A new three-point linearized conservative compact difference scheme based on reduction order method for the RLW equation

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Abstract

In this paper, a new fourth-order compact difference scheme based on the reduction order method is proposed for solving the regularized long wave (RLW) equation. The compact finite difference scheme is three-level and linear. The discrete mass and discrete energy, boundedness and uniqueness of the present compact scheme are proved. Convergence and stability of the compact scheme are also analyzed by using the discrete energy method. Our compact scheme has the rates of convergence of second-order in temporal direction and fourth-order in spatial direction, respectively. Numerical examples are carried out to verify the reliability of the theory analysis.

Key words: RLW equation, compact difference scheme, reduction order method, conservation, convergence

1. Introduction

In this paper, we consider the following initial boundary problem for the regularized long wave (RLW) equation [1]

$$u_t - \mu u_{xxt} + u_x + \delta u u_x = 0, \quad (x,t) \in [x_l, x_r] \times (0,T], \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in [x_l, x_r], \tag{2}$$

and the boundary condition

$$u(x_l,t) = u(x_r,t) = 0, \quad t \in (0,T], \tag{3}$$

where μ and δ are non-negative constants, $u_0(x)$ is a known smooth function. It is significant to construct a conservative scheme for solving the nonlinear partial differential equation. The original differential equation problem (1)-(3) has the following conservation quantities

$$Q(t) = \int_{x_l}^{x_r} u(x,t)dx = \int_{x_l}^{x_r} u_0(x,t)dx = Q(0), \quad t > 0,$$

and

$$E(t) = \|u\|_{L^2}^2 + \mu\|u_x\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \mu\|(u_0)_x\|_{L^2}^2 = E(0), \quad t > 0,$$

where $Q(0)$ and $E(0)$ are two positive constants which relate to the initial condition.

The RLW equation is also known as Benjamin-Bona-Mahony equation [2], which was first formulated by Peregrine [3] as an alternative to the Korteweg-de-Vries (KdV) equation to describe the behavior of the undular bore and as a model for long water waves of small but finite amplitude, generated in a

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uniform open channel by a wavemaker at one end [2]. Many important nonlinear physical phenomena, such as shallow water and ionic waves, can be described by the RLW equation, which is the one of the most important nonlinear wave equations.

It is difficult to find the exact solution of RLW equation because of the nonlinear term uu_x. Thus, the numerical methods of the initial and boundary conditions for the RLW equation have become the focus of the investigators. Various numerical techniques including the Galerkin method [4, 5, 6], the pseudo-spectral method [7], the variational iteration method [8] have been used to solve the RLW equation. Specially, the study of the finite difference scheme for the initial-boundary value problem of the RLW has attracted great attention. EL-Danaf et al. [9] solved the generalized RLW equation using the finite difference method, which shows the linearized scheme is unconditionally by the fourier stability analysis. The local truncation errors of three schemes are $O(\tau + h)$, $O(\tau + h^2)$, $O(\tau^2 + h^2)$, respectively. A energy conservative finite difference scheme with local truncation error $O(\tau^2 + h^2)$ for a Cauchy problem of the generalized regularized long-wave (GRLW) equation was considered in [10]. A Crank-Nicolson-type finite difference scheme for solving the BBM equation was presented in [11], which has second-order convergence in discrete H^1-norm. Zheng [12] studied a conservative Crank-Nicolson finite difference scheme with the Richardson extrapolation technique for solving the RLW equation. Wang et al.[13] proposed two conservative fourth-order compact finite difference schemes for the RLW eqaution, which are fourth-order in spatial direction and second-order in temporal direction. In [14], a new compact finite difference scheme for solving the GRLW equation was analyzed and the rate of convergence of the scheme is of order $O(\tau^2 + h^4)$.

The main goals of this paper are to construct a high-order accurate, linearized and conservative compact difference scheme for solving RLW equation, which needs only three mesh point along the x-direction based on the reduction order method. A novel fourth-order compact operator is applied to approximate the strong nonlinear term uu_x. The method is different from those in [14, 13, 12]. The optimal convergence order $O(\tau^2 + h^4)$ and stability in discrete L^∞-, L^2- and H^1-norms are completely overcome by using the sophisticated discrete energy method. Conservations of discrete mass and energy, boundedness and uniquely solvability are given in detail.

The rest of this paper is arranged as follows. In Section 2, we introduce some notations and lemmas. The linearized conservative compact difference scheme based on reduction method is proposed for the RLW equation in Section 3. The discrete mass and energy conservation of the compact difference scheme are given in Section 4. Furthermore, the boundedness and uniquely solvability of the compact difference scheme are proved in Sections 5. Convergence and stability of the compact difference scheme are analyzed by using the discrete energy method in Section 6. The scheme is proved to be convergent with second-order in time and fourth-order in space, respectively. In Section 7, numerical experiments are provided to verify the reliability of theoretical analysis by simulating the collision of solitary waves. Finally, some concluded remarks are given in Section 8.

2. Notations and lemmas

In order to solve the problem (1)-(3), we first divide the domain $[x_l, x_r] \times [0, T]$. Taking positive integers J and N and letting $h = (x_r - x_l)/J, \tau = T/N$, where h and τ are space-step and time-step, respectively. Denote $u = \{u_j^n | 0 \le j \le J, 0 \le n \le N\}$ as the discrete grid function on $\Omega_{h,\tau} = \{(x_j, t_n) | x_j = x_l + jh, t_n = n\tau, 0 \le j \le J, 0 \le n \le N\}$, we further define the following difference notations

$$u_j^{n+\frac{1}{2}} = \frac{1}{2}(u_j^{n+1} + u_j^n), \quad \overline{u}_j^n = \frac{1}{2}(u_j^{n+1} + u_j^{n-1}), \quad (u_j^n)_t = \frac{1}{\tau}(u_j^{n+1} - u_j^n),$$
$$(u_j^n)_x = \frac{1}{h}(u_{j+1}^n - u_j^n), \quad (u_j^n)_{\hat{t}} = \frac{1}{2\tau}(u_j^{n+1} - u_j^{n-1}), \quad (u_j^n)_{\bar{x}} = \frac{1}{h}(u_j^n - u_{j-1}^n),$$
$$(u_j^n)_{\hat{x}} = \frac{1}{2h}(u_{j+1}^n - u_{j-1}^n).$$

Let

$$\mathbb{Z}_h^0 = \{u | u = (u_j), u_0 = u_J = 0, j = -1, 0, 1, \ldots, J, J+1\}$$

be the discrete Sobolev space. For any grid functions $u, v \in \mathbb{Z}_h^0$, we introduce the discrete inner product and norms

$$\langle u, v \rangle = h \sum_{j=1}^{J-1} u_j v_j, \quad \|u\|^2 = \langle u, u \rangle, \quad \|u\|_\infty = \max_{1 \le j \le J-1} |u_j|,$$

and define the function Ψ as follows

$$\Psi(u_j, v_j) = \frac{1}{3}[u_j (v_j)_{\hat{x}} + (u_j v_j)_{\hat{x}}], \quad 1 \le j \le J-1.$$

The constant C that appears in the passage is a positive constant independent of τ and h.

Lemma 1. *[15] For any grid functions $u, v \in \mathbb{Z}_h^0$, we have*

$$\langle u_{\hat{x}}, v \rangle = -\langle u, v_{\hat{x}} \rangle, \quad \langle u_{x\bar{x}}, v \rangle = -\langle u_x, v_x \rangle, \quad \langle \Psi(u, v), v \rangle = 0.$$

When $u = v$, we have

$$\langle u_{\hat{x}}, u \rangle = 0, \quad \langle u_{x\bar{x}}, u \rangle = -\|u_x\|^2, \quad \langle u_{x\bar{x}}, u_{x\bar{x}} \rangle = \|u_{x\bar{x}}\|^2, \quad \langle u_{x\bar{x}}, u_{\hat{x}} \rangle = 0.$$

Lemma 2. *[16] For any grid function $u \in \mathbb{Z}_h^0$, we have*

$$\|u_x\| \le \frac{2}{h}\|u\|, \quad \|u\| \le \frac{L}{\sqrt{6}}\|u_x\|.$$

In addition, for any grid function $u \in \mathbb{Z}_h^0$ and arbitrary $\varepsilon > 0$, we have

$$\|u\|_\infty \le \varepsilon \|u_x\| + \frac{1}{4\varepsilon}\|u\|.$$

Lemma 3. *[17] Let $g(x) \in C^5[x_{j-1}, x_{j+1}]$ and $G(x_j) = g''(x_j), 1 \le j \le J-1$, we have*

$$\begin{aligned}
g'(x_j) &= (g(x_j))_{\hat{x}} - \frac{h^2}{6}(G(x_j))_{\hat{x}} + O(h^4), \\
g''(x_j) &= (g(x_j))_{x\bar{x}} - \frac{h^2}{12}(G(x_j))_{x\bar{x}} + O(h^4), \\
g(x_j)g'(x_j) &= \Psi(g(x_j), g(x_j)) - \frac{h^2}{2}\Psi(G(x_j), g(x_j)) + O(h^4).
\end{aligned}$$

Lemma 4. *[17, 18] For any grid functions $u^n, v^n, R \in \mathbb{Z}_h^0$ satisfying*

$$v_j^n = (u_j^n)_{x\bar{x}} - \frac{h^2}{12}(v_j^n)_{x\bar{x}} + R_j^n, \quad 1 \le j \le J-1,$$

then we have

$$\begin{aligned}
\langle v_{\hat{x}}^n, u^n \rangle &= \frac{h^2}{12}\langle v_{\hat{x}}^n, R^n \rangle - \langle R^n, u_{\hat{x}}^n \rangle, \\
\langle v_{\hat{t}}^n, 2\bar{u}^n \rangle &= -\|u_x^n\|_{\hat{t}}^2 - \frac{h^2}{12}\|v^n\|_{\hat{t}}^2 + \frac{h^4}{144}\|v_x^n\|_{\hat{t}}^2 + \frac{h^2}{12}\langle v_{\hat{t}}^n, 2\bar{R}^n \rangle + \langle R_{\hat{t}}^n, 2\bar{u}^n \rangle.
\end{aligned}$$

3. Construction of compact difference scheme

Let $v = u_{xx}$, then the problem (1)-(3) is equivalent to

$$u_t - \mu v_t + u_x + \delta u u_x = 0, \quad x \in [x_l, x_r], \quad t \in (0,T], \tag{4}$$
$$v = u_{xx}, \quad x \in [x_l, x_r], \quad t \in (0,T], \tag{5}$$
$$u(x,0) = u_0(x), \quad x \in [x_l, x_r], \tag{6}$$
$$u(x_l,t) = u(x_r,t) = 0, \quad v(x_l,t) = v(x_r,t) = 0, \quad t \in (0,T]. \tag{7}$$

Define the grid functions

$$u_j^n \approx U_j^n = u(x_j, t_n), \quad v_j^n \approx V_j^n = v(x_j, t_n), \quad 0 \le j \le J, \quad 0 \le n \le N.$$

