The time-domain scattering by the elastic shell in a two-layered unbounded structure

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June 2, 2022

Abstract

The subject of this research is the time-domain scattering problem for a three-dimensional layered elastic shell submerged in a two-layered fluid separated by an unbounded rough surface. The essence of the problem is to model the scattering interaction between the given elastic shell and some incident wave in a two-layered medium. Using the exact transparent boundary condition (TBC), we reformulate the unbounded scattering problem into an equivalent initial-boundary value problem. The well-posedness is proved for the problem by the Laplace transform and variational method in the s-domain. Moreover, we show that the reduced problem has a unique weak solution by the energy method. The priori estimates with explicit dependence on the time are derived for the acoustic pressure and the elastic displacement in the time domain.

Keywords: Elastic shell scattering, Unbounded rough surface, Wave equation, Navier equation, Transparent boundary condition, Variational method.

1 Introduction

The hollow elastic shell is a common structure for various underwater vehicles. The acoustic scattering of these targets is a concerned question in many fields for a long time. It can be categorized into the class of fluid-solid interaction scattering problems, and it is also the foundation for targets reconstruction in underwater acoustics. Recently, there has been an increasing interest in the mathematical study about the scattering of an underwater elastic shell in understanding their distinctive behaviors and wildly applications [1, 2, 31] and the references therein.

In the scattering problem of fluid-solid interaction, most often a linear elastic (viscoelastic) bounded obstacles that are surrounded by an inviscid compressible fluid are considered (see e.g. [3–7] and the references therein). In this case, the boundary integral equation method can be used to prove the well-posedness of the solution for smooth boundaries. The system of integral equations is solvable, using the theory of systems of multidimensional singular integral equations. For the time-harmonic acoustic-elastic interaction problems, there are a lot of available mathematical and numerical results see, e.g. [9–11]. The time-domain problems have received considerable attention due to their capability of capturing wide-band signals and modeling more

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general material and nonlinearity. However, comparing with the time-harmonic scattering problems, the time-domain problems are less studied due to the additional challenge of the temporal dependence. Some approaches are attempted to solve numerically the time-domain problems such as coupling of boundary element and finite element with different time quadratures [12–14]. The mathematical analysis of the time-dependent acoustic, elastic and electromagnetic scattering problems can be found in [15–17].

Among the various scattering phenomena, we are particularly interested in the case of layered elastic shells scatterer immersed in an unbounded structure and its associated scattering effects. The scattering behavior includes multiple scattering processes that are not only determined by the distribution and structure of the elastic materials in shells but also by the shape and properties of rough surfaces. In fact, the scattering properties of hollow, coated elastic shells, such as submarine, are more interesting and challenging. Mathematically, the wave propagation and interaction with elastic shells in layered media that described by coupling the Navier equations with the wave equations to govern the various physical phenomena. For example, Lame's parameters of the differential operator will affect the properties of wave propagation. A large number of investigations dealing with various aspects of the transient response of submerged and/or fluid-filled elastic shell structures with simple geometrical configuration (closed-form analytic solution for spherical or cylindrical shells) have been reported in literature, see, e.g. [18–20] and the references therein. Meanwhile, when an unbounded interface is considered, it would bring additional difficulties. In [21, 22], the integral equation method, and variation method are used to prove the existence of solution in elastic wave scattering by unbounded rough surfaces. In [23, 24], a homogeneous obstacle composite acoustic (electromagnetic) scattering and inverse scattering problems have been considered. The well-posedness is proved by using the integral equation method for the scattering problem, and the obstacle and the infinite rough surface can be uniquely determined by the measured wave fields is also obtained.

In this paper, we consider a time-domain acoustic wave incident onto an interface of the two-layered medium from above. The medium above the surface is supposed to be filled with homogeneous fluid with a constant mass density (air), whereas the region below is occupied by another homogeneous fluid (water) containing submerged the layered elastic shells with general shape. The acoustic wave passes through the interface from the air into the water, then it acts on the elastic shells, an elastic wave is incited inside the shells, while the acoustic wave is scattered in the fluid. This leads to the fluid-solid interaction on the shells between acoustic and elastic waves. We reduce the scattering problem into an initial boundary value problem by using the exact Transparent boundary condition (TBC). The well-posedness and stability are established for the reduced problem. To the best of our knowledge, our results are the first theoretical result about the analysis of composite scattering problems for general shape elastic shells in two-layered medium wgeneralith unbounded air-liquid interface.

The paper is organized as follows. In Section 2, we formulate the scattering problem for the three-dimensional layered elastic shell immersed in an unbounded structure. Introducing an exact transparent boundary condition and suitable interface conditions, we study an initial boundary value problem for the coupling of the acoustic and elastic wave equations. The related Sobolev spaces and useful lemmas are given. In Section 3, the existence and uniqueness of the reduced scattering problem are established based on the Laplace transform and variational method in both the frequency and time domains. Furthermore, a priori estimates with explicit dependence on the time are obtained for the acoustic pressure and the elastic displacement with detailed analysis by the energy method.

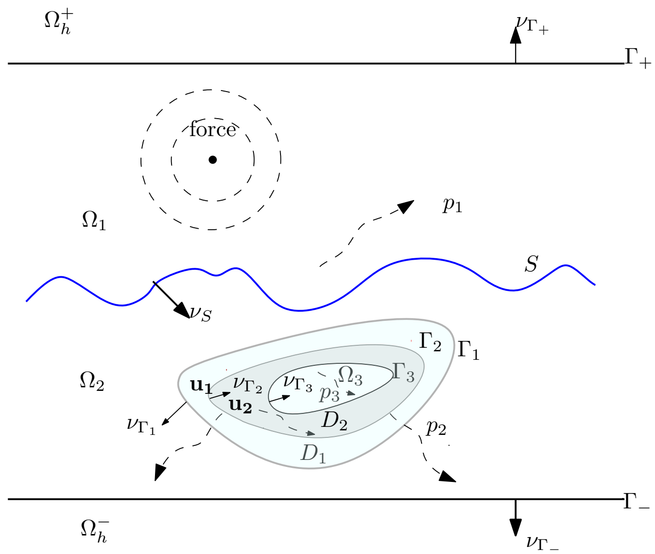

Figure 1: Geometry of the scattering problems

⟨pg⟩

2 Scattering Problems

on:formulation⟩ This section first establishes a mathematical model for the three-dimensional time-domain scattering by the elastic shell in a two-layered unbounded structure. The model problem is equivalently converted into an initial boundary value problem with TBC in a bounded domain. Some useful lemmas and notations for the model problem are also introduced.

2.1 Mathematical model

Figure 1 shows the geometry of the scattering problem. The unbounded rough surface $S = \{\boldsymbol{x} = (\boldsymbol{r}, x_3) \in \mathbb{R}^3 |\ x_3 = f(\boldsymbol{r}) \in W^{1,\infty}(\mathbb{R}^2)\}$ divides the whole space $\mathbb{R}^3$ into the upper half space $\Omega_f^+ = \{\boldsymbol{x} \in \mathbb{R}^3 |\ x_3 > f(\boldsymbol{r})\}$ and the lower half space $\Omega_f = \{\boldsymbol{x} \in \mathbb{R}^3 |\ x_3 < f(\boldsymbol{r})\}$. Let $\boldsymbol{\nu}_S$ denote the unit normal vector to the boundary S directed into the exterior of Ω_f^+. The elastic shell $D = D_1 \cup \Gamma_2 \cup D_2 \cup \Gamma_3 \cup \Omega_3$ is composed of the bounded domains D_j $(j = 1, 2)$ and Ω_3 with Lipschitz boundaries $\Gamma_1, \Gamma_2, \Gamma_3$, where D_j $(j = 1, 2)$ are filled with the homogeneous isotropic elastic material with Lamé moduli $\lambda_j + \frac{2}{3}\mu_j > 0$ and $\mu_j > 0$, Poisson's ratio $-1 < \nu_j < \frac{1}{2}$, and mass density $\rho_j > 0$. It embedded in Ω_f, i.e. $D \subset\subset \Omega_f$. Denote by $\Gamma_\pm = \{\boldsymbol{x} \in \mathbb{R}^3 |\ x_3 = h_\pm\}$ the plane surface above the unbounded rough surface and below the obstacle, respectively.

Define $\Omega_f^- = \Omega_f \setminus \overline{D}$, $\Omega_1 = \{\boldsymbol{x} \in \mathbb{R}^3 : f(\boldsymbol{r}) < x_3 < h_+\}$, $\Omega_h^+ = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 > h_+\}$ and $\Omega_2 = \{\boldsymbol{x} \in \mathbb{R}^3 : h_- < x_3 < f(\boldsymbol{r})\} \setminus \overline{D}$, $\Omega_h^- = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 < h_-\}$. Assume that the open spaces Ω_f^+, Ω_f^- and Ω_3 are filled with the compressible inviscid fluid. We will describe the scattering problem by the elastic shell in a two-layered unbounded structure as:

$$\begin{cases} (\Delta - \frac{1}{c_1^2}\partial_t^2 - \frac{\gamma_1}{c_1^2}\partial_t)p_1(\boldsymbol{x},t) = g(\boldsymbol{x},t) & \text{in} \quad \Omega_f^+,\ t>0, \\ (\Delta - \frac{1}{c_2^2}\partial_t^2 - \frac{\gamma_2}{c_2^2}\partial_t)p_2(\boldsymbol{x},t) = 0 & \text{in} \quad \Omega_f^-,\ t>0, \\ (\Delta - \frac{1}{c_3^2}\partial_t^2 - \frac{\gamma_3}{c_3^2}\partial_t)p_3(\boldsymbol{x},t) = 0 & \text{in} \quad \Omega_3,\ t>0, \\ (\mu_1\Delta + (\lambda_1+\mu_1)\nabla\nabla\cdot - \rho_1\partial_t^2)\boldsymbol{u}_1(\boldsymbol{x},t) = 0 & \text{in} \quad D_1,\ t>0, \\ (\mu_2\Delta + (\lambda_2+\mu_2)\nabla\nabla\cdot - \rho_2\partial_t^2)\boldsymbol{u}_2(\boldsymbol{x},t) = 0 & \text{in} \quad D_2,\ t>0, \\ p_1 = p_2, \qquad \partial_{\boldsymbol{\nu}_S}p_1 = \partial_{\boldsymbol{\nu}_S}p_2 & \text{on} \quad S,\ \ t>0, \\ \partial_{\boldsymbol{\nu}_{\Gamma_1}}p_2 = -\varrho_2\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t^2\boldsymbol{u}_1, \quad -p_2\ \boldsymbol{\nu}_{\Gamma_1} = \boldsymbol{\nu}_{\Gamma_1}\cdot\boldsymbol{\sigma}_1(\boldsymbol{u}_1) & \text{on} \quad \Gamma_1,\ t>0, \\ \boldsymbol{u}_1 = \boldsymbol{u}_2, \quad \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\boldsymbol{u}_1) = \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\boldsymbol{u}_2) & \text{on} \quad \Gamma_2,\ t>0, \\ \partial_{\boldsymbol{\nu}_{\Gamma_3}}p_3 = -\varrho_3\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t^2\boldsymbol{u}_2, \quad -p_3\ \boldsymbol{\nu}_{\Gamma_3} = \boldsymbol{\nu}_{\Gamma_3}\cdot\boldsymbol{\sigma}_2(\boldsymbol{u}_2) & \text{on} \quad \Gamma_3,\ t>0, \\ p_1|_{t=0} = \partial_t p_1|_{t=0} = 0 & \text{in} \quad \Omega_f^+, \\ p_2|_{t=0} = \partial_t p_2|_{t=0} = 0 & \text{in} \quad \Omega_f^-, \\ p_3|_{t=0} = \partial_t p_3|_{t=0} = 0 & \text{in} \quad \Omega_3, \\ \boldsymbol{u}_1|_{t=0} = \partial_t\boldsymbol{u}_1|_{t=0} = 0 & \text{in} \quad D_1, \\ \boldsymbol{u}_2|_{t=0} = \partial_t\boldsymbol{u}_2|_{t=0} = 0 & \text{in} \quad D_2, \end{cases} \tag{2.1}$$

?TMmodel?

where g is the force, which is assumed to have a compact support contained in $\Omega_1 \times (0,T)$ for any $T>0$. p_l $(l=1,2,3)$ is the pressure and $\boldsymbol{u}_j$ $(j=1,2)$ is the displacement vector in respective domain. ϱ_l $(l=1,2,3)>0$ is the constant density, c_l $(l=1,2,3)>0$ is the speed of sound and γ_l $(l=1,2,3)>0$ is the damping coefficient. Here $\boldsymbol{\sigma}_j(\boldsymbol{u}_j)$ $(j=1,2)$ is the symmetric stress tensor, defined by

$$\boldsymbol{\sigma}_j(\boldsymbol{u}_j) = \lambda_j \mathrm{tr}\,(\mathcal{E}(\boldsymbol{u}_j))\,\mathbf{I} + 2\mu_j\mathcal{E}(\boldsymbol{u}_j) = \lambda_j(\nabla\cdot\boldsymbol{u}_j)\mathbf{I} + \mu_j\big[\nabla\boldsymbol{u}_j + (\nabla\boldsymbol{u}_j)^\top\big], \tag{2.2}$$

?sig?

and satisfies

$$\nabla\cdot\boldsymbol{\sigma}_l(\boldsymbol{u}_j) = (\mu_j\Delta + (\lambda_j+\mu_j)\nabla\nabla\cdot)\boldsymbol{u}_j. \tag{2.3}$$

?nsig?

