \frac{6K_4K_5}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\mathbb{E}\left\|\int_0^\rho \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\left[\mathcal{H}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\mathcal{H}\left(s,z(s),z\left(s-\tau_1\right),z\left(s-\tau_2\right)\right)\right]ds\right\|^2 \\
&+\frac{6K_4K_5}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)}\mathbb{E}\left\|\int_0^\rho \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0,\mathfrak{A}_1,\mathfrak{A}_2;\rho-s\right)\left[\widetilde{\Delta}\left(s,x(s),x\left(s-\tau_1\right),x\left(s-\tau_2\right)\right)-\widetilde{\Delta}\left(s,z(s),z\left(s-\tau_1\right),z\left(s-\tau_2\right)\right)\right]dw(s)\right\|^2 \\
\leq& \frac{6K_3K_4K_5\Gamma(2\alpha-1)}{\lambda}\left\|\hat{x}-\hat{z}\right\|_\lambda^2 (b\mathcal{L}_1+\mathcal{L}_2).
\end{aligned}
$$

From the results of J_1, J_2, and J_3, we get the following

$$
\begin{aligned}
&\mathbb{E}\frac{\left\|\boldsymbol{\phi}_1(x)(\rho)-(\boldsymbol{\phi}_1 z)(\rho)\right\|^2}{\mathcal{M}_{2\alpha-1}\left(\lambda\rho^{2\alpha-1}\right)} \\
\leq& J_1+J_2+J_3 \\
\leq& \frac{3b\Gamma(2\alpha-1)}{\lambda}\mathcal{L}_1K_3\left\|\hat{x}-\hat{z}\right\|_\lambda^2+\frac{3\Gamma(2\alpha-1)}{\lambda}\mathcal{L}_2K_3\left\|\hat{x}-\hat{z}\right\|_\lambda^2 \\
&+\frac{6K_3K_4K_5\Gamma(2\alpha-1)}{\lambda}\left\|\hat{x}-\hat{z}\right\|_\lambda^2(b\mathcal{L}_1+\mathcal{L}_2) \\
=& \frac{3\Gamma(2\alpha-1)K_3}{\lambda}(1+2K_4K_5)(b\mathcal{L}_1+\mathcal{L}_2)\left\|\hat{x}-\hat{z}\right\|_\lambda^2 \\
=& K\left\|\hat{x}-\hat{z}\right\|_\lambda^2 .
\end{aligned}
$$

Since $K<1$, by (H_3), $\boldsymbol{\phi}$ is a contraction mapping on Θ_k and so $\boldsymbol{\phi}$ has an unique fixed point $x\in\Theta_k$ with $\mathfrak{u}_x(\cdot)\in\mathcal{U}_{ad}$, which is the solution of (2). The proof is completed and we can conclude that system (2) is relatively exactly controllable.

□

4 | EXAMPLES

4.1 | Example 4.1

Considering the neutral stochastic system with two different delays, firstly, we will talk about the linear case

Let $\alpha = 0.6, \tau_1 = 1, \tau_2 = 0.5$. Then, $\tau = \max\left\{\tau_1, \tau_2\right\} = 1, b = 1$,

$$\begin{cases} \left({}^{C}\mathfrak{D}_{0^+}^{0.6}x\right)(\rho) = \mathfrak{A}_0 x(\rho) + \mathfrak{A}_1 x(\rho-1) + \mathfrak{A}_2\left({}^{C}\mathfrak{D}_{0^+}^{0.6}x\right)(\rho-0.5) + f(\rho) + \mathfrak{B}\mathfrak{u}(\rho) + \widetilde{\Delta}(\rho)\frac{dw}{d\rho}, \quad \rho \in [0,1], \\ x(\rho) = \varphi(\rho) \in \mathbb{C}\left([-1,0], \mathbb{R}^2\right), \quad -1 \le \rho \le 0, \quad \tau := \max\left\{\tau_1, \tau_2\right\}, \quad \tau_1, \tau_2 > 0, \end{cases} \tag{17}$$