Considering Eq. (4) at the point $(x_j, t_{\frac{1}{2}})$ and Eqs. (4)-(5) at the point (x_j, t_n), respectively, and applying Lemma 3, we have

$$(U_j^0)_t - \mu (V_j^0)_t + (U_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(V_j^{\frac{1}{2}})_{\hat{x}} + \delta \Psi(U_j^0, U_j^{\frac{1}{2}}) - \frac{\delta h^2}{2}\Psi(V_j^0, U_j^{\frac{1}{2}}) = P_j^0,$$
$$1 \le j \le J-1, \quad 1 \le n \le N-1, \tag{8}$$
$$(U_j^n)_{\hat{t}} - \mu (V_j^n)_{\hat{t}} + (\bar{U}_j^n)_{\hat{x}} - \frac{h^2}{6}(\bar{V}_j^n)_{\hat{x}} + \delta \Psi(U_j^n, \bar{U}_j^n) - \frac{\delta h^2}{2}\Psi(V_j^n, \bar{U}_j^n) = P_j^n,$$
$$1 \le j \le J-1, \quad 1 \le n \le N-1, \tag{9}$$
$$V_j^n = (U_j^n)_{x\bar{x}} - \frac{h^2}{12}(V_j^n)_{x\bar{x}} + R_j^n, \quad 1 \le j \le J-1, \quad 0 \le n \le N, \tag{10}$$
$$(U_j^0) = u_0(x_j), \quad 0 \le j \le J, \tag{11}$$
$$U_0^n = U_J^n = 0, \quad V_0^n = V_J^n = 0, \quad 0 \le n \le N, \tag{12}$$

and there exists a constant C such that

$$|P_j^0| \le C(\tau^2 + h^4), \quad |P_j^n| \le C(\tau^2 + h^4), \quad |R_j^n| \le C(\tau^2 + h^4), \quad |R_{\hat{t}}^n| \le C(\tau^2 + h^4).$$

Ignoring the small terms P_j^0, P_j^n, R_j^n and replacing the grid function U_j^n, V_j^n with u_j^n, v_j^n, we construct a linearized four-order compact finite difference scheme for Eqs. (4)-(7) as follows

$$(u_j^0)_t - \mu (v_j^0)_t + (u_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(v_j^{\frac{1}{2}})_{\hat{x}} + \delta \Psi(u_j^0, u_j^{\frac{1}{2}}) - \frac{\delta h^2}{2}\Psi(v_j^0, u_j^{\frac{1}{2}}) = 0,$$
$$1 \le j \le J-1, \quad 1 \le n \le N-1, \tag{13}$$
$$(u_j^n)_{\hat{t}} - \mu (v_j^n)_{\hat{t}} + (\bar{u}_j^n)_{\hat{x}} - \frac{h^2}{6}(\bar{v}_j^n)_{\hat{x}} + \delta \Psi(u_j^n, \bar{u}_j^n) - \frac{\delta h^2}{2}\Psi(v_j^n, \bar{u}_j^n) = 0,$$
$$1 \le j \le J-1, \quad 1 \le n \le N-1, \tag{14}$$
$$v_j^n = (u_j^n)_{x\bar{x}} - \frac{h^2}{12}(v_j^n)_{x\bar{x}}, \quad 1 \le j \le J-1, \quad 0 \le n \le N, \tag{15}$$
$$u_j^0 = u_0(x_j), \quad 0 \le j \le J, \tag{16}$$
$$u_0^n = u_J^n = 0, \quad v_0^n = v_J^n = 0, \quad 0 \le n \le N. \tag{17}$$

Define

$$\mathbf{u}^n = (u_1^n, u_2^n, \ldots, u_{J-1}^n,)^T, \quad \mathbf{v}^n = (v_1^n, v_2^n, \ldots, v_{J-1}^n,)^T, \quad 0 \le n \le N,$$
$$k_1 = \frac{1}{2\tau}, \quad k_2 = \frac{1}{12h}, \quad k_3 = \frac{h}{24}.$$

From Eqs. (15)-(16), we get $\mathbf{u}^0$ and $\mathbf{v}^0$. Let $a = 2k_1$, then we can calculate $\mathbf{u}^1$ and $\mathbf{v}^1$ for the first level base on the following matrix-vector form

$$\begin{bmatrix} A_1 & B_1 \\ -E & D \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{v}^1 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ E & -D \end{bmatrix} \begin{bmatrix} \mathbf{u}^0 \\ \mathbf{v}^0 \end{bmatrix}.$$

Similarly, let $a = k_1$, then solving $\mathbf{u}^n$ and $\mathbf{v}^n$ $(n \geq 1)$ depends on the following matrix-vector form

$$\begin{bmatrix} A_1 & B_1 \\ -E & D \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{v}^{n+1} \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ E & -D \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n-1} \\ \mathbf{v}^{n-1} \end{bmatrix},$$

where A_1, A_2, B_1, B_2, E, D are $J-1 \times J-1$ matrices as follows:

$$A_1 = \begin{bmatrix} a & a_{2,1} & 0 & \cdots & 0 \\ a_{3,2} & a & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{3,J-2} & a & a_{2,J-2} \\ 0 & \cdots & 0 & a_{3,J-1} & a \end{bmatrix}, \quad A_2 = \begin{bmatrix} a & -a_{2,1} & 0 & \cdots & 0 \\ -a_{3,2} & a & -a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -a_{3,J-2} & a & -a_{2,J-2} \\ 0 & \cdots & 0 & -a_{3,J-1} & a \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -\mu a & -k_3 & 0 & \cdots & 0 \\ k_3 & -\mu a & -k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & k_3 & -\mu a & -k_3 \\ 0 & \cdots & 0 & k_3 & -\mu a \end{bmatrix}, \quad B_2 = \begin{bmatrix} -\mu a & k_3 & 0 & \cdots & 0 \\ -k_3 & -\mu a & k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -k_3 & -\mu a & k_3 \\ 0 & \cdots & 0 & -k_3 & -\mu a \end{bmatrix},$$

$$E = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}, \quad D = \frac{1}{12}\begin{bmatrix} 10 & 1 & 0 & \cdots & 0 \\ 1 & 10 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 10 & 1 \\ 0 & \cdots & 0 & 1 & 10 \end{bmatrix},$$

where

$$a_{2,j} = 3k_2 + \delta k_2(u_j^n + u_{j+1}^n) - \delta k_3(v_j^n + v_{j+1}^n), \quad a_{3,j} = -3k_2 - \delta k_2(u_j^n + u_{j-1}^n) + \delta k_3(v_j^n + v_{j-1}^n).$$

4. Conservative laws

Theorem 1. *Suppose that $\{u_j^n, v_j^n | 0 \leq j \leq J, 0 \leq n \leq N\}$ are the numerical solution of scheme (13)-(17), then the compact scheme (13)-(17) satisfies the discrete mass and energy conservation in the senses*

$$\begin{aligned} Q^{n+1} := & \frac{h}{2}\sum_{j=1}^{J-1}(u_j^{n+1} + u_j^n) - \frac{\mu h}{2}\sum_{j=1}^{J-1}(v_j^{n+1} + v_j^n) + \frac{\delta h\tau}{6}\sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_{\hat{x}} \\ & - \frac{\delta h^3 \tau}{12}\sum_{j=1}^{J-1}(u_j^n)_{x\bar{x}}(u_j^{n+1})_{\hat{x}} - \frac{\delta h^5 \tau}{144}(v_j^n)_{\hat{x}} v_j^{n+1} - \frac{\delta h^7 \tau}{1728}(v_j^n)_{\hat{x}}(v_j^{n+1})_{x\bar{x}} \\ = & Q^n = \ldots = Q^1 = Q^0, \quad 0 \leq n \leq N-1, \end{aligned} \tag{18}$$

$$\begin{aligned} E^{n+1} := & \frac{1}{2}(\|u^{n+1}\|^2 + \|u^n\|^2) + \frac{\mu}{2}(\|u_x^{n+1}\|^2 + \|u_x^n\|^2) + \frac{\mu h^2}{24}(\|v^{n+1}\|^2 + \|v^n\|^2) \\ & - \frac{\mu h^4}{288}(\|v_x^{n+1}\|^2 + \|v_x^n\|^2) \\ = & E^n = \ldots = E^1 = E^0, \quad 0 \leq n \leq N-1, \end{aligned} \tag{19}$$

where

$$
\begin{aligned}
Q^0 &= h\sum_{j=1}^{J-1}(u_j^0-\mu v_j^0)+\frac{\delta h\tau}{6}\sum_{j=1}^{J-1}u_j^0(u_j^0)_{\hat{x}}-\frac{\delta h^3\tau}{12}\sum_{j=1}^{J-1}(u_j^0)_{x\bar{x}}(u_j^0)_{\hat{x}}\\
&\quad -\frac{\delta h^5\tau}{144}(v_j^0)_{\hat{x}}v_j^0-\frac{\delta h^7\tau}{1728}(v_j^0)_{\hat{x}}(v_j^0)_{x\bar{x}},\\
E^0 &= \|u^0\|^2+\mu\|u_x^0\|^2+\frac{\mu h^2}{12}\|v^0\|^2-\frac{\mu h^4}{144}\|v_x^0\|^2.
\end{aligned}
$$

Proof. Multiplying Eq. (14) with h and summing up j from 1 to J, we have

$$
\frac{h}{2\tau}\sum_{j=1}^{J-1}(u_j^{n+1}-u_j^{n-1})-\frac{\mu h}{2\tau}\sum_{j=1}^{J-1}(v_j^{n+1}-v_j^{n-1})+\delta h\sum_{j=1}^{J-1}\Psi(u_j^n,\bar{u}_j^n)-\frac{\delta h^3}{2}\sum_{j=1}^{J-1}\Psi(v_j^n,\bar{u}_j^n)=0.
$$