Noting that the unbounded structures, the usual SilverâĂŞMüler radiation condition is no longer valid. We will introduce an exact time-domain transparent boundary condition (TBC) to establish an initial boundary value problem to ensure uniqueness. For the function $p(\boldsymbol{x},t)$, define the Laplace transform by

$$\check{p}(\boldsymbol{x},s) = \mathscr{L}(p)(\boldsymbol{x},s) = \int_0^\infty e^{-st}p(\boldsymbol{x},t)\mathrm{d}t. \tag{2.4}$$

?LA?

where $s = s_1 + \mathrm{i}s_2$ with $s_1>0, s_2\in\mathbb{R}$. It follows from the integration by parts that

$$\int_0^t p(\boldsymbol{x},\tau)\mathrm{d}\tau = \mathscr{L}^{-1}(s^{-1}\check{p}(\boldsymbol{x},s)), \tag{2.5}$$

?LA-1?

where $\mathscr{L}^{-1}$ is the inverse Laplace transform. Therefore, from the inverse Laplace transform we have

$$p(\boldsymbol{x},t) = \mathscr{F}^{-1}(e^{s_1 t}\mathscr{L}(p)(\boldsymbol{x},s_1+\mathrm{i}s_2)), \tag{2.6}$$

?LA-2?

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2. For the Laplace transform, the Plancherel or Parseval identity (cf. [29, (2.46)]) is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\check{p}(\boldsymbol{x},s)\check{q}(\boldsymbol{x},s)\mathrm{d}s_2 = \int_0^{\infty} e^{-2s_1 t}p(\boldsymbol{x},t)q(\boldsymbol{x},t)\mathrm{d}t \qquad \forall\ s_1 > \sigma_0, \tag{2.7}$$
LA-3

where $\check{p} = \mathscr{L}(p), \check{q} = \mathscr{L}(q)$, and $\sigma_0 > 0$ is the abscissa of convergence for the Laplace transform of p and q. Since the force g is supported in Ω_1 and the medium is homogeneous in Ω_h^+, hence, the Helmholtz equation with respect to p_1 is reduced to

$$\left(\Delta - \frac{1}{c_1^2}\partial_t^2 - \frac{\gamma_1}{c_1^2}\partial_t\right)p_1(\boldsymbol{x},t) = 0 \qquad \text{in } \Omega_h^+. \tag{2.8}$$
p1Leq-1

It follows from (2.8) and those essential of Laplace operator $\mathscr{L}$ that

$$\Delta\check{p}_1(\boldsymbol{x},s) - \frac{s^2+\gamma_1 s}{c_1^2}\check{p}_1(\boldsymbol{x},s) = 0 \qquad \text{in } \Omega_h^+. \tag{2.9}$$
p1Leq

For any $s = s_1 + \mathrm{i}s_2$ with $s_1 > 0, s_2 \in \mathbb{R}$, taking the Fourier transform of (2.9) with respect to $\boldsymbol{r} = (x_1, x_2)$ yields

$$\begin{cases} \frac{\mathrm{d}^2\hat{\check{p}}_1(\boldsymbol{\xi},x_3,s)}{\mathrm{d}x_3^2} - \left(\frac{s^2+\gamma_1 s}{c_1^2} + |\boldsymbol{\xi}|^2\right)\hat{\check{p}}_1(\boldsymbol{\xi},x_3,s) = 0, & x_3 > h_+, \\ \hat{\check{p}}_1(\boldsymbol{\xi},x_3,s) = \hat{\check{p}}_1(\boldsymbol{\xi},h_+,s), & x_3 = h_+, \end{cases} \tag{2.10}$$
p1LF

where $\boldsymbol{\xi} = (\xi_1,\xi_2)^\top \in \mathbb{R}^2$ and $\hat{\check{p}}_1(\boldsymbol{\xi},x_3,s) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\check{p}_1(\boldsymbol{r},x_3,s)e^{-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{r}$. Then, solving (2.10) and using the bounded outgoing wave condition, we obtain

$$\hat{\check{p}}_1(\boldsymbol{\xi},x_3,s) = \hat{\check{p}}_1(\boldsymbol{\xi},h_+,s)e^{-\beta_+(\boldsymbol{\xi})(x_3-h_+)}, \qquad x_3 > h_+, \tag{2.11}$$
?p1OGC?

where $\beta_+^2(\boldsymbol{\xi}) = \frac{s^2+\gamma_1 s}{c_1^2} + |\boldsymbol{\xi}|^2$ with $\Re(\beta_+(\boldsymbol{\xi})) > 0$. Thus, we have

$$\check{p}_1(\boldsymbol{x},s) = \check{p}_1(\boldsymbol{r},x_3,s) = \int_{\mathbb{R}^2}\hat{\check{p}}_1(\boldsymbol{\xi},h_+,s)e^{-\beta_+(\boldsymbol{\xi})(x_3-h_+)}e^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{\xi}. \tag{2.12}$$
p1SL

After introducing the boundary operator $\mathscr{B}_+$ and taking the normal derivative of (2.12) on Γ^+, we can get

$$\partial_{\boldsymbol{\nu}_{\Gamma+}}\check{p}_1(\boldsymbol{x},s) = -\int_{\mathbb{R}^2}\beta_+(\boldsymbol{\xi})\hat{\check{p}}_1(\boldsymbol{\xi},h_+,s)e^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{\xi} = \mathscr{B}_+(\check{p}_1)(\boldsymbol{x},s) \qquad \text{on } \Gamma_+, \tag{2.13}$$
?p1SLnd?

which leads to a transparent boundary condition on Γ_+

$$\partial_{\boldsymbol{\nu}_{\Gamma_+}}\check{p}_1 = \mathscr{B}_+(\check{p}_1) \qquad \text{on } \Gamma_+, \tag{2.14}$$
?p1B+?

Similarly, it can be seen that

$$\partial_{\boldsymbol{\nu}_{\Gamma_-}}\check{p}_2(\boldsymbol{x},s) = \int_{\mathbb{R}^2}\beta_-(\boldsymbol{\xi})\hat{\check{p}}_2(\boldsymbol{\xi},h_-,s)e^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{\xi} = \mathscr{B}_-(\check{p}_2)(\boldsymbol{x},s) \qquad \text{on } \Gamma_-, \tag{2.15}$$
?p1SLnd1?

where $\beta_-^2(\boldsymbol{\xi}) = \frac{s^2+\gamma_2 s}{c_2^2} + |\boldsymbol{\xi}|^2$ with $\Re(\beta_-(\boldsymbol{\xi})) > 0$. Thus, we have established the following initial boundary value problem:

$$\begin{cases} (\Delta - \frac{1}{c_1^2}\partial_t^2 - \frac{\gamma_1}{c_1^2}\partial_t)p_1(\boldsymbol{x},t) = g(\boldsymbol{x},t) & \text{in} \quad \Omega_1,\ t>0, \\ (\Delta - \frac{1}{c_l^2}\partial_t^2 - \frac{\gamma_l}{c_l^2}\partial_t)p_l(\boldsymbol{x},t) = 0 & \text{in} \quad \Omega_l,\ t>0,\ l=2,3, \\ (\mu_j\Delta + (\lambda_j+\mu_j)\nabla\nabla\cdot - \rho_j\partial_t^2)\boldsymbol{u}_j(\boldsymbol{x},t) = 0 & \text{in} \quad D_j,\ t>0,\ j=1,2, \\ p_1 = p_2, \qquad \partial_{\boldsymbol{\nu}_S} p_1 = \partial_{\boldsymbol{\nu}_S} p_2 & \text{on} \quad S,\ \ t>0, \\ \partial_{\boldsymbol{\nu}_{\Gamma_1}} p_2 = -\varrho_2 \boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t^2\boldsymbol{u}_1, \quad -p_2\ \boldsymbol{\nu}_{\Gamma_1} = \boldsymbol{\nu}_{\Gamma_1}\cdot\boldsymbol{\sigma}_1(\boldsymbol{u}_1) & \text{on} \quad \Gamma_1,\ t>0, \\ \boldsymbol{u}_1 = \boldsymbol{u}_2, \quad \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\boldsymbol{u}_1) = \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\boldsymbol{u}_2) & \text{on} \quad \Gamma_2,\ t>0, \\ \partial_{\boldsymbol{\nu}_{\Gamma_3}} p_3 = -\varrho_3 \boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t^2\boldsymbol{u}_2, \quad -p_3\ \boldsymbol{\nu}_{\Gamma_3} = \boldsymbol{\nu}_{\Gamma_3}\cdot\boldsymbol{\sigma}_2(\boldsymbol{u}_2) & \text{on} \quad \Gamma_3,\ t>0, \\ p_l|_{t=0} = \partial_t p_l|_{t=0} = 0 & \text{in} \quad \Omega_l,\ l=1,2,3, \\ \boldsymbol{u}_j|_{t=0} = \partial_t \boldsymbol{u}_j|_{t=0} = 0 & \text{in} \quad D_j,\ j=1,2, \\ \partial_{\boldsymbol{\nu}_{\Gamma+}} p_1 = \mathscr{T}_+ p_1 & \text{on} \quad \Gamma_+,\ t>0, \\ \partial_{\boldsymbol{\nu}_{\Gamma-}} p_2 = \mathscr{T}_- p_2 & \text{on} \quad \Gamma_-,\ t>0, \end{cases} \tag{2.16}$$ TMB

where $\mathscr{T}_\pm$ is the time-domain TBC operator on $\Gamma_\pm$ which satisfies $\mathscr{T}_\pm = \mathscr{L}^{-1}\circ\mathscr{B}_\pm\circ\mathscr{L}$, and $\mathscr{B}_\pm$ is the Dirichlet-to-Neumann (DtN) operator on $\Gamma_\pm$ in the s-domain.