where $\mathfrak{A}_0 = \begin{pmatrix} 0.5 & 0.7 \\ 0.6 & 0.8 \end{pmatrix} \in \mathbb{R}_+^{2\times 2}$, $\mathfrak{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}_+^{2\times 2}$ and $\mathfrak{A}_2 = \begin{pmatrix} 0.6 & 0.3 \\ 0.5 & 0.4 \end{pmatrix} \in \mathbb{R}_+^{2\times 2}$, $\varphi(\rho) = \begin{pmatrix} 0.5\rho + 0.9 \\ 0.2\rho + 0.4 \end{pmatrix} \in \mathbb{R}_+^2$, $\widetilde{\Delta}(\rho) = \begin{pmatrix} \rho \\ 2\rho \end{pmatrix} \in \mathbb{R}_+^2$. The Grammian matrix of system (17) is

$$G_0^1 = \int_0^1 \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; 1-s\right)\mathfrak{B}\mathfrak{B}^*\left[\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; 1-s\right)\right]^* ds := G_{11} + G_{12}.$$

The delayed Mittag-Leffler type matrix functions $\mathscr{E}_{0.6,0.5}^{1,0.5}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; (\cdot)\right)$ is

$$\mathscr{E}_{0.6,0.5}^{1,0.5}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho\right) := \begin{cases} \Theta, & -1 \le \rho < 0, \\ I, & \rho = 0, \\ \sum_{k=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} Q_{k+1}\left(m_1, 0.5m_2\right)\frac{(\rho - m_1 - 0.5m_2)_+^{0.5k-0.5}}{\Gamma(0.6k+0.5)}, & \rho \in \mathbb{R}_+, \end{cases}$$

where

$$G_{11} = \int_0^{0.5} \mathscr{E}_{0.6,0.5}^{1,0.5}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; 1-s\right)\mathfrak{B}\mathfrak{B}^*\left[\mathscr{E}_{0.6,0.5}^{1,0.5}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; 1-s\right)\right]^* ds,$$

$$G_{12} = \int_{0.5}^{1} \mathscr{E}_{0.6,0.5}^{1,0.5}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; 1-s\right)\mathfrak{B}\mathfrak{B}^*\left[\mathscr{E}_{0.6,0.5}^{1,0.5}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; 1-s\right)\right]^* ds.$$

By simple calculation, we can obtain

$$G_{11} = \begin{pmatrix} 1.0666 & 1.2435 \\ 1.2435 & 1.4498 \end{pmatrix},$$

$$G_{12} = \begin{pmatrix} 4.9385 & 0.6639 \\ 0.6639 & 1.5940 \end{pmatrix}.$$

So

$$G_0^1 = \begin{pmatrix} 6.0051 & 1.8774 \\ 1.8774 & 3.0438 \end{pmatrix},$$

and its inverse

$$\left[G_0^1\right]^{-1} = \begin{pmatrix} 0.2063 & -0.1273 \\ -0.1273 & 0.4070 \end{pmatrix},$$

and we can find that G_0^1 is positive definitely, so system (17) is relatively controllable on $[0, 1]$.

4.2 | Example 4.2

Considering the relatively exactly controllable of the following nonlinear case

$$\begin{cases} \left({}^{C}\mathfrak{D}_{0^+}^{0.6}x\right)(\rho) = & \mathfrak{A}_0 x(\rho) + \mathfrak{A}_1 x(\rho-1) + \mathfrak{A}_2\left({}^{C}\mathfrak{D}_{0^+}^{0.6}x\right)(\rho-0.5) + \mathfrak{B}\mathfrak{u}(\rho) + \mathcal{H}(\rho, x(\rho), x(\rho-1), x(\rho-0.5)) \\ & +\widetilde{\Delta}(\rho, x(\rho), x(\rho-1), x(\rho-0.5))\frac{dw}{d\rho}, \quad \rho \in [0,1], \\ x(\rho) = \varphi(\rho) \in & \mathbb{C}\left([-1,0], \mathbb{R}^2\right), -1 \le \rho \le 0, \quad \tau := \max\left\{\tau_1, \tau_2\right\}, \quad \tau_1, \tau_2 > 0, \end{cases} \tag{18}$$

and we let $\mathfrak{A}_0 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix} \in \mathbb{R}_+^{2\times 2}$, $\mathfrak{A}_1 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix} \in \mathbb{R}_+^{2\times 2}$, $\mathfrak{A}_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix} \in \mathbb{R}_+^{2\times 2}$, and $\mathfrak{B} = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix} \in \mathbb{R}_+^{2\times 2}$,