Applying Lemma 1 and Eq. (15) and taking $R^n=0$, we have

$$
\begin{aligned}
\delta h\sum_{j=1}^{J-1}\Psi(u_j^n,\bar{u}_j^n) &= \frac{\delta h}{3}\sum_{j=1}^{J-1}[u_j^n(\bar{u}_j^n)_{\hat{x}}+(u_j^n\bar{u}_j^n)_{\hat{x}}]\\
&= \frac{\delta h}{6}\sum_{j=1}^{J-1}[u_j^n(u_j^{n+1})_{\hat{x}}+u_j^n(u_j^{n-1})_{\hat{x}}]=\frac{\delta h}{6}\sum_{j=1}^{J-1}[u_j^n(u_j^{n+1})_{\hat{x}}-u_j^{n-1}(u_j^n)_{\hat{x}}],
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\delta h^3}{2}\sum_{j=1}^{J-1}\Psi(v_j^n,\bar{u}_j^n) &= \frac{\delta h^3}{6}\sum_{j=1}^{J-1}[v_j^n(\bar{u}_j^n)_{\hat{x}}+(v_j^n\bar{u}_j^n)_{\hat{x}}]\\
&= \frac{\delta h^3}{6}\sum_{j=1}^{J-1}[(u_j^n)_{x\bar{x}}(\bar{u}_j^n)_{\hat{x}}-\frac{h^2}{12}(v_j^n)_{x\bar{x}}(\bar{u}_j^n)_{\hat{x}}]\\
&= \frac{\delta h^3}{6}\sum_{j=1}^{J-1}(u_j^n)_{x\bar{x}}(\bar{u}_j^n)_{\hat{x}}+\frac{\delta h^5}{72}\sum_{j=1}^{J-1}(v_j^n)_{\hat{x}}[\bar{v}_j^n+\frac{h^2}{12}(\bar{v}_j^n)_{x\bar{x}}]\\
&= \frac{\delta h^3}{12}\sum_{j=1}^{J-1}(u_j^n)_{x\bar{x}}(u_j^{n+1})_{\hat{x}}+\frac{\delta h^5}{144}\sum_{j=1}^{J-1}(v_j^n)_{\hat{x}}v_j^{n+1}+\frac{\delta h^7}{1728}\sum_{j=1}^{J-1}(v_j^n)_{\hat{x}}(v_j^{n+1})_{x\bar{x}}\\
&\quad -\frac{\delta h^3}{12}\sum_{j=1}^{J-1}(u_j^{n-1})_{x\bar{x}}(u_j^n)_{\hat{x}}-\frac{\delta h^5}{144}\sum_{j=1}^{J-1}(v_j^{n-1})_{\hat{x}}v_j^n-\frac{\delta h^7}{1728}\sum_{j=1}^{J-1}(v_j^{n-1})_{\hat{x}}(v_j^n)_{x\bar{x}},
\end{aligned}
$$

then we have

$$
Q^{n+1}=Q^n,\quad 1\le n\le N-1. \tag{20}
$$

Multiplying Eq. (13) with h and summing up j from 1 to $J-1$, we have

$$
\frac{h}{\tau}\sum_{j=1}^{J-1}(u_j^1-u_j^0)-\frac{\mu h}{\tau}\sum_{j=1}^{J-1}(v_j^1-v_j^0)+\delta h\sum_{j=1}^{J-1}\Psi(u_j^0,\bar{u}_j^{\frac{1}{2}})-\frac{\delta h^3}{2}\sum_{j=1}^{J-1}\Psi(v_j^0,\bar{u}_j^{\frac{1}{2}})=0.
$$

Thus, we have

$$
\begin{aligned}
&h\sum_{j=1}^{J-1}(u_j^1+u_j^0)-\mu h\sum_{j=1}^{J-1}(v_j^1+v_j^0)+\frac{\delta h\tau}{3}\sum_{j=1}^{J-1}u_j^0(u_j^1)_{\hat{x}}-\frac{\delta h^3\tau}{6}\sum_{j=1}^{J-1}(u_j^0)_{x\bar{x}}(u_j^1)_{\hat{x}}\\
&\quad-\frac{\delta h^5\tau}{72}(v_j^0)_{\hat{x}}v_j^1-\frac{\delta h^7\tau}{864}(v_j^0)_{\hat{x}}(v_j^1)_{x\bar{x}}\\
&=2h\sum_{j=1}^{J-1}(u_j^0-\mu v_j^0)+\frac{\delta h\tau}{3}\sum_{j=1}^{J-1}u_j^0(u_j^0)_{\hat{x}}-\frac{\delta h^3\tau}{6}\sum_{j=1}^{J-1}(u_j^0)_{x\bar{x}}(u_j^0)_{\hat{x}}\\
&\quad-\frac{\delta h^5\tau}{72}(v_j^0)_{\hat{x}}v_j^0-\frac{\delta h^7\tau}{864}(v_j^0)_{\hat{x}}(v_j^0)_{x\bar{x}},
\end{aligned}
$$

and hence $Q^1=Q^0$.

Taking the inner product of Eq. (14) with $2\bar{u}^n$, applying Lemma 1, we have

$$\|u^n\|_{\hat{t}}^2-\mu\langle v_{\hat{t}}^n,2\bar{u}^n\rangle-\frac{h^2}{6}\langle v_{\hat{x}}^n,2\bar{u}^n\rangle=0.$$

According to Lemma 4 and taking $R=0$, we have

$$\|u^n\|_{\hat{t}}^2+\mu\|u_x\|_{\hat{t}}^2+\frac{\mu h^2}{12}\|v\|_{\hat{t}}^2-\frac{\mu h^4}{144}\|v_x\|_{\hat{t}}^2=0,$$

and hence

$$E^{n+1}=E^n,\quad 1\le n\le N-1. \tag{21}$$

Taking the inner product of Eq. (13) with $2u^{\frac{1}{2}}$, we have

$$\|u^0\|_t^2+\mu\|u_x^0\|_t^2+\frac{\mu h^2}{12}\|v^0\|_t^2-\frac{\mu h^4}{144}\|v_x^0\|_t^2=0,$$

which gives

$$
\begin{aligned}
&\|u^1\|^2+\|u^0\|^2+\mu\|u_x^1\|^2+\mu\|u_x^0\|^2-\frac{\mu h^2}{12}(\|v^1\|^2+\|v^0\|^2)-\frac{\mu h^4}{144}(\|v_x^1\|^2+\|v_x^0\|^2)\\
&=2(\|u^0\|^2+\mu\|u_x^0\|^2)+\frac{\mu h^2}{6}\|v^0\|^2-\frac{\mu h^4}{72}\|v_x^0\|^2,
\end{aligned}
$$

which is $E^1=E^0$. This completes the proof.

5. Boundedness and uniqueness

Theorem 2. *Suppose that $u_0(x)\in H_0^1[x_l,x_r]$ and $u(x,t)\in C_{x,t}^{5,3}([x_l,x_r]\times(0,T])$, then the numerical solution $\{u_j^n|0\le j\le J,0\le n\le N\}$ of the compact difference scheme (13)-(17) satisfies*

$$\|u^n\|\le C,\quad \|u_x^n\|\le C,\quad \|u^n\|_\infty\le C,\quad 0\le n\le N.$$

Proof. Assume that there exists a positive constant $\widehat{c}$ such that

$$\max_{x\in\Omega}\left\{|u(x,t)|,\left|\frac{\partial u(x,t)}{\partial x}\right|,\left|\frac{\partial^2u(x,t)}{\partial x^2}\right|,|v(x,t)|,\left|\frac{\partial v(x,t)}{\partial x}\right|,\left|\frac{\partial^2v(x,t)}{\partial x^2}\right|\right\}\le\widehat{c}.$$

From Eq. (19) and Lemma 2, we have

$$E^0=E^{n+1}\ge\frac{1}{2}(\|u^{n+1}\|^2+\mu\|u_x^{n+1}\|^2)+\frac{\mu h^2}{36}\|v^{n+1}\|^2,\quad 0\le n\le N-1.$$

There exists a positive constant h_0 such that when $h \le h_0$ and $E^0 \le (1+\mu+\frac{\mu h_0^2}{12})\hat{c}^2 \le C$, we have

$$\|u^n\| \le \sqrt{2E^0}, \quad \|u_x^n\| \le \sqrt{\frac{2E^0}{\mu}}, \quad \|v^n\| \le \frac{6}{h}\sqrt{\frac{E^0}{\mu}}, \quad 1 \le n \le N.$$

Thus, according to Lemma 2, we obtain

$$\|u^n\|_\infty \le \frac{\sqrt{L}}{2}\|u_x^n\| \le \sqrt{\frac{LE^0}{2\mu}}, \quad 1 \le n \le N.$$

This completes the proof.

Theorem 3. *The compact difference scheme (13)-(17) is uniquely solvable.*

Proof. We can easily know that u^0, v^0 have been determined from Eqs. (15)-(16). The first level u^1, v^1 are computed by Eqs. (13) and (15). Now, we consider its homogenous system

$$\frac{1}{\tau}(u_j^1 - \mu v_j^1) + \frac{1}{2}(u_j^1)_{\hat{x}} - \frac{h^2}{12}(v_j^1)_{\hat{x}} + \frac{\delta}{2}\Psi(u_j^0, u_j^1) - \frac{\delta h^2}{4}\Psi(v_j^0, u_j^1) = 0, \quad 1 \le j \le J-1, \tag{22}$$

$$v_j^1 = (u_j^1)_{x\bar{x}} - \frac{h^2}{12}(v_j^1)_{x\bar{x}}, \quad 1 \le j \le J-1. \tag{23}$$

Taking the inner product of Eq. (22) with u^1 and applying Lemma 1, we have

$$\frac{1}{\tau}\|u^1\|^2 - \frac{\mu}{\tau}\langle v^1, u^1\rangle - \frac{h^2}{12}\langle v_{\hat{x}}^1, u^1\rangle = 0.$$

Using Lemmas 2, 4 and taking $R = 0$, we have

$$0 = \frac{1}{\tau}\left(\|u^1\|^2 - \mu\langle v^1, u^1\rangle\right) \ge \frac{1}{\tau}\left(\|u^1\|^2 + \mu\|u_x^1\|^2 + \frac{\mu h^2}{18}\|v^1\|^2\right). \tag{24}$$

Thus, we get $\|u^1\| = \|v^1\| = 0$, which implies that u^1 and v^1 have been determined by Eqs. (13) and (15) uniquely.

Now, we suppose that u^k, v^k with $0 \le k \le n$ have been determined uniquely. Since u^{n+1}, v^{n+1} are computed by Eqs. (14)-(15), we consider its homogenous system

$$\frac{1}{2\tau}(u_j^{n+1} - \mu v_j^{n+1}) + \frac{1}{2}(u_j^{n+1})_{\hat{t}} - \frac{h^2}{12}(v_j^{n+1})_{\hat{x}} + \frac{\delta}{2}\Psi(u_j^n, u_j^{n+1}) - \frac{\delta h^2}{4}\Psi(v_j^n, u_j^{n+1}) = 0, \quad 1 \le j \le J-1, \tag{25}$$

$$v_j^{n+1} = (u_j^{n+1})_{x\bar{x}} - \frac{h^2}{12}(v_j^{n+1})_{x\bar{x}}, \quad 1 \le j \le J-1. \tag{26}$$

Taking the inner product of Eq. (25) with u^{n+1}, and applying Lemma 1, we have

$$\frac{1}{2\tau}\|u^{n+1}\|^2 - \frac{\mu}{2\tau}\langle v^{n+1}, u^{n+1}\rangle - \frac{h^2}{12}\langle v_{\hat{x}}^{n+1}, u^{n+1}\rangle = 0.$$

According to Lemma 4, we have

$$0 = \frac{1}{2\tau}(\|u^{n+1}\|^2 - \mu\langle v^{n+1}, u^{n+1}\rangle) \ge \frac{1}{2\tau}(\|u^{n+1}\|^2 + \mu\|u_x^{n+1}\|^2 + \frac{\mu h^2}{18}\|v^{n+1}\|^2).$$

Thus, we have $\|u^{n+1}\| = \|v^{n+1}\| = 0$, which implies that Eqs. (14)-(15) determine u^{n+1}, v^{n+1} uniquely. This completes the proof.