Taking the Laplace transform of (2.16) and using the initial conditions, we obtain the following problem in the s-domain:

$$\begin{cases} (\Delta - \frac{s^2+\gamma_1 s}{c_1^2})\check{p}_1 = \check{g} & \text{in} \quad \Omega_1, \\ (\Delta - \frac{s^2+\gamma_l s}{c_l^2})\check{p}_l = 0 & \text{in} \quad \Omega_1,\ l=2,3, \\ (\mu_j\Delta + (\lambda_j+\mu_j)\nabla\nabla\cdot - \rho_j s^2)\check{\boldsymbol{u}}_j = 0 & \text{in} \quad D_j,\ j=1,2, \\ \check{p}_1 = \check{p}_2, \qquad \partial_{\boldsymbol{\nu}_S}\check{p}_1 = \partial_{\boldsymbol{\nu}_S}\check{p}_2 & \text{on} \quad S, \\ \partial_{\boldsymbol{\nu}_{\Gamma_1}}\check{p}_2 = -\varrho_2 s^2\boldsymbol{\nu}_{\Gamma_1}\cdot\check{\boldsymbol{u}}_1, \quad -\check{p}_2\ \boldsymbol{\nu}_{\Gamma_1} = \boldsymbol{\nu}_{\Gamma_1}\cdot\boldsymbol{\sigma}_1(\check{\boldsymbol{u}}_1) & \text{on} \quad \Gamma_1, \\ \check{\boldsymbol{u}}_1 = \check{\boldsymbol{u}}_2, \quad \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\check{\boldsymbol{u}}_1) = \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\check{\boldsymbol{u}}_2) & \text{on} \quad \Gamma_2, \\ \partial_{\boldsymbol{\nu}_{\Gamma_3}}\check{p}_3 = -\varrho_3 s^2\boldsymbol{\nu}_{\Gamma_3}\cdot\check{\boldsymbol{u}}_2, \quad -\check{p}_3\ \boldsymbol{\nu}_{\Gamma_3} = \boldsymbol{\nu}_{\Gamma_3}\cdot\boldsymbol{\sigma}_2(\check{\boldsymbol{u}}_2) & \text{on} \quad \Gamma_3, \\ \partial_{\boldsymbol{\nu}_{\Gamma+}}\check{p}_1 = \mathscr{B}_+\check{p}_1 & \text{on} \quad \Gamma_+, \\ \partial_{\boldsymbol{\nu}_{\Gamma-}}\check{p}_2 = \mathscr{B}_-\check{p}_2 & \text{on} \quad \Gamma_-. \end{cases} \tag{2.17}$$ TDB

2.2 Sobolev spaces and some Lemmas

In this subsection, we introduce some notations, Sobolev spaces and useful Lemmas. For any $\alpha\in\mathbb{R}$, define

$$H^\alpha(\Gamma_\pm) := \left\{ q\in L^2(\Gamma_\pm) \middle| \int_{\mathbb{R}^2} (1+|\boldsymbol{\xi}|^2)^\alpha |\hat{q}(\boldsymbol{\xi},h_\pm)|^2 \mathrm{d}\boldsymbol{\xi} < \infty \right\}, \tag{2.18}$$?SSH1?

where $\hat{q}(\boldsymbol{\xi},h_\pm) = \frac{1}{2\pi}\int_{\mathbb{R}^2} q(\boldsymbol{r},h_\pm)e^{-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{r}$. The dual space of $H^\alpha(\Gamma_\pm)$ is the space $H^{-\alpha}(\Gamma_\pm)$ under the dual paring $\langle\cdot,\cdot\rangle_{\Gamma_\pm}$ defined by

$$\langle p,q\rangle_{\Gamma_\pm} = \int_{\Gamma_\pm} [p(\boldsymbol{r},h_\pm)\overline{q(\boldsymbol{r},h_\pm)}]\mathrm{d}\boldsymbol{r} = \int_{\mathbb{R}^2} [\hat{p}(\boldsymbol{\xi},h_\pm)\overline{\hat{q}(\boldsymbol{\xi},h_\pm)}]\mathrm{d}\boldsymbol{\xi}. \tag{2.19}$$?DP1?

Define $H^{\frac{1}{2}}(S) := \left\{q : S \to \mathbb{C} \middle| \; q(\boldsymbol{r}, f(\boldsymbol{r})) \in H^{\frac{1}{2}}(\mathbb{R}^2)\right\}$. For $l = 1,2,3$, denote by $H^{\frac{1}{2}}(\Gamma_l)$ and $H^1(\Omega_l)$ the standard Sobolev spaces. Let $H^{\frac{1}{2}}(\Gamma_l)^3$ $(l = 1,2,3)$ and $H^1(D_j)^3$ $(j = 1,2)$ be the Cartesian product spaces equipped with the corresponding 2-norms of $H^{\frac{1}{2}}(\Gamma_l)$ and $H^1(D_j)$, respectively. For any $\boldsymbol{v}_j = (v_{j1}, v_{j2}, v_{j3})^\top \in H^1(D_j)^3$, define

$$\|\nabla \boldsymbol{v}_j\|_{L^2(D_j)^{3\times 3}} = \left[\sum_{k=1}^{3} \int_{D_j} |\nabla v_{jk}(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}\right]^{\frac{1}{2}}, \qquad j = 1,2. \tag{2.20}$$?Nv?

Define $\tilde{\Omega}_1 = \{\boldsymbol{x} \in \mathbb{R}^3 : f_+ < x_3 < h_+\} \subset \Omega_1$, where $f_+ = \sup_{\boldsymbol{r}\in\mathbb{R}^2} f(\boldsymbol{r})$. Using Fourier coefficient, we can get an explicit characterization of the norm in $H^1(\tilde{\Omega}_1)$ as follows:

$$\|q\|^2_{H^1(\tilde{\Omega}_1)} = \int_{f_+}^{h_+} \int_{\mathbb{R}^2} \left[(1 + |\boldsymbol{\xi}|^2)|\hat{q}(\boldsymbol{\xi}, x_3)|^2 + |\partial_{x_3}\hat{q}(\boldsymbol{\xi}, x_3)|^2\right] \mathrm{d}\boldsymbol{\xi}\mathrm{d}x_3, \tag{2.21}$$?FH1?

obviously, it satisfies $\|q\|^2_{H^1(\tilde{\Omega}_1)} \le \|q\|^2_{H^1(\Omega_1)}$. Then, from Lemma 2.3 in [8] and Lemma 2.2 in [16] we readily obtain the following trace regularity results

⟨Le1⟩ **Lemma 2.1.** *There exist some positive constants γ_j $(j = 1,2,3,4)$ such that*

$$\|q_1\|_{H^{\frac{1}{2}}(\Gamma_+)} \le \gamma_1 \|q_1\|_{H^1(\Omega_1)}, \qquad \|q_2\|_{H^{\frac{1}{2}}(\Gamma_-)} \le \gamma_2 \|q_2\|_{H^1(\Omega_2)},$$
$$\|q_1\|_{H^{\frac{1}{2}}(S)} \le \gamma_3 \|q_1\|_{H^1(\Omega_1)}, \qquad \|q_2\|_{H^{\frac{1}{2}}(S)} \le \gamma_4 \|q_2\|_{H^1(\Omega_2)}.$$

With the aid of the trace theorem, we have

⟨Le2⟩ **Lemma 2.2.** *There exist some positive constants $\tilde{\gamma}_j$ $(j = 1,2,3)$ such that*

$$\|\boldsymbol{v}_1\|_{H^{\frac{1}{2}}(\Gamma_1)^3} \le \tilde{\gamma}_1 \|\boldsymbol{v}_1\|_{H^1(D_1)^3}, \quad \|\boldsymbol{v}_2\|_{H^{\frac{1}{2}}(\Gamma_3)^3} \le \tilde{\gamma}_2 \|\boldsymbol{v}_2\|_{H^1(D_2)^3}, \quad \|q_3\|_{H^{\frac{1}{2}}(\Gamma_3)} \le \tilde{\gamma}_3 \|q_3\|_{H^1(\Omega_3)}.$$

Proceeding as in the proof of the previous Lemma [16, Lemma 2.4, 2.5, 3.3], we can derive the following lemmas

⟨Le3⟩ **Lemma 2.3.** *Let $s = s_1 + \mathrm{i}s_2, s_1 \ge \sigma_0 > 0, s_2 \in \mathbb{R}$. The DtN operator $\mathscr{B}_\pm(s) : H^{\frac{1}{2}}(\Gamma_\pm) \to H^{-\frac{1}{2}}(\Gamma_\pm)$ is continuous, i.e.,*

$$\|\mathscr{B}_+\check{p}_1\|^2_{H^{-\frac{1}{2}}(\Gamma_+)} \le \gamma |s|^2 \|\check{p}_1\|^2_{H^{\frac{1}{2}}(\Gamma_+)}, \qquad \|\mathscr{B}_-\check{p}_2\|^2_{H^{-\frac{1}{2}}(\Gamma_-)} \le \gamma |s|^2 \|\check{p}_2\|^2_{H^{\frac{1}{2}}(\Gamma_-)},$$

where $\gamma = \max\{c_1^{-2}(1 + \gamma_1\sigma_0^{-1}), c_2^{-2}(1 + \gamma_2\sigma_0^{-1}), \sigma_0^{-2}\}$.

⟨Le4⟩ **Lemma 2.4.** *Let $s = s_1 + \mathrm{i}s_2, s_1 > 0, s_2 \in \mathbb{R}$. Then we have*

$$-\Re\langle s^{-1}\mathscr{B}_+\check{p}_1, \check{p}_1\rangle_{\Gamma_+} \ge 0, \qquad -\Re\langle s^{-1}\mathscr{B}_-\check{p}_2, \check{p}_2\rangle_{\Gamma_-} \ge 0.$$

⟨Le6⟩ **Lemma 2.5.** *Given $\eta_j \ge 0$ and $p_j \in H^1(\Omega_j)$ $(j = 1,2)$, we have*

$$\Re \int_{\Gamma_+} \int_0^{\eta_1} \left(\int_0^t \mathscr{T}_+ p_1(\cdot, \tau)\mathrm{d}\tau\right) \bar{p}_1(\cdot, t)\mathrm{d}t\mathrm{d}\boldsymbol{r} \le 0,$$
$$\Re \int_{\Gamma_-} \int_0^{\eta_2} \left(\int_0^t \mathscr{T}_- p_2(\cdot, \tau)\mathrm{d}\tau\right) \bar{p}_2(\cdot, t)\mathrm{d}t\mathrm{d}\boldsymbol{r} \le 0.$$

⟨Le5⟩ **Lemma 2.6.** *For $p_j \in H^1(\Omega_j)$ $(j=1,2)$, we have*

$$\Re\int_0^t\int_{\Gamma_+}(\mathscr{T}_+p_1)(\partial_t\bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau \le 0, \qquad \Re\int_0^t\int_{\Gamma_-}(\mathscr{T}_-p_2)(\partial_t\bar{p}_2)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau \le 0.$$