$$\mathcal{H}(\rho, y(\rho), y(\rho-1), y(\rho-0.5)) = \begin{pmatrix} \frac{e^{0.8\rho-2}}{3}\left(y_1(\rho)+y_1(\rho-1)+y_1(\rho-0.5)\right) \\ \frac{e^{0.8\rho-2}}{3}\left(y_2(\rho)+y_2(\rho-1)+y_2(\rho-0.5)\right) \end{pmatrix},$$

$$\tilde{\Delta}(\rho, y(\rho), y(\rho-1), y(\rho-0.5)) = \begin{pmatrix} \frac{e^{0.4\rho-2}}{2}\left(y_1(\rho)+y_1(\rho-1)+y_1(\rho-0.5)\right) \\ \frac{e^{0.4\rho-2}}{2}\left(y_2(\rho)+y_2(\rho-1)+y_2(\rho-0.5)\right) \end{pmatrix}, \varphi(\rho) = \begin{pmatrix} \rho \\ 2\rho \end{pmatrix}.$$

The controllability Grammian matrix is

$$G_0^1 = \int_0^1 \mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; 1-s\right)\mathfrak{B}\mathfrak{B}^*\left[\mathscr{E}_{\alpha,\alpha}^{\tau_1,\tau_2}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; 1-s\right)\right]^* ds := G_{21} + G_{22}.$$

The delayed Mittag-Leffler type matrix functions $\mathscr{E}_{0.6,0.5}^{1,0.5}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; (\cdot)\right)$ *is*

$$\mathscr{E}_{0.6,0.5}^{1,0.5}\left(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2; \rho\right) := \begin{cases} \Theta, & -1 \le \rho < 0, \\ I, & \rho = 0, \\ \sum\limits_{k=0}^{\infty}\sum\limits_{m_1=0}^{\infty}\sum\limits_{m_2=0}^{\infty} Q_{k+1}\left(m_1, 0.5m_2\right)\frac{(\rho-m_1-0.5m_2)_+^{0.6k-0.5}}{\Gamma(0.6k+0.5)}, & \rho \in \mathbb{R}_+. \end{cases}$$

By simple calculation, we get the controllability Grammian matrix

$$G_{21} = \begin{pmatrix} 2.2596\times 10^{-4} & 0 \\ 0 & 2.3476\times 10^{-6} \end{pmatrix},$$

$$G_{22} = \begin{pmatrix} 0.5002 & 0 \\ 0 & 0.1211 \end{pmatrix}.$$

So

$$G_0^1 = \begin{pmatrix} 0.5005 & 0 \\ 0 & 0.1211 \end{pmatrix},$$

and

$$\left[G_0^1\right]^{-1} = \begin{pmatrix} 1.9114 & 0 \\ 0 & 0.8868 \end{pmatrix}.$$

We can obtain that G_0^1 is positive definitely, so system (18) is relatively exactly controllable on [0,1]. Thus, we have

$$\left\langle G_0^1 x, x\right\rangle = 1.9114x_1^2 + 0.8868x_2^2 \ge h\|x\|^2,$$

where $0 < h \le 0.8868$, and we can let $\lambda = 1$, then we can easily obtain $K_3 = 0.8782$, $K_4 = 0.5005$, $K_5 = 0.2505$. Letting

$$\begin{aligned} K &= \frac{3\Gamma(2\alpha-1)K_3}{\lambda}\left(1+2K_4K_5\right)\left(b\mathcal{L}_1+\mathcal{L}_2\right) \\ &= 3\times 4.5814\times 0.8782\times(1+0.2508)\times(0.0302+0.0306) \\ &= 0.9198 < 1, \end{aligned}$$

which implies that all the conditions in Theorem1 are satisfied. So system (18) is relatively exactly controllable.

5 | CONCLUSIONS

This paper considers the relative exact controllability of fractional neutral stochastic system with two incommensurate constant delays. With the applying of controllability Grammian matrix, a result of relatively exact controllability of linear part FSDS is obtained. The nonlinear part of relatively exact controllability is given by using the Banach contraction principle. Controllability criterions for linear and nonlinear systems are also established, respectively.

In the forthcoming papers, we will focus on fuzzy fractional delay system and study the controllability of the addressed system. For this fuzzy fractional system, it is different from traditional fractional differential equations because of its variables and parameters are uncertain. But until now, there are few authors paying attention to it.

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CONFLICT OF INTEREST

Authors declare that they have no conflict of interest.

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