6. Convergence and stability

Theorem 4. *Suppose that $u_0(x) \in H_0^1[x_l, x_r], u(x,t) \in C_{x,t}^{5,3}([x_l, x_r] \times (0,T])$, then the solution of the difference scheme (13)-(17) converges to the solution of Eqs. (4)-(7) and the convergence order is $O(\tau^2 + h^4)$ in discrete norms $\|\cdot\|_\infty$ and $\|\cdot\|$.*

Proof. Let $e_j^n = U_j^n - u_j^n, \eta = V_j^n - v_j^n$ for $0 \le j \le J$, $0 \le n \le N$. Then, we obtain the following error system as

$$(e_j^0)_t - \mu(\eta_j^0)_t + (e_j^{\frac{1}{2}})_{\hat{x}} - \frac{h^2}{6}(\eta_j^{\frac{1}{2}})_{\hat{x}} + \delta\Psi(U_j^0, U_j^{\frac{1}{2}}) - \delta\Psi(u_j^0, u_j^{\frac{1}{2}}) - \frac{\delta h^2}{2}[\Psi(V_j^0, U_j^{\frac{1}{2}}) - \Psi(v_j^0, u_j^{\frac{1}{2}})] = P_j^0,$$
$$1 \le j \le J-1, \tag{27}$$
$$(e_j^n)_{\hat{t}} - \mu(\eta_j^n)_{\hat{t}} + (\bar{e}_j^n)_{\hat{x}} - \frac{h^2}{6}(\bar{\eta}_j^n)_{\hat{x}} + \delta\Psi(U_j^n, \bar{U}_j^n) - \delta\Psi(u_j^n, \bar{u}_j^n) - \frac{\delta h^2}{2}[\Psi(V_j^n, \bar{U}_j^n) - \Psi(v_j^n, \bar{u}_j^n)] = P_j^n,$$
$$1 \le j \le J-1, \quad 1 \le n \le N-1, \tag{28}$$
$$\eta_j^n = (e_j^n)_{x\bar{x}} - \frac{h^2}{12}(\eta_j^n)_{x\bar{x}} + R_j^n, \quad 1 \le j \le J-1, \quad 0 \le n \le N, \tag{29}$$
$$e_j^0 = 0, \quad 0 \le j \le J, \tag{30}$$
$$e_0^n = e_J^n = 0, \quad \eta_0^n = \eta_J^n = 0, \quad 0 \le n \le N. \tag{31}$$

Followings Eqs. (27) and (30), we have

$$\frac{1}{\tau}(e_j^1 - \mu\eta_j^1) + \frac{1}{2}(e_j^1)_{\hat{x}} - \frac{h^2}{12}(\eta_j^1)_{\hat{x}} + \delta\Psi(U_j^0, e_j^1) - \frac{\delta h^2}{2}\Psi(V_j^0, e_j^1) = P_j^0, \quad 1 \le j \le J-1. \tag{32}$$

Taking the inner product of Eq. (32) with e^1, and applying Lemma 1, we have

$$\frac{1}{\tau}\|e^1\|^2 - \frac{\mu}{\tau}\langle \eta^1, e^1 \rangle - \frac{h^2}{12}\langle \eta_{\hat{x}}^1, e^1 \rangle = \langle P^0, e^1 \rangle. \tag{33}$$

From Lemma 4, we have

$$\langle \eta, e^1 \rangle = -\|e_x^1\|^2 - \frac{h^2}{12}\|\eta^1\|^2 + \frac{h^4}{144}\|\eta_x\|^2 + \frac{h^2}{12}\langle R^1, \eta^1 \rangle + \langle R^1, e^1 \rangle, \tag{34}$$

and

$$\langle \eta_{\hat{x}}^1, e^1 \rangle = \frac{h^2}{12}\langle \eta_{\hat{x}}^1, R^1 \rangle - \langle R^1, e_{\hat{x}}^1 \rangle. \tag{35}$$

Thus, from Eqs. (33)-(35), we get

$$\begin{aligned} &\|e^1\|^2 + \mu\|e_x^1\|^2 + \frac{\mu h^2}{12}\|\eta^1\|^2 - \frac{\mu h^4}{144}\|\eta_x^1\|^2 \\ = \;& \frac{\mu h^2}{12}\langle R^1, \eta^1 \rangle + \mu\langle R^1, e^1 \rangle + \frac{\tau h^4}{144}\langle \eta_{\hat{x}}^1, R^1 \rangle - \frac{\tau h^2}{12}\langle R^1, e_{\hat{x}}^1 \rangle + \tau\langle P^0, e^1 \rangle. \end{aligned} \tag{36}$$

Applying Lemma 2, we obtain

$$\|e^1\|^2 + \mu\|e_x^1\|^2 + \frac{\mu h^2}{18}\|\eta^1\|^2 \le \|e^1\|^2 + \mu\|e_x^1\|^2 + \frac{\mu h^2}{12}\|\eta^1\|^2 - \frac{\mu h^4}{144}\|\eta_x^1\|^2.$$

Using Young inequality, Eq. (36) can be rewritten as

$$
\begin{aligned}
&\|e^1\|^2+\mu\|e^1_x\|^2+\frac{\mu h^2}{18}\|\eta^1\|^2\\
\le\ &\frac{\mu h^2}{12}\|R^1\|\|\eta^1\|+\mu\|R^1\|\|e^1\|+\frac{\tau h^4}{144}\|\eta^1_x\|\|R^1\|+\frac{\tau h^2}{12}\|R^1\|\|e^1_x\|+\tau\|P^0\|\|e^1\|\\
\le\ &\frac{\mu h^2}{24}(\|R^1\|^2+\|\eta^1\|^2)+\frac{1}{4}\|e^1\|^2+4\mu^2\|R^1\|^2+\frac{h^4}{144}\left(\frac{\mu}{4}\|\eta^1_x\|^2+\frac{4\tau^2}{\mu}\|R^1\|^2\right)\\
&+\frac{\mu}{2}\|e^1_x\|^2+\frac{\tau^2h^4}{6\mu}\|R^1\|^2+\frac{1}{4}\|e^1\|^2+4\tau^2\|P^0\|^2\\
\le\ &\frac{1}{2}\left(\|e^1\|^2+\mu\|e^1_x\|^2\right)+\frac{7\mu h^2}{144}\|\eta^1\|^2+\left(\frac{\mu h^2}{24}+4\mu^2+\frac{7\tau^2h^4}{36\mu}\right)\|R^1\|^2+4\tau^2\|P^0\|^2,
\end{aligned}
$$

which yields

$$
\|e^1\|^2+\mu\|e^1_x\|^2+\frac{\mu h^2}{72}\|\eta^1\|^2\le\left(\frac{\mu h^2}{12}+8\mu^2+\frac{7\tau^2h^4}{18\mu}\right)\|R^1\|^2+8\tau^2\|P^0\|^2\le C(\tau^2+h^4)^2.
$$

Thus, we obtain

$$
\widetilde{c}(\|e^1\|^2+\|e^1_x\|^2+\|\eta^1\|^2)\le\|e^1\|^2+\mu\|e^1_x\|^2+\frac{\mu h^2}{72}\|\eta^1\|^2\le C(\tau^2+h^4)^2,
$$

where $\widetilde{c}=\min\left\{1,\mu,\frac{\mu h^2}{72}\right\}$. Hence, we have $\|e^1\|\le C(\tau^2+h^4),\|e^1_x\|\le C(\tau^2+h^4),\|\eta^1\|\le C(\tau^2+h^4)$, which concludes $\|e^1\|_\infty\le C(\tau^2+h^4)$ by Lemma 2.

Now, we suppose that

$$
\|e^k\|_\infty\le C(\tau^2+h^4),\quad \|\eta^k\|\le C(\tau^2+h^4),\quad 0\le k\le n.
$$

Taking the inner product of Eq. (28) with $2\bar{e}^n$, we have

$$
\begin{aligned}
&\|e^n\|^2_{\hat{t}}-\mu\langle\eta^n_{\hat{t}},2\bar{e}^n\rangle-\frac{h^2}{6}\langle\bar{\eta}^n_{\hat{t}},2\bar{e}^n\rangle+\delta\langle\Psi(U^n,\bar{U}^n)-\Psi(u^n,\bar{u}^n),2\bar{e}^n\rangle\\
&-\frac{\delta h^2}{2}\langle\Psi(V^n,\bar{U}^n)-\Psi(v^n,\bar{u}^n),2\bar{e}^n\rangle=\langle P^n,2\bar{e}^n\rangle.
\end{aligned}\tag{37}
$$

Applying Lemma 4, we have

$$
\langle\eta^n_{\hat{t}},2\bar{e}^n\rangle=-\|e^n_x\|^2_{\hat{t}}-\frac{h^2}{12}\|\eta^n\|^2_{\hat{t}}+\frac{h^4}{144}\|\eta^n_x\|^2_{\hat{t}}+\frac{h^2}{12}\langle\eta^n_{\hat{t}},2\bar{R}^n\rangle+\langle R^n_{\hat{t}},2\bar{e}^n\rangle,\tag{38}
$$

$$
\langle\bar{\eta}^n_{\hat{t}},2\bar{e}^n\rangle=\frac{h^2}{12}\langle\bar{\eta}^n_{\hat{x}},R^n\rangle-\langle R^n,\bar{e}^n_{\hat{x}}\rangle.\tag{39}
$$

Noticing

$$
\begin{aligned}
\Psi(V_j,U_j)-\Psi(v_j,u_j)&=\frac{1}{3}[V_j(U_j)_{\hat{x}}+(V_jU_j)_{\hat{x}}-v_j(u_j)_{\hat{x}}+(v_ju_j)_{\hat{x}}]\\
&=\frac{1}{3}[(v_j+\eta_j)(U_j)_{\hat{x}}+((v_j+\eta_j)U_j)_{\hat{x}}-v_j(u_j)_{\hat{x}}+(v_ju_j)_{\hat{x}}]\\
&=\frac{1}{3}[v_j(e_j)_{\hat{x}}+\eta_j(U_j)_{\hat{x}}+(v_je_j+\eta_jU_j)_{\hat{x}}]\\
&=\frac{1}{3}[(v_j+v_{j+1})(e_j)_{\hat{x}}+(v_j)_{\hat{x}}e_{j-1}+(\eta_j+\eta_{j+1})(U_j)_{\hat{x}}+(\eta_j)_{\hat{x}}U_{j-1}],
\end{aligned}
$$

using Lemmas 1, 2 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\langle\Psi(U^n,\bar U^n)-\Psi(u^n,\bar u^n),2\bar e^n\rangle &= \frac{2}{3}\langle e^n\bar U^n_{\hat x}+(e^n\bar U)_{\hat x},\bar e^n\rangle\\ &\le \frac{2}{3}(2\|\bar U^n_{\hat x}\|_\infty\|e^n\|+\|\bar U^n\|_\infty\|e^n_x\|)\|\bar e^n\|\\ &\le \frac{2\widehat c}{3}(2\|e^n\|+\|e^n_x\|)\|\bar e^n\|,\\ &\le \widehat c\left(\|e^n\|^2+\|e^n_x\|^2+\|\bar e^n\|^2\right),\end{aligned}\tag{40}$$

and

$$\begin{aligned}\langle\Psi(V^n,\bar U^n)-\Psi(v^n,\bar u^n),2\bar e^n\rangle &= \frac{2}{3}\langle \eta^n\bar U^n_{\hat x}+(\eta^n\bar U)_{\hat x},\bar e^n\rangle\\ &\le \frac{2}{3}(2\|\bar U^n_{\hat x}\|_\infty\|\eta^n\|+\|\bar U^n\|_\infty\|\eta^n_x\|)\|\bar e^n\|\\ &\le \frac{2\widehat c}{3}(2\|\eta^n\|+\|\eta^n_x\|)\|\bar e^n\|,\\ &\le \widehat c\left(\|\eta^n\|^2+\|\eta^n_x\|^2+\|\bar e^n\|^2\right),\end{aligned}\tag{41}$$

$$\langle P^n,2\bar x^n\rangle=\langle P^n,e^{n+1}+e^{n-1}\rangle\le\|P^n\|^2+\frac12\left(\|e^{n+1}\|^2+\|e^{n-1}\|^2\right).\tag{42}$$