Proof. Let $\tilde{p}_1$ be the extension of p_1 with respect to τ in $\mathbb{R}$ such that $\tilde{p}_1 = 0$ outside the interval $[0,t]$. By the Parseval identity (2.7) and Lemma 2.4, we get

$$\begin{aligned}
&\Re\int_0^t e^{-2s_1\tau}\langle\mathscr{T}_+p_1, \partial_t p_1\rangle_{\Gamma_+}\mathrm{d}\tau = \Re\int_0^t e^{-2s_1\tau}\int_{\Gamma_+}(\mathscr{T}_+p_1)(\partial_t\bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&= \Re\int_0^\infty e^{-2s_1\tau}\int_{\Gamma_+}(\mathscr{T}_+\tilde{p}_1)(\partial_t\bar{\tilde{p}}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau = \Re\int_{\Gamma_+}\int_0^\infty e^{-2s_1\tau}(\mathscr{T}_+\tilde{p}_1)(\partial_t\bar{\tilde{p}}_1)\mathrm{d}\tau\mathrm{d}\boldsymbol{r}\\
&= \frac{1}{2\pi}\Re\int_{\Gamma_+}\int_{-\infty}^{\infty}\mathscr{L}(\mathscr{T}_+\tilde{p}_1)\mathscr{L}(\partial_t\bar{\tilde{p}}_1)\mathrm{d}s_2\mathrm{d}\boldsymbol{r} = \frac{1}{2\pi}\Re\int_{\Gamma_+}\int_{-\infty}^{\infty}\left[\mathscr{B}_+\circ\mathscr{L}(\tilde{p}_1)\right]\left[\mathscr{L}(\partial_t\bar{\tilde{p}}_1)\right]\mathrm{d}s_2\mathrm{d}\boldsymbol{r}\\
&= \frac{1}{2\pi}\Re\int_{\Gamma_+}\int_{-\infty}^{\infty}\mathscr{B}_+\check{\tilde{p}}_1(\bar{s}\bar{\check{\tilde{p}}}_1)\mathrm{d}s_2\mathrm{d}\boldsymbol{r} = \frac{1}{2\pi}\Re\int_{-\infty}^{\infty}\bar{s}\int_{\Gamma_+}\mathscr{B}_+\check{\tilde{p}}_1\bar{\check{\tilde{p}}}_1\mathrm{d}\boldsymbol{r}\mathrm{d}s_2\\
&= \frac{1}{2\pi}\Re\int_{-\infty}^{\infty}|s|^2\int_{\Gamma_+}s^{-1}\mathscr{B}_+\check{\tilde{p}}_1\bar{\check{\tilde{p}}}_1\mathrm{d}\boldsymbol{r}\mathrm{d}s_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}|s|^2\Re\langle s^{-1}\mathscr{B}_+\check{\tilde{p}}_1, \check{\tilde{p}}_1\rangle_{\Gamma_+}\mathrm{d}s_2 \le 0,
\end{aligned} \tag{2.22}$$ Le5-1

which yields after taking $s_1 \to 0$ in (2.22) that

$$\Re\int_0^t\int_{\Gamma_+}(\mathscr{T}_+p_1)(\partial_t\bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau = \Re\int_0^t\langle\mathscr{T}_+p_1, \partial_t p_1\rangle_{\Gamma_+}\mathrm{d}\tau \le 0. \tag{2.23}$$?Le5-2?

□

Similarly, we can prove that

⟨Le7⟩ **Lemma 2.7.** *For $p_j \in H^1(\Omega_j)$ $(j=1,2)$, we have*

$$\Re\int_0^t\int_{\Gamma_+}(\mathscr{T}_+\partial_t p_1)(\partial_t^2\bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau \le 0, \qquad \Re\int_0^t\int_{\Gamma_-}(\mathscr{T}_-\partial_t p_2)(\partial_t^2\bar{p}_2)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau \le 0.$$

3 The main results

ection:results⟩ Now we are ready to establish our main result on the scattering problem includes the well-posedness and stability in the s-domain and time domain.

3.1 Well-posedness in the s-domain

In this subsection, consider the reduced problem (2.17) in the s-domain. Multiplying $(\Delta - \frac{s^2+\gamma_1 s}{c_1^2})\check{p}_1 = \check{g}$ by the complex conjugate of a test function $\check{q}_1 \in H^1(\Omega_1)$. Taking the inner products, using the Lemma 2.2 in [8], the integration by parts and boundary conditions, which include the TBC condition $\partial_{\boldsymbol{\nu}_{\Gamma+}}\check{p}_1 = \mathscr{B}_+\check{p}_1$ on Γ_+, we arrive at

$$\int_{\Omega_1}\left(\nabla\check{p}_1\cdot\nabla\bar{\check{q}}_1 + \frac{s^2+\gamma_1 s}{c_1^2}\check{p}_1\bar{\check{q}}_1\right)\mathrm{d}\boldsymbol{x} - \langle\mathscr{B}_+\check{p}_1, \check{q}_1\rangle_{\Gamma_+} - \int_S\partial_{\boldsymbol{\nu}_S}\check{p}_1\bar{\check{q}}_1\mathrm{d}\boldsymbol{r} = -\int_{\Omega_1}\check{g}\bar{\check{q}}_1\mathrm{d}\boldsymbol{x}. \tag{3.1}$$ VP1

From continuous conditions on S in (2.17), we rewrite (3.1) in the form

$$\frac{1}{\varrho_2}\int_{\Omega_1}\left(\frac{1}{s}\nabla\check{p}_1\cdot\nabla\bar{\check{q}}_1+\frac{s+\gamma_1}{c_1^2}\check{p}_1\bar{\check{q}}_1\right)\mathrm{d}\boldsymbol{x}-\frac{1}{\varrho_2}\langle s^{-1}\mathscr{B}_+\check{p}_1,\check{q}_1\rangle_{\Gamma_+}-\frac{1}{\varrho_2 s}\int_S\partial_{\boldsymbol{\nu}_S}\check{p}_2\bar{\check{q}}_1\mathrm{d}\boldsymbol{r}$$
$$=-\frac{1}{\varrho_2 s}\int_{\Omega_1}\check{g}\bar{\check{q}}_1\mathrm{d}\boldsymbol{x},\qquad\forall\,\check{q}_1\in H^1(\Omega_1),\tag{3.2}$$ VP2

and similarly it can be shown that

$$\frac{1}{\varrho_2}\int_{\Omega_2}\left(\frac{1}{s}\nabla\check{p}_2\cdot\nabla\bar{\check{q}}_2+\frac{s+\gamma_2}{c_2^2}\check{p}_2\bar{\check{q}}_2\right)\mathrm{d}\boldsymbol{x}-\frac{1}{\varrho_2}\langle s^{-1}\mathscr{B}_-\check{p}_2,\check{q}_2\rangle_{\Gamma_-}+\frac{1}{\varrho_2 s}\int_S\partial_{\boldsymbol{\nu}_S}\check{p}_2\bar{\check{q}}_2\mathrm{d}\boldsymbol{r}$$
$$-s\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\check{\boldsymbol{u}}_1)\bar{\check{q}}_2\mathrm{d}s_{\boldsymbol{x}}=0,\qquad\forall\,\check{q}_2\in H^1(\Omega_2),\tag{3.3}$$?VP3?

$$\frac{1}{\varrho_3}\int_{\Omega_3}\left(\frac{1}{s}\nabla\check{p}_3\cdot\nabla\bar{\check{q}}_3+\frac{s+\gamma_3}{c_3^2}\check{p}_3\bar{\check{q}}_3\right)-s\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\check{\boldsymbol{u}}_2)\bar{\check{q}}_3\mathrm{d}s_{\boldsymbol{x}}=0,\qquad\forall\,\check{q}_3\in H^1(\Omega_3).\tag{3.4}$$ VP4

For $j=1,2$, multiplying $(\mu_j\Delta+(\lambda_j+\mu_j)\,\nabla\nabla\cdot-\rho_j s^2)\check{\boldsymbol{u}}_j=0$ by the complex conjugate of a test function $\check{\boldsymbol{v}}_j\in H^1(D_j)^3$. Taking the inner products, using the integration by parts and boundary conditions, we arrive at

$$\int_{D_1}\left[\mu_1(\nabla\check{\boldsymbol{u}}_1:\nabla\bar{\check{\boldsymbol{v}}}_1)+(\lambda_1+\mu_1)(\nabla\cdot\check{\boldsymbol{u}}_1)(\nabla\cdot\bar{\check{\boldsymbol{v}}}_1)+\rho_1 s^2(\check{\boldsymbol{u}}_1\cdot\bar{\check{\boldsymbol{v}}}_1)\right]\mathrm{d}\boldsymbol{x}$$
$$-\int_{\Gamma_1}\bar{\check{\boldsymbol{v}}}_1\cdot(\boldsymbol{\nu}_{\Gamma_1}\cdot\boldsymbol{\sigma}_1(\check{\boldsymbol{u}}_1))\mathrm{d}s_{\boldsymbol{x}}-\int_{\Gamma_2}\bar{\check{\boldsymbol{v}}}_1\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\check{\boldsymbol{u}}_1))\mathrm{d}s_{\boldsymbol{x}}=0,\quad\forall\,\check{\boldsymbol{v}}_1\in H^1(D_1)^3,\tag{3.5}$$ VP5

and

$$\int_{D_2}\left[\mu_2(\nabla\check{\boldsymbol{u}}_2:\nabla\bar{\check{\boldsymbol{v}}}_2)+(\lambda_2+\mu_2)(\nabla\cdot\check{\boldsymbol{u}}_2)(\nabla\cdot\bar{\check{\boldsymbol{v}}}_2)+\rho_2 s^2(\check{\boldsymbol{u}}_2\cdot\bar{\check{\boldsymbol{v}}}_2)\right]\mathrm{d}\boldsymbol{x}$$
$$+\int_{\Gamma_2}\bar{\boldsymbol{v}}_2\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\check{\boldsymbol{u}}_2))\mathrm{d}s_{\boldsymbol{x}}-\int_{\Gamma_3}\bar{\check{\boldsymbol{v}}}_2\cdot(\boldsymbol{\nu}_{\Gamma_3}\cdot\boldsymbol{\sigma}_2(\check{\boldsymbol{u}}_2))\mathrm{d}s_{\boldsymbol{x}}=0,\quad\forall\,\check{\boldsymbol{v}}_2\in H^1(D_2)^3.\tag{3.6}$$ VP6

By combining (3.5), (3.6) and the continuous conditions on Γ_j ($j=1,2,3$) in (2.17), we obtain

$$\int_{D_1}\bar{s}\left[\mu_1(\nabla\check{\boldsymbol{u}}_1:\nabla\bar{\check{\boldsymbol{v}}}_1)+(\lambda_1+\mu_1)(\nabla\cdot\check{\boldsymbol{u}}_1)(\nabla\cdot\bar{\check{\boldsymbol{v}}}_1)+\rho_1 s^2(\check{\boldsymbol{u}}_1\cdot\bar{\check{\boldsymbol{v}}}_1)\right]\mathrm{d}\boldsymbol{x}$$
$$+\bar{s}\int_{\Gamma_1}(\check{p}_2\boldsymbol{\nu}_{\Gamma_1})\cdot\bar{\check{\boldsymbol{v}}}_1\mathrm{d}s_{\boldsymbol{x}}-\bar{s}\int_{\Gamma_2}\bar{\check{\boldsymbol{v}}}_1\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\check{\boldsymbol{u}}_2))\mathrm{d}s_{\boldsymbol{x}}=0,\quad\forall\,\check{\boldsymbol{v}}_1\in H^1(D_1)^3,\tag{3.7}$$ VP7

and

$$\int_{D_2}\bar{s}\left[\mu_2(\nabla\check{\boldsymbol{u}}_2:\nabla\bar{\check{\boldsymbol{v}}}_2)+(\lambda_2+\mu_2)(\nabla\cdot\check{\boldsymbol{u}}_2)(\nabla\cdot\bar{\check{\boldsymbol{v}}}_2)+\rho_2 s^2(\check{\boldsymbol{u}}_2\cdot\bar{\check{\boldsymbol{v}}}_2)\right]\mathrm{d}\boldsymbol{x}$$
$$+\bar{s}\int_{\Gamma_2}\bar{\check{\boldsymbol{v}}}_2\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\check{\boldsymbol{u}}_2))\mathrm{d}s_{\boldsymbol{x}}+\bar{s}\int_{\Gamma_3}(\check{p}_3\boldsymbol{\nu}_{\Gamma_3})\cdot\bar{\check{\boldsymbol{v}}}_2\mathrm{d}s_{\boldsymbol{x}}=0,\quad\forall\,\check{\boldsymbol{v}}_2\in H^1(D_2)^3.\tag{3.8}$$ VP8

We assume that $\check{q}_1 = \check{q}_2$ on S and $\check{\boldsymbol{v}}_1 = \check{\boldsymbol{v}}_2$ on Γ_2. Then, adding these equations (3.2)-(3.4) and (3.7)-(3.8) to obtain an equivalent variational problem: to find $\check{p}_l \in H^1(\Omega_l)$ $(l = 1, 2, 3)$ and $\check{\boldsymbol{u}}_j \in H^1(D_j)^3$ $(j = 1, 2)$ such that

$$a(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2; \check{q}_1, \check{q}_2, \check{q}_3, \check{\boldsymbol{v}}_1, \check{\boldsymbol{v}}_2) = -\frac{1}{\varrho_2 s} \int_{\Omega_1} \check{g}\bar{\check{q}}_1 \mathrm{d}\boldsymbol{x}, \tag{3.9}$$ VP10

where the sesquilinear form

$$\begin{aligned}
&a(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2; \check{q}_1, \check{q}_2, \check{q}_3, \check{\boldsymbol{v}}_1, \check{\boldsymbol{v}}_2) \\
&= \frac{1}{\varrho_2} \int_{\Omega_1} \left(\frac{1}{s} \nabla \check{p}_1 \cdot \nabla \bar{\check{q}}_1 + \frac{s + \gamma_1}{c_1^2} \check{p}_1 \bar{\check{q}}_1 \right) \mathrm{d}\boldsymbol{x} - \frac{1}{\varrho_2} \langle s^{-1} \mathscr{B}_+ \check{p}_1, \check{q}_1 \rangle_{\Gamma_+} - \frac{1}{\varrho_2} \langle s^{-1} \mathscr{B}_- \check{p}_2, \check{q}_2 \rangle_{\Gamma_-} \\
&\quad + \frac{1}{\varrho_2} \int_{\Omega_2} \left(\frac{1}{s} \nabla \check{p}_2 \cdot \nabla \bar{\check{q}}_2 + \frac{s + \gamma_2}{c_2^2} \check{p}_2 \bar{\check{q}}_2 \right) \mathrm{d}\boldsymbol{x} + \frac{1}{\varrho_3} \int_{\Omega_3} \left(\frac{1}{s} \nabla \check{p}_3 \cdot \nabla \bar{\check{q}}_3 + \frac{s + \gamma_3}{c_3^2} \check{p}_3 \bar{\check{q}}_3 \right) \\
&\quad + \int_{D_1} \bar{s} \Big[\mu_1 (\nabla \check{\boldsymbol{u}}_1 : \nabla \bar{\check{\boldsymbol{v}}}_1) + (\lambda_1 + \mu_1)(\nabla \cdot \check{\boldsymbol{u}}_1)(\nabla \cdot \bar{\check{\boldsymbol{v}}}_1) + \rho_1 s^2 (\check{\boldsymbol{u}}_1 \cdot \bar{\check{\boldsymbol{v}}}_1) \Big] \mathrm{d}\boldsymbol{x} \\
&\quad + \int_{D_2} \bar{s} \Big[\mu_2 (\nabla \check{\boldsymbol{u}}_2 : \nabla \bar{\check{\boldsymbol{v}}}_2) + (\lambda_2 + \mu_2)(\nabla \cdot \check{\boldsymbol{u}}_2)(\nabla \cdot \bar{\check{\boldsymbol{v}}}_2) + \rho_2 s^2 (\check{\boldsymbol{u}}_2 \cdot \bar{\check{\boldsymbol{v}}}_2) \Big] \mathrm{d}\boldsymbol{x} \\
&\quad - \int_{\Gamma_1} [s(\boldsymbol{\nu}_{\Gamma_1} \cdot \check{\boldsymbol{u}}_1) \bar{\check{q}}_2 - \bar{s}(\check{p}_2 \boldsymbol{\nu}_{\Gamma_1}) \cdot \check{\boldsymbol{v}}_1] \mathrm{d}s_{\boldsymbol{x}} - \int_{\Gamma_3} [s(\boldsymbol{\nu}_{\Gamma_3} \cdot \check{\boldsymbol{u}}_2) \bar{\check{q}}_3 - \bar{s}(\check{p}_3 \boldsymbol{\nu}_{\Gamma_3}) \cdot \bar{\check{\boldsymbol{v}}}_2] \mathrm{d}s_{\boldsymbol{x}}.
\end{aligned} \tag{3.10}$$ VP11

⟨EUT⟩ **Theorem 3.1.** *The variational problem* (3.9) *has a unique weak solution* $(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\Omega_3) \times H^1(D_1)^3 \times H^1(D_2)^3$ *satisfying*

$$\sum_{l=1}^{3} \left[\|\nabla \check{p}_l\|^2_{L^2(\Omega_l)^3} + \|s\check{p}_l\|^2_{L^2(\Omega_l)} \right] \leq M_1 \frac{(1 + |s|)^2}{s_1^2} \|\check{g}\|^2_{L^2(\Omega_1)}, \tag{3.11}$$ EUT01

$$\sum_{j=1}^{2} \left[\|\nabla \check{\boldsymbol{u}}_j\|^2_{L^2(D_j)^{3\times 3}} + \|\nabla \cdot \check{\boldsymbol{u}}_j\|^2_{L^2(D_j)} + \|s\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)^3} \right] \leq M_2 \frac{(1 + |s|)^2}{|s|^2 s_1^2} \|\check{g}\|^2_{L^2(\Omega_1)}, \tag{3.12}$$ EUT02

where M_1 *and* M_2 *are positive constants.*

Proof. Using the CauchyâĂŞSchwarz inequality, Lemma 2.1, Lemma 2.2 and Lemma 2.3, show that there exists a positive constant M such that

$$
\begin{aligned}
&|a(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2; \check{q}_1, \check{q}_2, \check{q}_3, \check{\boldsymbol{v}}_1, \check{\boldsymbol{v}}_2)| \\
&\le \frac{1}{\varrho_2 |s|} \|\nabla \check{p}_1\|_{L^2(\Omega_1)^3} \|\nabla \check{q}_1\|_{L^2(\Omega_1)^3} + \frac{|s| + \gamma_1}{\varrho_2 c_1^2} \|\check{p}_1\|_{L^2(\Omega_1)} \|\check{q}_1\|_{L^2(\Omega_1)} \\
&\quad + \frac{1}{\varrho_2 |s|} \|\nabla \check{p}_2\|_{L^2(\Omega_2)^3} \|\nabla \check{q}_2\|_{L^2(\Omega_2)^3} + \frac{|s| + \gamma_2}{\varrho_2 c_2^2} \|\check{p}_2\|_{L^2(\Omega_2)} \|\check{q}_2\|_{L^2(\Omega_2)} \\
&\quad + \frac{1}{\varrho_3 |s|} \|\nabla \check{p}_3\|_{L^2(\Omega_3)^3} \|\nabla \check{q}_3\|_{L^2(\Omega_3)^3} + \frac{|s| + \gamma_3}{\varrho_3 c_3^2} \|\check{p}_3\|_{L^2(\Omega_3)} \|\check{q}_3\|_{L^2(\Omega_3)} \\
&\quad + |s| \Big[\mu_1 \|\nabla \check{\boldsymbol{u}}_1\|_{L^2(D_1)^{3\times 3}} \|\nabla \check{\boldsymbol{v}}_1\|_{L^2(D_1)^{3\times 3}} + (\lambda_1 + \mu_1) \|\nabla \cdot \check{\boldsymbol{u}}_1\|_{L^2(D_1)} \|\nabla \cdot \check{\boldsymbol{v}}_1\|_{L^2(D_1)} \Big] \\
&\quad + |s| \Big[\mu_2 \|\nabla \check{\boldsymbol{u}}_2\|_{L^2(D_2)^{3\times 3}} \|\nabla \boldsymbol{v}_2\|_{L^2(D_2)^{3\times 3}} + (\lambda_2 + \mu_2) \|\nabla \cdot \check{\boldsymbol{u}}_2\|^2_{L^2(D_2)} \|\nabla \cdot \check{\boldsymbol{v}}_2\|_{L^2(D_2)} \Big] \\
&\quad + |s|^3 \Big[\rho_1 \|\check{\boldsymbol{u}}_1\|_{L^2(D_1)^3} \|\check{\boldsymbol{v}}_1\|_{L^2(D_1)^3} + \rho_2 \|\check{\boldsymbol{u}}_2\|_{L^2(D_2)^3} \|\check{\boldsymbol{v}}_2\|_{L^2(D_2)^3} \Big] \\
&\quad + \frac{1}{\varrho_2 |s|} \Big[\|\mathscr{B}_+ \check{p}_1\|_{H^{-\frac{1}{2}}(\Gamma_+)} \|\check{q}_1\|_{H^{\frac{1}{2}}(\Gamma_+)} + \|\mathscr{B}_- \check{p}_2\|_{H^{-\frac{1}{2}}(\Gamma_-)} \|\check{q}_2\|_{H^{\frac{1}{2}}(\Gamma_-)} \Big] \\
&\quad + |s| \Big[\|\boldsymbol{\nu}_{\Gamma_1} \cdot \check{\boldsymbol{u}}_1\|_{L^2(\Gamma_1)} \|\check{q}_2\|_{L^2(\Gamma_1)} + \|\boldsymbol{\nu}_{\Gamma_1} \cdot \check{\boldsymbol{v}}_1\|_{L^2(\Gamma_1)} \|\check{p}_2\|_{L^2(\Gamma_1)} \Big] \\
&\quad + |s| \Big[\|\boldsymbol{\nu}_{\Gamma_3} \cdot \check{\boldsymbol{u}}_2\|_{L^2(\Gamma_3)} \|\check{q}_3\|_{L^2(\Gamma_3)} + \|\boldsymbol{\nu}_{\Gamma_1} \cdot \boldsymbol{v}_2\|_{L^2(\Gamma_3)} \|\check{p}_3\|_{L^2(\Gamma_3)} \Big] \\
&\le C_1 \|\check{p}_1\|_{H^1(\Omega_1)} \|\check{q}_1\|_{H^1(\Omega_1)} + C_2 \|\check{p}_2\|_{H^1(\Omega_2)} \|\check{q}_2\|_{H^1(\Omega_2)} + C_3 \|\check{p}_3\|_{H^1(\Omega_3)} \|\check{q}_3\|_{H^1(\Omega_3)} \\
&\quad + C_4 \|\check{\boldsymbol{u}}_1\|_{H^1(D_1)^3} \|\check{\boldsymbol{v}}_1\|_{H^1(D_1)^3} + C_5 \|\check{\boldsymbol{u}}_2\|_{H^1(D_2)^3} \|\check{\boldsymbol{v}}_2\|_{H^1(D_2)^3} \\
&\quad + C_6 \|\check{p}_1\|_{H^{\frac{1}{2}}(\Gamma_+)} \|\check{q}_1\|_{H^{\frac{1}{2}}(\Gamma_+)} + C_7 \|\check{p}_2\|_{H^{\frac{1}{2}}(\Gamma_-)} \|\check{q}_2\|_{H^{\frac{1}{2}}(\Gamma_-)} \\
&\quad + C_8 \|\check{\boldsymbol{u}}_1\|_{H^{\frac{1}{2}}(\Gamma_1)^3} \|\check{q}_2\|_{H^{\frac{1}{2}}(\Gamma_1)} + C_9 \|\check{\boldsymbol{v}}_1\|_{H^{\frac{1}{2}}(\Gamma_1)^3} \|\check{p}_2\|_{H^{\frac{1}{2}}(\Gamma_1)} \\
&\quad + C_{10} \|\check{\boldsymbol{u}}_2\|_{H^{\frac{1}{2}}(\Gamma_3)^3} \|\check{q}_3\|_{H^{\frac{1}{2}}(\Gamma_3)} + C_{11} \|\check{\boldsymbol{v}}_2\|_{H^{\frac{1}{2}}(\Gamma_3)^3} \|\check{p}_3\|_{H^{\frac{1}{2}}(\Gamma_3)} \\
&\le C_1 \|\check{p}_1\|_{H^1(\Omega_1)} \|\check{q}_1\|_{H^1(\Omega_1)} + C_2 \|\check{p}_2\|_{H^1(\Omega_2)} \|\check{q}_2\|_{H^1(\Omega_2)} + C_3 \|\check{p}_3\|_{H^1(\Omega_3)} \|\check{q}_3\|_{H^1(\Omega_3)} \\
&\quad + C_4 \|\check{\boldsymbol{u}}_1\|_{H^1(D_1)^3} \|\check{\boldsymbol{v}}_1\|_{H^1(D_1)^3} + C_5 \|\check{\boldsymbol{u}}_2\|_{H^1(D_2)^3} \|\check{\boldsymbol{v}}_2\|_{H^1(D_2)^3} \\
&\quad + \tilde{C}_6 \|\check{p}_1\|_{H^1(\Omega_1)} \|\check{q}_1\|_{H^1(\Omega_1)} + \tilde{C}_7 \|\check{p}_2\|_{H^1(\Omega_2)} \|q_2\|_{H^1(\Omega_2)} \\
&\quad + \tilde{C}_8 \|\check{\boldsymbol{u}}_1\|_{H^1(D_1)^3} \|\check{q}_2\|_{H^1(\Omega_2)} + \tilde{C}_9 \|\check{\boldsymbol{v}}_1\|_{H^1(D_1)^3} \|\check{p}_2\|_{H^1(\Omega_2)} \\
&\quad + \tilde{C}_{10} \|\check{\boldsymbol{u}}_2\|_{H^1(D_2)^3} \|\check{q}_3\|_{H^1(\Omega_3)} + \tilde{C}_{11} \|\check{\boldsymbol{v}}_2\|_{H^1(D_2)^3} \|\check{p}_3\|_{H^1(\Omega_3)} \\
&= (C_1 + \tilde{C}_6) \|\check{p}_1\|_{H^1(\Omega_1)} \|\check{q}_1\|_{H^1(\Omega_1)} + (C_2 + \tilde{C}_7) \|\check{p}_2\|_{H^1(\Omega_2)} \|\check{q}_2\|_{H^1(\Omega_2)} + C_3 \|\check{p}_3\|_{H^1(\Omega_3)} \|\check{q}_3\|_{H^1(\Omega_3)} \\
&\quad + C_4 \|\check{\boldsymbol{u}}_1\|_{H^1(D_1)^3} \|\check{\boldsymbol{v}}_1\|_{H^1(D_1)^3} + C_5 \|\check{\boldsymbol{u}}_2\|_{H^1(D_2)^3} \|\check{\boldsymbol{v}}_2\|_{H^1(D_2)^3} \\
&\quad + \tilde{C}_8 \|\check{\boldsymbol{u}}_1\|_{H^1(D_1)^3} \|\check{q}_2\|_{H^1(\Omega_2)} + \tilde{C}_9 \|\boldsymbol{v}_1\|_{H^1(D_1)^3} \|\check{p}_2\|_{H^1(\Omega_2)} \\
&\quad + \tilde{C}_{10} \|\check{\boldsymbol{u}}_2\|_{H^1(D_2)^3} \|\check{q}_3\|_{H^1(\Omega_3)} + \tilde{C}_{11} \|\boldsymbol{v}_2\|_{H^1(D_2)^3} \|\check{p}_3\|_{H^1(\Omega_3)} \\
&\le M \big(\|\check{p}_1\|_{H^1(\Omega_1)} \|\check{q}_1\|_{H^1(\Omega_1)} + \|\check{p}_2\|_{H^1(\Omega_2)} \|\check{q}_2\|_{H^1(\Omega_2)} + \|\check{p}_3\|_{H^1(\Omega_3)} \|\check{q}_3\|_{H^1(\Omega_3)} \\
&\qquad\quad + \|\check{\boldsymbol{u}}_1\|_{H^1(D_1)^3} \|\boldsymbol{v}_1\|_{H^1(D_1)^3} + \|\check{\boldsymbol{u}}_2\|_{H^1(D_2)^3} \|\check{\boldsymbol{v}}_2\|_{H^1(D_2)^3} \\
&\qquad\quad + \|\check{\boldsymbol{u}}_1\|_{H^1(D_1)^3} \|q_2\|_{H^1(\Omega_2)} + \|\check{\boldsymbol{v}}_1\|_{H^1(D_1)^3} \|\check{p}_2\|_{H^1(\Omega_2)} \\
&\qquad\quad + \|\check{\boldsymbol{u}}_2\|_{H^1(D_2)^3} \|\check{q}_3\|_{H^1(\Omega_3)} + \|\check{\boldsymbol{v}}_2\|_{H^1(D_2)^3} \|\check{p}_3\|_{H^1(\Omega_3)} \big),
\end{aligned}
\tag{3.13}
$$