Adding Eqs. (38)-(42) into Eq. (37), we obtain

$$\begin{aligned}&\|e^n\|^2+\mu\|e^n_x\|^2_{\hat t}+\frac{\mu h^2}{12}\|\eta^n\|^2_{\hat t}-\frac{\mu h^4}{144}\|\eta^n_x\|^2_{\hat t}\\ \le\ &\frac{\mu h^2}{12}\langle\eta^n_{\hat t},2\bar R^n\rangle+\mu\langle R^n_{\hat t},2\bar e^n\rangle+\frac{h^4}{72}\langle\bar\eta^n_{\hat x},R^n\rangle-\frac{h^2}{6}\langle R^n,\bar e^n_{\hat x}\rangle\\ &+\delta\widehat c\left(\|\bar e^n\|^2+\|e^n\|^2+\|\bar e^n_{\hat x}\|^2\right)+\frac{\delta\widehat c h^2}{2}\left(\|\eta^n\|^2+\|\eta^n_x\|^2+\|\bar e^n\|^2\right)+\|P^n\|^2+\frac12\left(\|e^{n+1}\|^2+\|e^{n-1}\|^2\right)\\ \le\ &C\left(\|e^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2+\|e^{n+1}_x\|^2+\|e^n_x\|^2+\|e^{n-1}_x\|^2+\|\eta^{n+1}\|^2+\|\eta^n\|^2+\|\eta^{n-1}\|^2\right.\\ &\left.+\|R^n\|^2+\|R^n_{\hat t}\|^2+\|P^n\|^2\right).\end{aligned}\tag{43}$$

Let

$$\mathcal A^n=\|e^{n+1}\|^2+\|e^n\|^2+\mu\left(\|e^{n+1}_x\|^2+\|e^n_x\|^2\right)+\frac{\mu h^2}{12}\left(\|\eta^{n+1}\|^2+\|\eta^n\|^2\right)-\frac{\mu h^4}{144}\left(\|\eta^{n+1}_x\|^2+\|\eta^n_x\|^2\right).$$

According to Lemma 2, we have

$$\begin{aligned}&\bar c\left(\|e^{n+1}\|^2+\|e^n\|^2+\|e^{n+1}_x\|^2+\|e^n_x\|^2+\|\eta^{n+1}\|^2+\|\eta^n\|^2\right)\\ \le\ &\|e^{n+1}\|^2+\|e^n\|^2+\mu\left(\|e^{n+1}_x\|^2+\|e^n_x\|^2\right)+\frac{\mu h^2}{18}\left(\|\eta^{n+1}\|^2+\|\eta^n\|^2\right)\le\mathcal A^n.\end{aligned}$$

where $\bar c=\min\left\{1,\mu,\frac{\mu h^2}{18}\right\}$. Summing up Eq. (43) from 1 to n, we have

$$\mathcal A^n\le\mathcal A^0+C\tau\sum_{i=0}^{n+1}\left(\|e^i\|^2+\|e^i_x\|^2+\|\eta^i\|^2\right)+\tau\sum_{i=1}^{n}\left(\|P^i\|^2+\|R^i\|^2+\|R^i_{\hat t}\|^2\right).\tag{44}$$

Note that

$$\tau\sum_{i=1}^{n}\left(\|P^i\|^2+\|R^i\|^2+\|R^i_{\hat t}\|^2\right)\le n\tau\max_{1\le i\le n}\left(\|P^i\|^2+\|R^i\|^2+\|R^i_{\hat t}\|^2\right)\le T\cdot C(\tau^2+h^4)^2,$$

and

$$\mathcal{A}^0 = \|e^1\|^2 + \mu\|e_x^1\|^2 + \frac{\mu h^2}{12}\|\eta^1\|^2 - \frac{\mu h^4}{144}\|\eta_x^1\|^2 \leq \|e^1\|^2 + \mu\|e_x^1\|^2 + + \frac{\mu h^2}{9}\|\eta^1\|^2 \leq C(\tau^2 + h^4)^2.$$

Then we have

$$\begin{aligned}&(\bar{c} - C\tau)(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|\eta^{n+1}\|^2 + \|\eta^n\|^2)\\ \leq\ & C\tau \sum_{i=0}^{n-1}(\|e^i\|^2 + \|e_x^i\|^2 + \|\eta^i\|^2) + C(\tau^2 + h^4)^2.\end{aligned}$$

If τ and h are sufficiently small such that $\bar{c} - C\tau \geq 1/2$, we obtain

$$\begin{aligned}&\|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|\eta^{n+1}\|^2 + \|\eta^n\|^2\\ \leq\ & 2C\tau \sum_{i=0}^{n-1}(\|e^i\|^2 + \|e_x^i\|^2 + \|\eta^i\|^2) + 2C(\tau^2 + h^4)^2.\end{aligned}$$

According to the discrete Gronwall inequality, we get

$$\begin{aligned}&\|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|\eta^{n+1}\|^2 + \|\eta^n\|^2\\ \leq\ & 2C(\tau^2 + h^4)^2 \cdot e^{2CT} \leq C(\tau^2 + h^4)^2.\end{aligned}$$

Hence, we obtain $\|e^{n+1}\| \leq C(\tau^2 + h^4)$, $\|e_x^{n+1}\| \leq C(\tau^2 + h^4)$ and $\|\eta^{n+1}\| \leq C(\tau^2 + h^4)$, which concludes $\|e^{n+1}\|_\infty \leq C(\tau^2 + h^4)$ by Lemma 2. This completes the proof.

Theorem 5. *Suppose that $u_0(x) \in H_0^1[x_l, x_r], u(x,t) \in C_{x,t}^{5,3}([x_l, x_r] \times (0, T])$, then the solution u^n of the difference scheme (13)-(17) is stable with respect to the initial conditions in discrete norm $\|\cdot\|_\infty$.*

Proof. Assume that $\{w_j^n, z_j^n | 0 \leq j \leq J, 0 \leq n \leq N\}$ is the numerical solution of the following system

$$\begin{aligned}&(w_j^n)_{\hat{t}} - \mu(z_j^n)_{\hat{t}} + (\bar{w}_j^n)_{\hat{x}} - \frac{h^2}{6}(\bar{z}_j^n)_{\hat{x}} + \delta\Psi(w_j^n, \bar{w}_j^n) - \frac{\delta h^2}{2}\Psi(z_j^n, \bar{z}_j^n) = 0,\\ &\qquad 1 \leq j \leq J-1, \quad 1 \leq n \leq N-1,\end{aligned} \tag{45}$$

$$z_j^n = (w_j^n)_{x\bar{x}} - \frac{h^2}{12}(z_j^n)_{x\bar{x}}, \quad 1 \leq j \leq J-1, \quad 0 \leq n \leq N, \tag{46}$$

$$w_j^0 = u_0(x_j) + \varepsilon(x_j), \quad 0 \leq j \leq J, \tag{47}$$

$$w_0^n = w_J^n = 0, \quad z_0^n = z_J^n = 0, \quad 0 \leq n \leq N, \tag{48}$$

where the initial condition is chosen to be $u_0(x) + \varepsilon(x)$ and $\varepsilon(x)$ is a perturbation function. Setting $\xi_j^n = u_j^n - w_j^n$, $\eta_j^n = v_j^n - z_j^n$, and substituting Eqs. (45)-(48) from Eqs. (13)-(17), we obtain

$$\begin{aligned}&(\xi_j^n)_{\hat{t}} - \mu(\eta_j^n)_{\hat{t}} + (\bar{\xi}_j^n)_{\hat{x}} - \frac{h^2}{6}(\bar{\eta}_j^n)_{\hat{x}} + \delta\Psi(u_j^n, \bar{u}_j^n) - \delta\Psi(w_j^n, \bar{w}_j^n)\\ &\qquad -\frac{\delta h^2}{2}\Psi(v_j^n, \bar{v}_j^n) + \frac{\delta h^2}{2}\Psi(z_j^n, \bar{z}_j^n) = 0,\\ &\eta_j^n = (\xi_j^n)_{x\bar{x}} - \frac{h^2}{12}(\eta_j^n)_{x\bar{x}},\\ &\xi_j^0 = -\varepsilon(x_j),\\ &\xi_0^n = \xi_J^n = 0, \quad \eta_0^n = \eta_J^n = 0.\end{aligned}$$

Similar to the proof of Theorem 4, we can conclude that

$$\|\xi^n\|_\infty \leq C\|\varepsilon\|_\infty.$$

This indicates that ξ^n is controlled by the initial condition $\varepsilon(x)$, implying that the scheme (13)-(17) is stable. This completes the proof.

7. Numerical examples

In this section, numerical examples are presented to verify the correction of theoretical analysis. For convenience, we denote the errors and convergence orders as

$$Er_{\infty}^{n}(h,\tau) = \|U^n(h,\tau) - u^n(h,\tau)\|_{\infty} = \max_{1\le j\le J-1} |U_j^n - u_j^n|,$$

$$Er_2^n(h,\tau) = \|U^n(h,\tau) - u^n(h,\tau)\|_2 = \sqrt{h\sum_{j=1}^{J-1}(U_j^n - u_j^n)^2},$$

$$Erx_2^n(h,\tau) = \|U_x^n(h,\tau) - u_x^n(h,\tau)\|_2 = \sqrt{\frac{1}{h}\sum_{j=1}^{J-1}(e_{j+1}^n - e_j^n)^2},$$

$$Order1 = \log_2\left(\frac{Er_{\infty}^n(h,\tau)}{Er_{\infty}^n(h,\frac{\tau}{2})}\right), Order2 = \log_2\left(\frac{Er_{\infty}^n(h,\tau)}{Er_{\infty}^n(\frac{h}{2},\frac{\tau}{4})}\right), Order3 = \log_2\left(\frac{Er_2^n(h,\tau)}{Er_2^n(h,\frac{\tau}{2})}\right),$$

$$Order4 = \log_2\left(\frac{Er_2^n(h,\tau)}{Er_2^n(\frac{h}{2},\frac{\tau}{4})}\right), Order5 = \log_2\left(\frac{Erx_2^n(h,\tau)}{Erx_2^n(h,\frac{\tau}{2})}\right), Order6 = \log_2\left(\frac{Erx_2^n(h,\tau)}{Erx_2^n(\frac{h}{2},\frac{\tau}{4})}\right),$$

where $e_j^n = U_j^n - u_j^n$, U_j^n and u_j^n represent the exact solution and the numerical solution, respectively. Furthermore, $Order1$, $Order3$ and $Order5$ denote the temporal convergence orders and $Order2$, $Order4$ and $Order6$ denote the spatial convergence orders.