?EUTO?

which implies that the sesquilinear form is bounded.

Letting $\check{q}_l = \check{p}_l$ $(l = 1, 2, 3)$ and $\check{\boldsymbol{v}}_j = \check{\boldsymbol{u}}_j$ $(j = 1, 2)$ in (3.10), then we get

$$
\begin{aligned}
& a(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2; \check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2) \\
&= \frac{1}{\varrho_2} \int_{\Omega_1} \left(\frac{1}{s} |\nabla \check{p}_1|^2 + \frac{s+\gamma_1}{c_1^2} |\check{p}_1|^2 \right) \mathrm{d}\boldsymbol{x} - \frac{1}{\varrho_2} \langle s^{-1} \mathscr{B}_+ \check{p}_1, \check{p}_1 \rangle_{\Gamma_+} - \frac{1}{\varrho_2} \langle s^{-1} \mathscr{B}_- \check{p}_2, \check{p}_2 \rangle_{\Gamma_-} \\
&\quad + \frac{1}{\varrho_2} \int_{\Omega_2} \left(\frac{1}{s} |\nabla \check{p}_2|^2 + \frac{s+\gamma_2}{c_2^2} |\check{p}_2|^2 \right) \mathrm{d}\boldsymbol{x} + \frac{1}{\varrho_3} \int_{\Omega_3} \left(\frac{1}{s} |\nabla \check{p}_3|^2 + \frac{s+\gamma_3}{c_3^2} |\check{p}_3|^2 \right) \\
&\quad + \int_{D_1} \bar{s} \Big[\mu_1 |\nabla \check{\boldsymbol{u}}_1|^2 + (\lambda_1 + \mu_1) |\nabla \cdot \check{\boldsymbol{u}}_1|^2 + \rho_1 s^2 |\check{\boldsymbol{u}}_1|^2 \Big] \mathrm{d}\boldsymbol{x} \\
&\quad + \int_{D_2} \bar{s} \Big[\mu_2 |\nabla \check{\boldsymbol{u}}_2|^2 + (\lambda_2 + \mu_2) |\nabla \cdot \check{\boldsymbol{u}}_2|^2 + \rho_2 s^2 |\check{\boldsymbol{u}}_2|^2 \Big] \mathrm{d}\boldsymbol{x} \\
&\quad - 2\mathrm{i} \int_{\Gamma_1} \Im [s \overline{\check{p}}_2 (\boldsymbol{\nu}_{\Gamma_1} \cdot \check{\boldsymbol{u}}_1)] \mathrm{d}s_{\boldsymbol{x}} - 2\mathrm{i} \int_{\Gamma_3} \Im [s \overline{\check{p}}_3 (\boldsymbol{\nu}_{\Gamma_3} \cdot \check{\boldsymbol{u}}_2)] \mathrm{d}s_{\boldsymbol{x}}.
\end{aligned} \tag{3.14}
$$
EUT1

It follows from lemma 2.4 and (3.14) that

$$
\begin{aligned}
& \Re [a(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2; \check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2)] \\
&\geq \frac{s_1}{\varrho_2 |s|^2} \|\nabla \check{p}_1\|^2_{L^2(\Omega_1)^3} + \frac{s_1}{\varrho_2 (c_1 |s|)^2} \|s \check{p}_1\|^2_{L^2(\Omega_1)} \\
&\quad + \frac{s_1}{\varrho_2 |s|^2} \|\nabla \check{p}_2\|^2_{L^2(\Omega_2)^3} + \frac{s_1}{\varrho_2 (c_2 |s|)^2} \|s \check{p}_2\|^2_{L^2(\Omega_2)} \\
&\quad + \frac{s_1}{\varrho_3 |s|^2} \|\nabla \check{p}_3\|^2_{L^2(\Omega_3)^3} + \frac{s_1}{\varrho_3 (c_3 |s|)^2} \|s \check{p}_3\|^2_{L^2(\Omega_3)} \\
&\quad + s_1 \big[\mu_1 \|\nabla \check{\boldsymbol{u}}_1\|^2_{L^2(D_1)^{3\times 3}} + (\lambda_1 + \mu_1) \|\nabla \cdot \check{\boldsymbol{u}}_1\|^2_{L^2(D_1)} + \rho_1 \|s \check{\boldsymbol{u}}_1\|^2_{L^2(D_1)^3} \big] \\
&\quad + s_1 \big[\mu_2 \|\nabla \check{\boldsymbol{u}}_2\|^2_{L^2(D_2)^{3\times 3}} + (\lambda_2 + \mu_2) \|\nabla \cdot \check{\boldsymbol{u}}_2\|^2_{L^2(D_2)} + \rho_2 \|s \check{\boldsymbol{u}}_2\|^2_{L^2(D_2)^3} \big] \\
&= \frac{s_1}{\varrho_2 |s|^2} \left[\|\nabla \check{p}_1\|^2_{L^2(\Omega_1)^3} + \frac{1}{c_1^2} \|s \check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla \check{p}_2\|^2_{L^2(\Omega_2)^3} + \frac{1}{c_2^2} \|s \check{p}_2\|^2_{L^2(\Omega_2)} \right] \\
&\quad + \frac{s_1}{\varrho_3 |s|^2} \left[\|\nabla \check{p}_3\|^2_{L^2(\Omega_3)^3} + \frac{1}{c_3^2} \|s \check{p}_3\|^2_{L^2(\Omega_3)} \right] \\
&\quad + s_1 \left[\mu_1 \|\nabla \check{\boldsymbol{u}}_1\|^2_{L^2(D_1)^{3\times 3}} + (\lambda_1 + \mu_1) \|\nabla \cdot \check{\boldsymbol{u}}_1\|^2_{L^2(D_1)} + \rho_1 \|s \check{\boldsymbol{u}}_1\|^2_{L^2(D_1)^3} \right] \\
&\quad + s_1 \left[\mu_2 \|\nabla \check{\boldsymbol{u}}_2\|^2_{L^2(D_2)^{3\times 3}} + (\lambda_2 + \mu_2) \|\nabla \cdot \check{\boldsymbol{u}}_2\|^2_{L^2(D_2)} + \rho_2 \|s \check{\boldsymbol{u}}_2\|^2_{L^2(D_2)^3} \right].
\end{aligned} \tag{3.15}
$$
EUT2

Then, combining the above inequalities with the help of the Lax-Milgram theorem, we find that the variational problem (3.9) has a unique weak solution $(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2)$, here $\check{p}_l \in H^1(\Omega_l)$ $(l = 1, 2, 3)$ and $\check{\boldsymbol{u}}_j \in H^1(D_j)^3$ $(j = 1, 2)$. Moreover, we have from (3.9) that

$$
\begin{aligned}
&|a(\check{p}_1,\check{p}_2,\check{p}_3,\check{\boldsymbol{u}}_1,\check{\boldsymbol{u}}_2;\check{p}_1,\check{p}_2,\check{p}_3,\check{\boldsymbol{u}}_1,\check{\boldsymbol{u}}_2)| = \left| -\frac{1}{\varrho_2 s}\int_{\Omega_1} \check{g}\bar{\check{p}}_1 \mathrm{d}\boldsymbol{x}\right| \\
&\le \frac{1}{\varrho_2|s|^2}\|\check{g}\|_{L^2(\Omega_1)}\|s\check{p}_1\|_{L^2(\Omega_1)} \le \frac{1}{\varrho_2|s|^2}\|\check{g}\|_{L^2(\Omega_1)}\|s\check{p}_1\|_{H^1(\Omega_1)} \\
&= \frac{1}{\varrho_2|s|^2}\left[\|\check{g}\|_{L^2(\Omega_1)}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|s\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)^{\frac{1}{2}}\right] \\
&\le \frac{1}{\varrho_2|s|^2}\left[(1+|s|)\|\check{g}\|_{L^2(\Omega_1)}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)^{\frac{1}{2}}\right]. \qquad (3.16)
\end{aligned}
$$

Case 1: $0 < c_1^2 < 1$**.** By Young's inequality, we have

$$
\begin{aligned}
&\frac{1}{\varrho_2|s|^2}\left[(1+|s|)\|\check{g}\|_{L^2(\Omega_1)}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)^{\frac{1}{2}}\right] \\
&\le \frac{(1+|s|)}{\varrho_2|s|^2}\left(\sqrt{\epsilon}\|\check{g}\|_{L^2(\Omega_1)}\right)\left(\frac{1}{\sqrt{\epsilon}}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)^{\frac{1}{2}}\right) \\
&\le \frac{(1+|s|)}{2\varrho_2|s|^2}\left(\epsilon\|\check{g}\|^2_{L^2(\Omega_1)} + \frac{1}{\epsilon}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)\right) \\
&= \frac{(1+|s|)}{2\varrho_2|s|^2}\left(\frac{2(1+|s|)}{s_1}\|\check{g}\|^2_{L^2(\Omega_1)} + \frac{s_1}{2(1+|s|)}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)\right) \\
&= \frac{(1+|s|)^2}{\varrho_2|s|^2 s_1}\|\check{g}\|^2_{L^2(\Omega_1)} + \frac{s_1}{4\varrho_2|s|^2}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big). \qquad (3.17)
\end{aligned}
$$

Case 2: $c_1^2 \ge 1$**.** By Young's inequality, we have

$$
\begin{aligned}
&\frac{1}{\varrho_2|s|^2}\left[(1+|s|)\|\check{g}\|_{L^2(\Omega_1)}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)^{\frac{1}{2}}\right] \\
&\le \frac{(1+|s|)}{\varrho_2|s|^2}\left(\sqrt{\epsilon}\|\check{g}\|_{L^2(\Omega_1)}\right)\left(\frac{1}{\sqrt{\epsilon}}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)^{\frac{1}{2}}\right) \\
&\le \frac{(1+|s|)}{2\varrho_2|s|^2}\left(\epsilon\|\check{g}\|^2_{L^2(\Omega_1)} + \frac{1}{\epsilon}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)\right) \\
&= \frac{(1+|s|)}{2\varrho_2|s|^2}\left(\frac{2(1+|s|)c_1^2}{s_1}\|\check{g}\|^2_{L^2(\Omega_1)} + \frac{s_1}{2(1+|s|)c_1^2}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big)\right) \\
&= \frac{(1+|s|)^2c_1^2}{\varrho_2|s|^2 s_1}\|\check{g}\|^2_{L^2(\Omega_1)} + \frac{s_1}{4\varrho_2|s|^2c_1^2}\big(\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3}\big). \qquad (3.18)
\end{aligned}
$$

Combing (3.15), (3.16) and (3.17) , for the case $0 < c_1^2 < 1$, we yield that

$$
\begin{aligned}
&\frac{s_1}{\varrho_2|s|^2}\left[\frac{3}{4}\|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3} + \left(\frac{1}{c_1^2}-\frac{1}{4}\right)\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_2\|^2_{L^2(\Omega_2)^3} + \frac{1}{c_2^2}\|s\check{p}_2\|^2_{L^2(\Omega_2)}\right] \\
&\quad + \frac{s_1}{\varrho_3|s|^2}\left[\|\nabla\check{p}_3\|^2_{L^2(\Omega_3)^3} + \frac{1}{c_3^2}\|s\check{p}_3\|^2_{L^2(\Omega_3)}\right] \\
&\quad + s_1\left[\mu_1\|\nabla\check{\boldsymbol{u}}_1\|^2_{L^2(D_1)^{3\times3}} + (\lambda_1+\mu_1)\|\nabla\cdot\check{\boldsymbol{u}}_1\|^2_{L^2(D_1)} + \rho_1\|s\check{\boldsymbol{u}}_1\|^2_{L^2(D_1)^3}\right] \\
&\quad + s_1\left[\mu_2\|\nabla\check{\boldsymbol{u}}_2\|^2_{L^2(D_2)^{3\times3}} + (\lambda_2+\mu_2)\|\nabla\cdot\check{\boldsymbol{u}}_2\|^2_{L^2(D_2)} + \rho_2\|s\check{\boldsymbol{u}}_2\|^2_{L^2(D_2)^3}\right] \\
&\leq \frac{(1+|s|)^2}{\varrho_2|s|^2 s_1}\|\check{g}\|^2_{L^2(\Omega_1)}.
\end{aligned} \tag{3.19}
$$
?EUT6?

Combing (3.15), (3.16) and (3.18), for the case $c_1^2 \geq 1$, we yield that

$$
\begin{aligned}
&\frac{s_1}{\varrho_2|s|^2}\left[\left(1-\frac{1}{4c_1^2}\right)\|\nabla\check{p}_1\|^2_{L^2(\Omega_1)^3} + \frac{3}{4c_1^2}\|s\check{p}_1\|^2_{L^2(\Omega_1)} + \|\nabla\check{p}_2\|^2_{L^2(\Omega_2)^3} + \frac{1}{c_2^2}\|s\check{p}_2\|^2_{L^2(\Omega_2)}\right] \\
&\quad + \frac{s_1}{\varrho_3|s|^2}\left[\|\nabla\check{p}_3\|^2_{L^2(\Omega_3)^3} + \frac{1}{c_3^2}\|s\check{p}_3\|^2_{L^2(\Omega_3)}\right] \\
&\quad + s_1\left[\mu_1\|\nabla\check{\boldsymbol{u}}_1\|^2_{L^2(D_1)^{3\times3}} + (\lambda_1+\mu_1)\|\nabla\cdot\check{\boldsymbol{u}}_1\|^2_{L^2(D_1)} + \rho_1\|s\check{\boldsymbol{u}}_1\|^2_{L^2(D_1)^3}\right] \\
&\quad + s_1\left[\mu_2\|\nabla\check{\boldsymbol{u}}_2\|^2_{L^2(D_2)^{3\times3}} + (\lambda_2+\mu_2)\|\nabla\cdot\check{\boldsymbol{u}}_2\|^2_{L^2(D_2)} + \rho_2\|s\check{\boldsymbol{u}}_2\|^2_{L^2(D_2)^3}\right] \\
&\leq \frac{(1+|s|)^2 c_1^2}{\varrho_2|s|^2 s_1}\|\check{g}\|^2_{L^2(\Omega_1)}.
\end{aligned} \tag{3.20}
$$
?EUt6?

These complete the proof. □

Proceeding as in the proof of the previous theorem 3.1, we can derive the following results.

?⟨SET⟩? **Theorem 3.2.** *The solution* $(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2)$ *of the variational problem* (3.9) *satisfying*

$$
\sum_{l=1}^{3}\left[\|s\nabla\check{p}_l\|^2_{L^2(\Omega_l)^3} + \|s^2\check{p}_l\|^2_{L^2(\Omega_l)}\right] \leq M_1\frac{(1+|s|)^2}{s_1^2}\|s\check{g}\|^2_{L^2(\Omega_1)}, \tag{3.21}
$$
?SET01?

and

$$
\sum_{j=1}^{2}\left[\|s\nabla\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)^{3\times3}} + \|s\nabla\cdot\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)} + \|s^2\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)^3}\right] \leq M_2\frac{(1+|s|)^2}{|s|^2 s_1^2}\|s\check{g}\|^2_{L^2(\Omega_1)}, \tag{3.22}
$$
?SET02?

or

$$
\sum_{j=1}^{2}\left[\|s^2\nabla\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)^{3\times3}} + \|s^2\nabla\cdot\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)} + \|s^3\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)^3}\right] \leq M_2\frac{(1+|s|)^2}{s_1^2}\|s\check{g}\|^2_{L^2(\Omega_1)}. \tag{3.23}
$$
?SET02-1?

3.2 Well-posedness in the time domain

We now consider the problem (2.16) in the time domain. Taking the partial derivative of the equations about $\boldsymbol{u}_j$ $(j = 1, 2)$ with respect to t in the problem (2.16), we obtain a new reduced problem:

$$
\begin{cases}
(\Delta - \frac{1}{c_1^2}\partial_t^2 - \frac{\gamma_1}{c_1^2}\partial_t)p_1(\boldsymbol{x},t) = g(\boldsymbol{x},t) & \text{in } \ \Omega_1,\\
(\Delta - \frac{1}{c_l^2}\partial_t^2 - \frac{\gamma_l}{c_l^2}\partial_t)p_l(\boldsymbol{x},t) = 0 & \text{in } \ \Omega_l, \ t>0, \ l=2,3,\\
(\mu_j\Delta + (\lambda_j+\mu_j)\,\nabla\nabla\cdot - \rho_j\partial_t^2)\partial_t\boldsymbol{u}_j(\boldsymbol{x},t) = 0 & \text{in } \ D_j, \ t>0 \ j=1,2,\\
p_1 = p_2, \qquad \partial_{\boldsymbol{\nu}_S}p_1 = \partial_{\boldsymbol{\nu}_S}p_2 & \text{on } \ S, \ \ t>0,\\
\partial_{\boldsymbol{\nu}_{\Gamma_1}}p_2 = -\varrho_2\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t^2\boldsymbol{u}_1, \quad -(\partial_t p_2)\ \boldsymbol{\nu}_{\Gamma_1} = \boldsymbol{\nu}_{\Gamma_1}\cdot\boldsymbol{\sigma}_1(\partial_t\boldsymbol{u}_1) & \text{on } \ \Gamma_1, \ t>0,\\
\partial_t\boldsymbol{u}_1 = \partial_t\boldsymbol{u}_2, \quad \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\partial_t\boldsymbol{u}_1) = \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\partial_t\boldsymbol{u}_2) & \text{on } \ \Gamma_2, \ t>0,\\
\partial_{\boldsymbol{\nu}_{\Gamma_3}}p_3 = -\varrho_3\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t^2\boldsymbol{u}_2, \quad -(\partial_t p_3)\ \boldsymbol{\nu}_{\Gamma_3} = \boldsymbol{\nu}_{\Gamma_3}\cdot\boldsymbol{\sigma}_2(\partial_t\boldsymbol{u}_2) & \text{on } \ \Gamma_3, \ t>0,\\
p_l|_{t=0} = \partial_t p_l|_{t=0} = 0 & \text{in } \ \Omega_l, \ l=1,2,3,\\
\boldsymbol{u}_j|_{t=0} = \partial_t\boldsymbol{u}_j|_{t=0} = 0 & \text{in } \ D_j, \ j=1,2,\\
\partial_t^2\boldsymbol{u}_j|_{t=0} = \rho_j^{-1}((\mu_j\Delta + (\lambda_j+\mu_j)\,\nabla\nabla\cdot)\boldsymbol{u}_j(\boldsymbol{x},t))|_{t=0} = 0 & \text{in } \ D_j, \ j=1,2,\\
\partial_{\boldsymbol{\nu}_{\Gamma+}}p_1 = \mathscr{T}_+p_1 & \text{on } \ \Gamma_+, \ t>0,\\
\partial_{\boldsymbol{\nu}_{\Gamma-}}p_2 = \mathscr{T}_-p_2 & \text{on } \ \Gamma_-, \ t>0.
\end{cases}
\tag{3.24}
$$
TEUT1