Example 1. *We consider the following initial condition*

$$u(x,0) = 3d\text{sech}^2[k(x-x_0)].$$

The initial-boundary problem (1)-(3) has the exact solution as

$$u(x,t) = 3d\text{sech}^2[k(x-x_0-vt)], \quad v = 1+\delta d, \quad k = \frac{1}{2}\sqrt{\frac{\delta d}{\mu(1+\delta d)}}.$$

In this case, we took $\delta = 1$, $\mu = 1$, $d = 1$, $x_0 = 0$. Numerical traveling solutions at different times and its profile with $h = 0.125$, $\tau = h^2$ were showed in Fig. 1. We see that numerical solutions agree with the exact solutions very well. The absolute error comparison at $T = 1$ with $h = 0.1$ and $\tau = h^2$ were showed in Fig. 2. The comparisons of errors and convergence orders at $T = 1$ with $\tau = \frac{h}{4}$ and $\tau = h^2$ were reported in Table 1, Table 2 and Table 3, respectively. It is clear that the present difference scheme has second-order in the temporal direction and fourth-order in the spatial direction in discrete L^2, L^{∞}, and H^1-norms in Tables 1-3. Furthermore, the different scheme (13)-(17) has much higher convergence order and smaller errors than the schemes (Berikelashvili [19]; Shao [20]) in Fig. 2 and Tables 1-2. The values of discrete mass and discrete energy with $h = 0.1$ and $\tau = 0.01$ were presented at different times in Table 4. The absolute errors of long-time discrete conservation at different times with $h = 0.1$, $\tau = h^2$, $T = 500$ were plotted in Fig. 3. It is easy to see from Table 4 and Fig. 3 that the present difference scheme preserves the discrete conservative properties very well, even for long-time simulations.

Example 2. *We consider the following initial condition [21]*

$$u(x,0) = \sum_{i=1}^{2} 3d_i\text{sech}^2(k_i(x-x_i)), \quad d_i = 4k_i^2/(1-4k_i^2).$$

For simulation computations, we chose $\delta = \mu = 1$, and took the parameters $k_1 = 0.4$, $k_2 = 0.3$, $x_1 = 15$, $x_2 = 35$, $h = 0.125$, $\tau = h^2$, $T = 30$ with the region $0 \le x \le 120$. The interactions of two solitary waves were showed at difference times in Fig. 4. From Fig. 4, a higher solitary wave with larger amplitude is on the left of the other lower solitary wave with smaller amplitude. The higher wave moves and overtakes the lower wave as time goes on. Both waves eventually return to their original shapes.

Example 3. *We investigate the collision of three solitary waves with the different amplitudes and moving speeds. Considering the following initial condition of RLW equation [13]*

$$u(x,0) = \sum_{i=1}^{3} A_i \mathrm{sech}^2(K_i(x - x_i)), \quad A_i = 3d_i, \quad K_i = \frac{1}{2}\sqrt{\frac{d_i}{1+d_i}}, \quad v_i = 1 + d_i.$$

In order to simulation the collision of three solitary waves, we chose $\delta = \mu = 1$, $T = 20$ and $-80 \le x \le 80$. Let $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $x_1 = -10$, $x_2 = -20$, $x_3 = -30$, $h = 0.5$, $\tau = 0.05$ and the speeds $v_1 = 2$, $v_2 = 3$, $v_3 = 4$. The collision of three numerical solitary waves with different speed was displayed in Fig. 5. The speed corresponding to the highest amplitude to the lowest amplitude is $v_3 = 4$, $v_2 = 3$, $v_1 = 2$, respectively. It is evident that the faster waves with higher amplitudes catch up with the slow wave. With the change of time, the fastest wave on the far left runs ahead of the two slower waves. Finally, all the waves regain the original shapes.

Example 4. *In this numerical example, we consider the following Maxwellian initial condition of the RLW equation with different values* μ

$$u(x,0) = \exp(-(x-7)^2), \quad x \in [0, 40].$$

To analyze the influence of the different values μ, we took $\mu = 0.04, 0.01, 0.004, 0.001$, $h = 0.1$, $\tau = h^2$ and $\delta = 1$. The numerical solution curves for different values μ were shown in Figs. 6-9. From Figs. 6-9, we can see that the number of generated soliton waves and its amplitude are highly dependent on the value of μ. More high-amplitude waves are generated as μ reduces. In addition, the results of the simulations of multi-wave collisions and Maxwellian initial calculated by our scheme are consistent with those in [13].

8. Conclusions

In this paper, a new three-point three-level linearized conservative compact difference scheme based on the reduction order method for the RLW equation is presented. The conservation laws of discrete mass and energy are given and proved. Boundedness and uniquely solvability of the our scheme are proved and convergence and stability of the scheme are proved by using the discrete energy method. The scheme has the accuracy of second-order in time and fourth-order in space. Some physical motions in numerical experiments such as sine wave, multi-waves collision and Maxwellian initial condition are simulated. The results show that the presented compact scheme is reliable for solving the RLW equation.

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References

[1] A. Soliman, M. Hussien, Collocation solution for RLW equation with septic spline, Applied Mathematics and Computation 161 (2) (2005) 623–636.

[2] J. Bona, P. J. Bryant, A mathematical model for long waves generated by wavemakers in non-linear dispersive systems, Mathematical Proceedings of the Cambridge Philosophical Society 73 (2) (1973) 391–405.

[3] D. H. Peregrine, Calculations of the development of an undular bore, Journal of Fluid Mechanics 25 (2) (1966) 321–330.

[4] L. Wahlbin, Error estimates for a Galerkin method for a class of model equations for long waves, Numerische Mathematik 23 (4) (1974) 289–303.

[5] İdris. Dağ, B. Saka, D. Irk, Galerkin method for the numerical solution of the RLW equation using quintic B-splines, Journal of Computational and Applied Mathematics 190 (1-2) (2006) 532–547.

[6] M. Alexander, J. L. Morris, Galerkin methods applied to some model equations for non-linear dispersive waves, Journal of Computational Physics 30 (3) (1979) 428–451.

[7] K. Djidjeli, W. Price, E. Twizell, Q. Cao, A linearized implicit pseudo-spectral method for some model equations: the regularized long wave equations, Communications in numerical methods in engineering 19 (11) (2003) 847–863.

[8] J. H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering 167 (1-2) (1998) 57–68.

[9] T. S. El-Danaf, K. Raslan, K. K. Ali, New numerical treatment for the generalized regularized long wave equation based on finite difference scheme, Internatial Journal of Soft Computing and Engineering 4 (2014) 16–24.

[10] L. Zhang, A finite difference scheme for generalized regularized long-wave equation, Applied Mathematics and Computation 168 (2) (2005) 962–972.

[11] T. Achouri, N. Khiari, K. Omrani, On the convergence of difference schemes for the Benjamin–Bona–Mahony (BBM) equation, Applied mathematics and computation 182 (2) (2006) 999–1005.

[12] K. Zheng, J. Hu, High-order conservative Crank-Nicolson scheme for regularized long wave equation, Advances in Difference Equations 2013 (1) (2013) 1–12.

[13] B. Wang, T. Sun, D. Liang, The conservative and fourth-order compact finite difference schemes for regularized long wave equation, Journal of Computational and Applied Mathematics 356 (2019) 98–117.

[14] R. Akbari, R. Mokhtari, A new compact finite difference method for solving the generalized long wave equation, Numerical Functional Analysis and Optimization 35 (2) (2014) 133–152.

[15] A. Ghiloufi, K. Omrani, New conservative difference schemes with fourth-order accuracy for some model equation for nonlinear dispersive waves, Numerical Methods for Partial Differential Equations 34 (2) (2018) 451–500.

[16] Z. Sun, Numerical Methods of Partial Differential Equations, 2nd, Science Press, Beijing.

[17] Q. Zhang, L. Liu, Convergence and Stability in Maximum Norms of Linearized Fourth-Order Conservative Compact Scheme for Benjamin–Bona–Mahony–Burgers Equation, Journal of Scientific Computing 87 (2) (2021) 1–31.

[18] Y. He, X. Wang, R. Zhong, A new liearized fourth-order conservative compact difference scheme for the SRLW equation, Advances in Computational Mathematics 48 (27) (2022) 1–34.

[19] G. Berikelashvili, M. Mirianashvili, A one-parameter family of difference schemes for the regularized long-wave equation, Georgian Mathematical Journal 18 (2011) 639–667.

[20] X. Shao, G. Xue, C. Li, A conservative weighted finite difference scheme for regularized long wave equation, Applied Mathematics and Computation 219 (17) (2013) 9202–9209.

[21] K. Raslan, A computational method for the regularized long wave (RLW) equation, Applied Mathematics and Computation 167 (2) (2005) 1101–1118.

Table 1: The comparison results of error and convergence order in temporal direction at $T = 1$ with $\tau = \frac{h}{4}$ and $x \in [-20, 40]$ for Example 1.

Scheme		Er^n_∞	$Order1$	Er^n_2	$Order3$
Present Scheme	$h = 0.2$	1.6554e-03	*	3.3415e-03	*
	$h = 0.1$	4.1466e-04	1.9971	8.3471e-04	2.0011
	$h = 0.05$	1.0382e-04	1.9979	2.0878e-04	1.9993
Berikelashvili [19]	$h = 0.2$	1.7385e-03	*	3.7159e-03	*
	$h = 0.1$	4.3569e-04	1.9964	9.2836e-04	2.0010
	$h = 0.05$	1.0915e-04	1.9970	2.3223e-04	1.9991
Shao [20]	$h = 0.2$	1.4624e-02	*	3.1609e-02	*
	$h = 0.1$	6.5797e-03	1.1523	1.5051e-02	1.0705
	$h = 0.05$	3.2302e-03	1.0264	7.4038e-03	1.0235

Table 2: The comparison results of error and convergence order in spatial direction at $T = 1$ with $\tau = h^2$ and $x \in [-20, 40]$ for Example 1.

Scheme		Er^n_∞	$Order2$	Er^n_2	$Order4$
Present Scheme	$h = 0.25$	2.5821e-03	*	5.2260e-03	*
	$h = 0.125$	1.6346e-04	3.9815	3.3003e-04	3.9850
	$h = 0.0625$	1.0256e-05	3.9944	2.1407e-05	3.9464
Berikelashvili [19]	$h = 0.25$	2.7126e-03	*	5.8122e-03	*
	$h = 0.125$	9.4270e-04	1.5248	1.9003e-03	1.6129
	$h = 0.0625$	2.3555e-04	2.0008	4.7541e-04	1.9990
Shao [20]	$h = 0.25$	1.9225e-02	*	4.0752e-02	*
	$h = 0.125$	4.7092e-03	2.0294	1.0043e-02	2.0206
	$h = 0.0625$	1.1697e-03	2.0094	2.5014e-03	2.0054

Table 3: The results of error and convergence order in H^1-norm at $T = 1$ with $x \in [-20, 40]$ for Example 1.

	Erx^n_2	$Order5$		Erx^n_2	$Order6$
$h = 4\tau = 0.25$	3.8105e-03	*	$h = \sqrt{\tau} = 0.25$	3.8105e-03	*
$h = 4\tau = 0.125$	9.5096e-04	2.0026	$h = \sqrt{\tau} = 0.125$	2.4363e-04	3.9672
$h = 4\tau = 0.0625$	2.3763e-04	2.0006	$h = \sqrt{\tau} = 0.0625$	1.7446e-05	3.8037

Table 4: Discrete mass and discrete energy at different times with $h = 0.1$, $\tau = h^2$ and $x \in [-30, 80]$ for Example 1.

T	Q^n	E^n
5	16.970619378024942	37.335238395938731
10	16.970619377287399	37.335238400438300
15	16.970619377490898	37.335238400367615
20	16.970619377172916	37.335238400348437

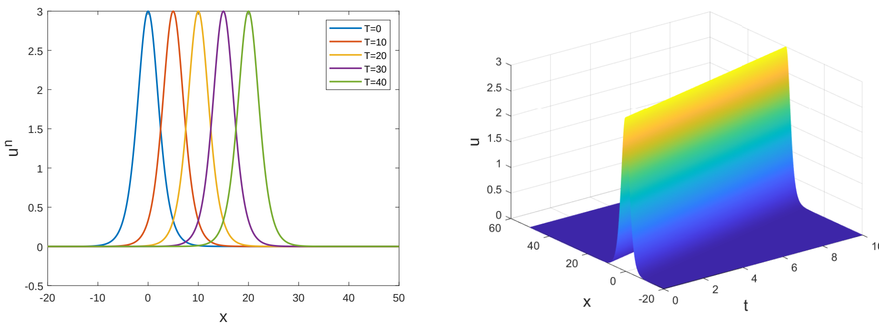

Figure 1: Numerical solutions at different times (left) and its profile (right) for Example 1.

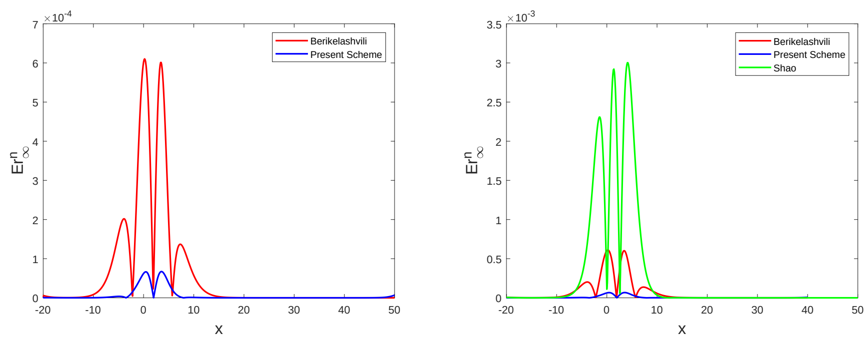

Figure 2: The comparison results of the absolute error at $T = 1$ with $h = 0.1$, $\tau = h^2$ for Example 1.

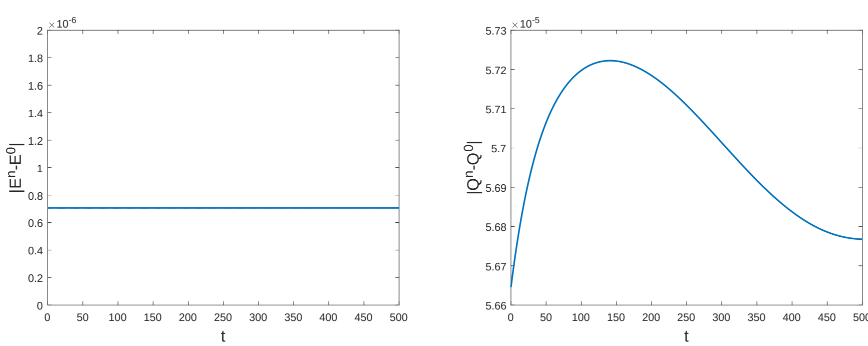

Figure 3: The absolute errors of long-time discrete conservation at different times with $h = 0.1$, $\tau = h^2$ and $x \in [-20, 50]$ for Example 1.

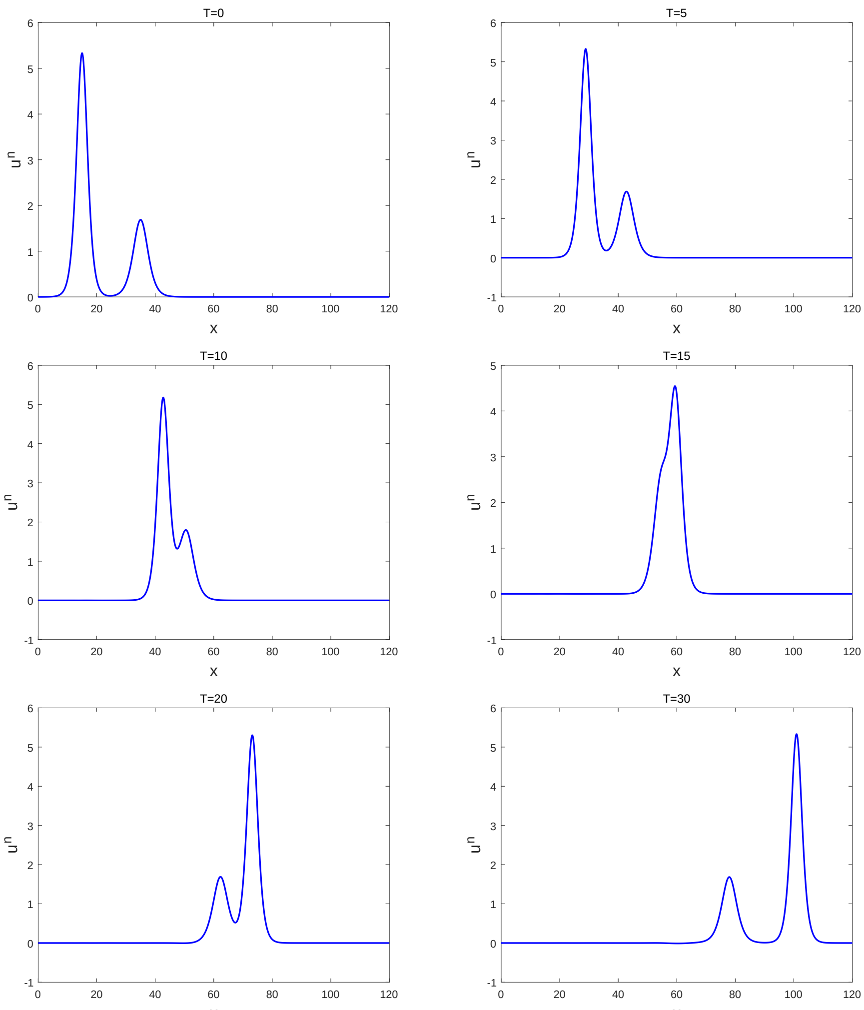

Figure 4: Catch-up collision of two solitons at different times for Example 2.

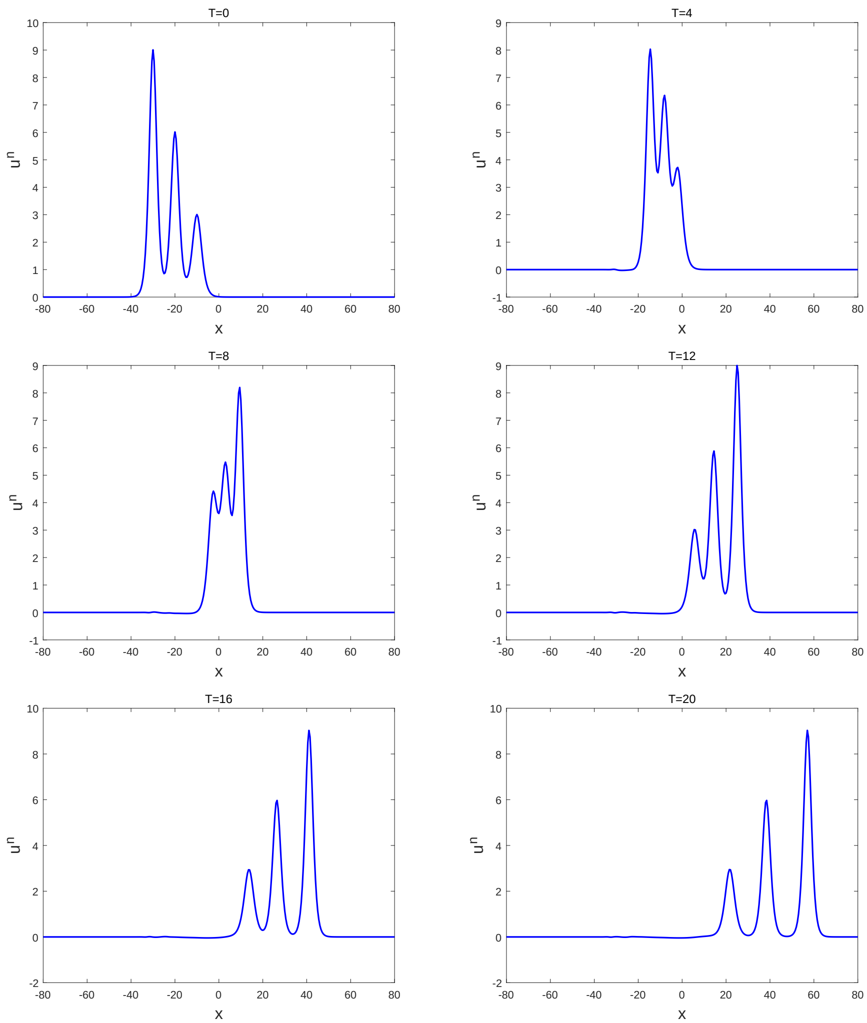

Figure 5: Collision of three solitons at different times for Example 3.

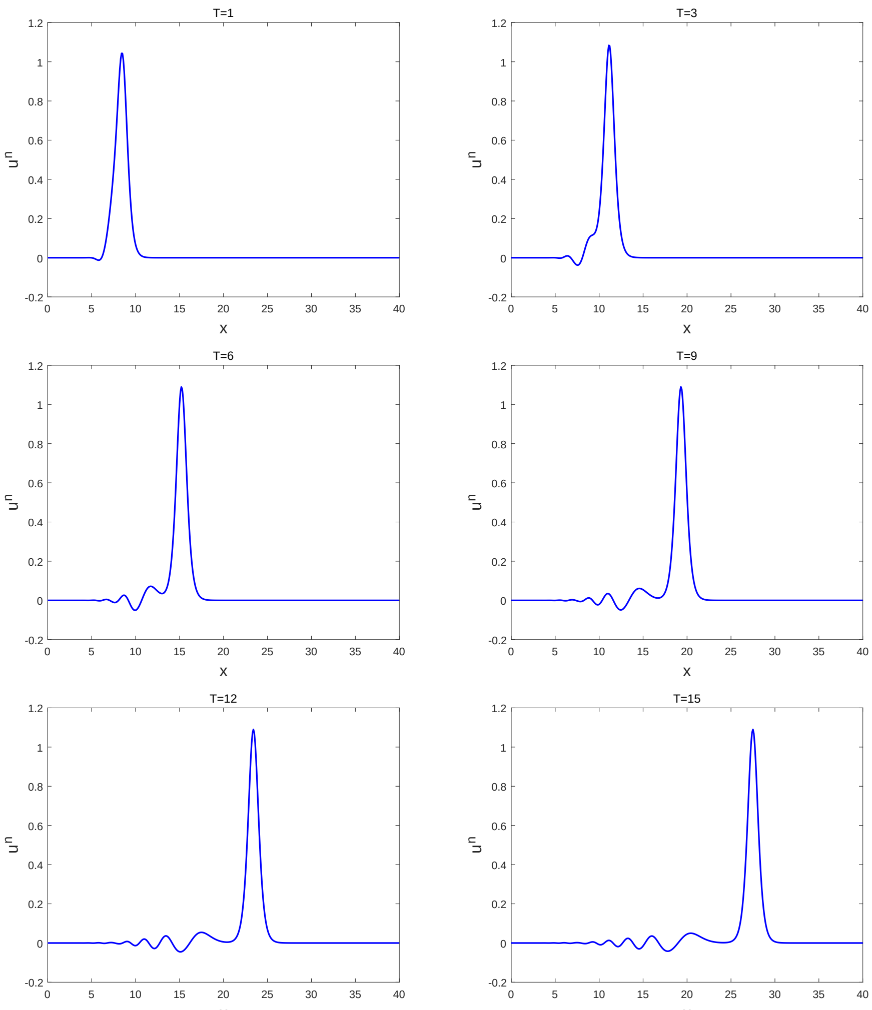

Figure 6: Numerical solution curves for the Maxwellian initial condition at different times with $\mu = 0.04$ for Example 4.

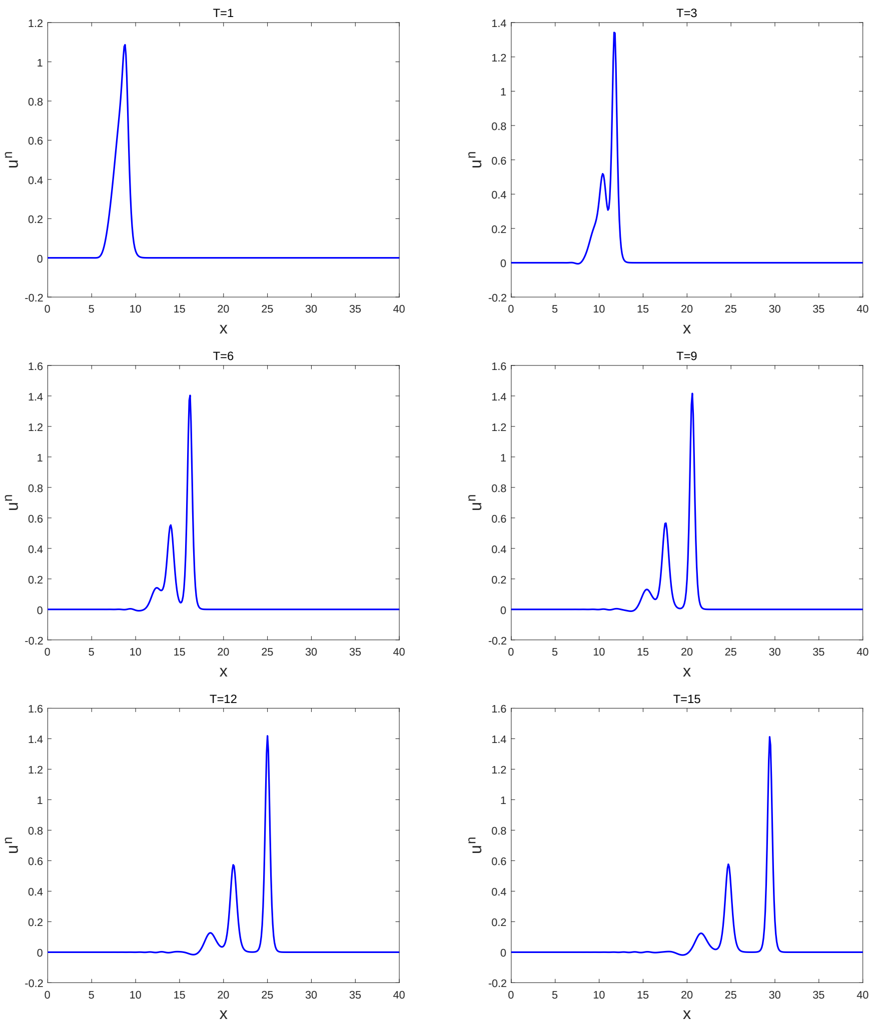

Figure 7: Numerical solution curves for the Maxwellian initial condition at different times with $\mu = 0.01$ for Example 4.

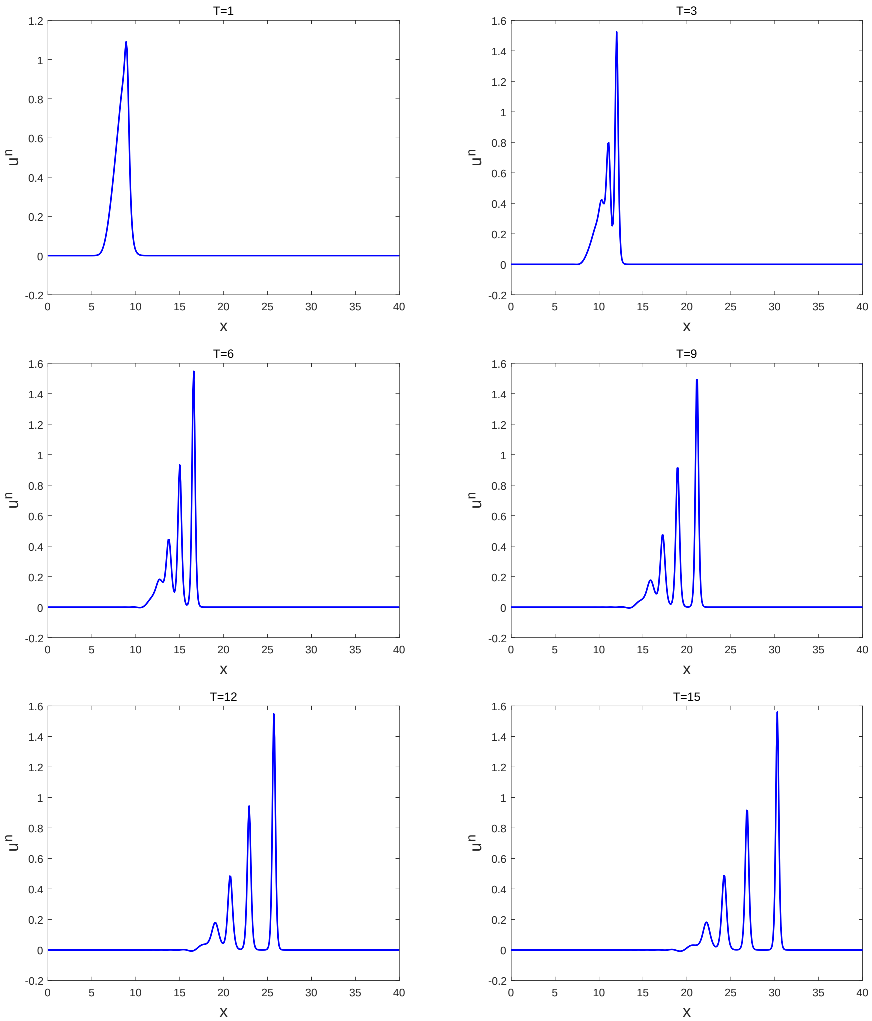

Figure 8: Numerical solution curves for the Maxwellian initial condition at different times with $\mu = 0.004$ for Example 4.

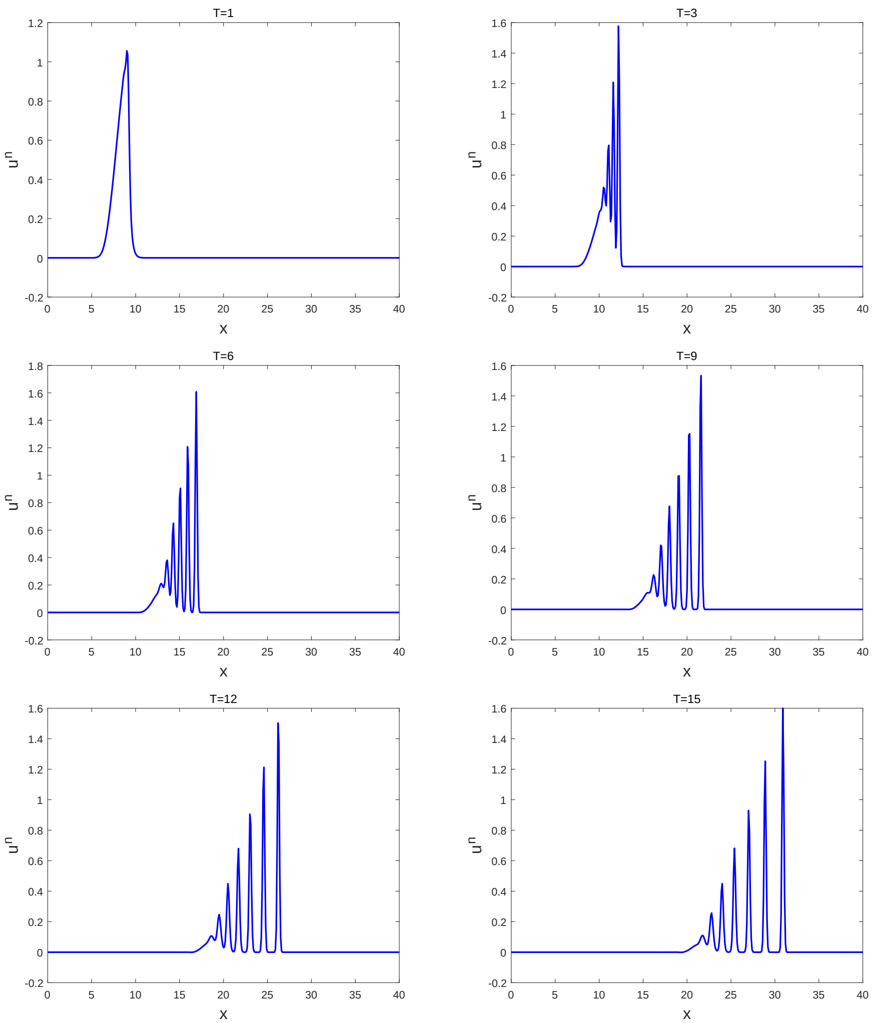

Figure 9: Numerical solution curves for the Maxwellian initial condition at different times with $\mu = 0.001$ for Example 4.