?⟨TEUT⟩? **Theorem 3.3.** *We assume that*

$$
g, \ \partial_t^2 g \in L^1(0,T;L^2(\Omega_1)), \qquad \partial_t g \in L^\infty(0,T;L^2(\Omega_1)). \tag{3.25}
$$
gsc-1

Then, the initial boundary value problem (2.16) *has a unique solution* $(p_1, p_2, p_3, \boldsymbol{u}_1, \boldsymbol{u}_2)$ *satisfying*

$$
\begin{aligned}
&p_l(\boldsymbol{x},t) \in L^2(0,T;H^1(\Omega_l)) \cap H^1(0,T;L^2(\Omega_l)), && l=1,2,3, \ T>0\\
&\boldsymbol{u}_j(\boldsymbol{x},t) \in L^2(0,T;H^1(D_j)^3) \cap H^1(0,T;L^2(D_j)^3), && j=1,2, \ T>0.
\end{aligned}
$$

Proof. From Theorem 3.1, noting that the solution $(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2)$ of the reduced problem (2.17) in the s-domain satisfy the estimates (3.11) and (3.12), respectively. It follows from [25] that $\check{p}_l$ $(l = 1, 2, 3)$ and $\check{\boldsymbol{u}}_j$ $(j = 1, 2)$ are holomorphic functions of s on the half-plane $\Re(s) > \sigma_0$. Hence, with the help of [16, Lemma 2.1], we see that the inverse Laplace transform of $(\check{p}_1, \check{p}_2, \check{p}_3, \check{\boldsymbol{u}}_1, \check{\boldsymbol{u}}_2)$ exists and is supported in $[0, \infty)$. Thus, we have proven that the initial boundary value problem (2.16) has a unique solution $(p_1, p_2, p_3, \boldsymbol{u}_1, \boldsymbol{u}_2)$.

Since for p_l $(l = 1, 2, 3)$ we have

$$
\begin{aligned}
&\int_0^T \sum_{l=1}^{3} \left(\|\nabla p_l\|^2_{L^2(\Omega_l)^3} + \|\partial_t p_l\|^2_{L^2(\Omega_l)} \right) \mathrm{d}t\\
&\le \int_0^T e^{-2s_1(t-T)} \sum_{l=1}^{3} \left(\|\nabla p_l\|^2_{L^2(\Omega_l)^3} + \|\partial_t p_l\|^2_{L^2(\Omega_l)} \right) \mathrm{d}t\\
&= e^{2s_1T} \int_0^T e^{-2s_1t} \sum_{l=1}^{3} \left(\|\nabla p_l\|^2_{L^2(\Omega_l)^3} + \|\partial_t p_l\|^2_{L^2(\Omega_l)} \right) \mathrm{d}t\\
&\le e^{2s_1T} \int_0^\infty e^{-2s_1t} \sum_{l=1}^{3} \left(\|\nabla p_l\|^2_{L^2(\Omega_l)^3} + \|\partial_t p_l\|^2_{L^2(\Omega_l)} \right) \mathrm{d}t,
\end{aligned}
\tag{3.26}
$$
TEUT01

using the Parseval identity (2.7), (3.11) and (3.25), shows that

$$
\begin{aligned}
&\int_0^\infty e^{-2s_1 t}\sum_{l=1}^{3}\left[\|\nabla p_l\|^2_{L^2(\Omega_1)^3}+\|\partial_t p_l\|^2_{L^2(\Omega_l)}\right]\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{l=1}^{3}\left[\|\nabla \check{p}_l\|^2_{L^2(\Omega_l)^3}+\|s\check{p}_l\|^2_{L^2(\Omega_l)}\right]\mathrm{d}s_2\\
&\leq \frac{M_1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{(1+|s|)^2}{s_1^2}\|\check{g}\|^2_{L^2(\Omega_1)}\right)\mathrm{d}s_2\leq \frac{M_1}{2\pi s_1^2}\int_{-\infty}^{\infty}\left(2(1+|s|^2)\|\check{g}\|^2_{L^2(\Omega_1)}\right)\mathrm{d}s_2\\
&=\frac{M_1}{\pi s_1^2}\int_{-\infty}^{\infty}\left(\|s\check{g}\|^2_{L^2(\Omega_1)}+\|\check{g}\|^2_{L^2(\Omega_1)}\right)\mathrm{d}s_2\\
&=\frac{M_1}{\pi s_1^2}\int_{-\infty}^{\infty}\left(\|\mathscr{L}(\partial_t g)\|^2_{L^2(\Omega_1)}+\|\mathscr{L}(g)\|^2_{L^2(\Omega_1)}\right)\mathrm{d}s_2\\
&=\frac{2M_1}{s_1^2}\int_0^\infty e^{-2s_1 t}\left(\|\partial_t g\|^2_{L^2(\Omega_1)}+\|g\|^2_{L^2(\Omega_1)}\right)\mathrm{d}t\\
&\leq C\int_0^T e^{-2s_1 t}\left(\|\partial_t g\|^2_{L^2(\Omega_1)}+\|g\|^2_{L^2(\Omega_1)}\right)\mathrm{d}t.
\end{aligned}
\tag{3.27}
$$
TEUT2

Then, combining (3.26) and (3.27), we find

$$
\int_0^T\sum_{l=1}^{3}\left(\|\nabla p_l\|^2_{L^2(\Omega_l)^3}+\|\partial_t p_l\|^2_{L^2(\Omega_l)}\right)\mathrm{d}t\leq C\int_0^T e^{-2s_1 t}\left(\|\partial_t g\|^2_{L^2(\Omega_1)}+\|g\|^2_{L^2(\Omega_1)}\right)\mathrm{d}t,\tag{3.28}
$$
TEUT3

where $C>0$ are some constants. Then, from (3.28), we obtain

$$
p_l(\boldsymbol{x},t)\in L^2(0,T;H^1(\Omega_l))\cap H^1(0,T;L^2(\Omega_l)),\qquad l=1,2,3.
$$

Similarly, for $\boldsymbol{u}_j$ $(j=1,2)$, we can obtain

$$
\begin{aligned}
&\int_0^T\sum_{j=1}^{2}\left(\|\partial_t\boldsymbol{u}_j\|^2_{L^2(D_j)^3}+\|\nabla\boldsymbol{u}_j\|^2_{L^2(D_j)^{3\times3}}+\|\nabla\cdot\boldsymbol{u}_j\|^2_{L^2(D_j)}\right)\mathrm{d}t\\
&\leq e^{2s_1T}\int_0^\infty e^{-2s_1 t}\sum_{j=1}^{2}\left(\|\partial_t\boldsymbol{u}_j\|^2_{L^2(D_j)^3}+\|\nabla\boldsymbol{u}_j\|^2_{L^2(D_j)^{3\times3}}+\|\nabla\cdot\boldsymbol{u}_j\|^2_{L^2(D_j)}\right)\mathrm{d}t.
\end{aligned}
\tag{3.29}
$$
TEUT5

and

$$\int_0^\infty e^{-2s_1 t}\sum_{j=1}^{2}\left[\|\partial_t \boldsymbol{u}_j\|^2_{L^2(D_j)^3}+\|\nabla \boldsymbol{u}_j\|^2_{L^2(D_j)^{3\times 3}}+\|\nabla\cdot \boldsymbol{u}_j\|^2_{L^2(D_j)}\right]\mathrm{d}t$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2}\left[\|s\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)^3}+\|\nabla\check{\boldsymbol{u}}_j\|^2_{L^2(D_1)^{3\times 3}}+\|\nabla\cdot\check{\boldsymbol{u}}_j\|^2_{L^2(D_j)}\right]\mathrm{d}s_2$$
$$\le \frac{M_2}{2\pi}\int_{-\infty}^{\infty}\left(\frac{(1+|s|)^2}{|s|^2 s_1^2}\|\check{g}\|^2_{L^2(\Omega_1)}\right)\mathrm{d}s_2 \le \frac{M_2}{2\pi s_1^4}\int_{-\infty}^{\infty}\left(2(1+|s|^2)\|\check{g}\|^2_{L^2(\Omega_1)}\right)\mathrm{d}s_2$$
$$=\frac{M_2}{\pi s_1^4}\int_{-\infty}^{\infty}\left(\|s\check{g}\|^2_{L^2(\Omega_1)}+\|\check{g}\|^2_{L^2(\Omega_1)}\right)\mathrm{d}s_2$$
$$=\frac{M_2}{2\pi s_1^4}\int_{-\infty}^{\infty}\left(\|\mathscr{L}(\partial_t g)\|^2_{L^2(\Omega_1)}+\|\mathscr{L}(g)\|^2_{L^2(\Omega_1)}\right)\mathrm{d}s_2$$
$$=\frac{2M_2}{s_1^4}\int_0^\infty e^{-2s_1 t}\left(\|\partial_t g\|^2_{L^2(\Omega_1)}+\|g\|^2_{L^2(\Omega_1)}\right)\mathrm{d}t$$
$$\le C\int_0^T e^{-2s_1 t}\left(\|\partial_t g\|^2_{L^2(\Omega_1)}+\|g\|^2_{L^2(\Omega_1)}\right)\mathrm{d}t. \tag{3.30}$$

TEUT5

Then, combining the last two inequalities (3.28) and (3.30), we get

$$\boldsymbol{u}_j(\boldsymbol{x},t)\in L^2(0,T;H^1(D_j)^3)\cap H^1(0,T;L^2(D_j)^3),\qquad j=1,2.$$

□

?⟨SE⟩? **Theorem 3.4.** *The solution $(p_1,p_2,p_3,\boldsymbol{u}_1,\boldsymbol{u}_2)$ of the initial boundary value problem* (2.16) *satisfies the following stability estimates:*

$$\max_{t\in[0,T]}\sum_{l=1}^{3}\left[\|\partial_t p_l\|^2_{L^2(\Omega_l)}+\|\nabla(\partial_t p_l)\|^2_{L^2(\Omega_l)^3}\right]$$
$$\le M_3\left[\|g\|^2_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|^2_{L^\infty(0,T;L^2(\Omega_1))}+\|\partial_t^2 g\|^2_{L^1(0,T;L^2(\Omega_1))}\right], \tag{3.31}$$

SE01

$$\max_{t\in[0,T]}\sum_{j=1}^{2}\left[\|\partial_t \boldsymbol{u}_j\|^2_{L^2(D_j)^3}+\|\nabla\cdot \boldsymbol{u}_j\|^2_{L^2(D_j)}+\|\nabla \boldsymbol{u}_j\|^2_{L^2(D_j)^{3\times 3}}\right]$$
$$\le M_4\left[\|g\|^2_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|^2_{L^\infty(0,T;L^2(\Omega_1))}+\|\partial_t^2 g\|^2_{L^1(0,T;L^2(\Omega_1))}\right], \tag{3.32}$$

SE02

where M_3 and M_4 are positiive constants.

Proof. First, for $\forall t\in[0,T]$, we define the energy function

$$E(t)=E_1(t)+E_2(t), \tag{3.33}$$

?SE1?

where

$$E_1(t)=\frac{1}{\varrho_2 c_1^2}\|\partial_t p_1\|^2_{L^2(\Omega_1)}+\frac{1}{\varrho_2}\|\nabla p_1\|^2_{L^2(\Omega_1)^3}+\frac{1}{\varrho_2 c_2^2}\|\partial_t p_2\|^2_{L^2(\Omega_2)}+\frac{1}{\varrho_2}\|\nabla p_2\|^2_{L^2(\Omega_2)^3}$$
$$+\frac{1}{\varrho_3 c_3^2}\|\partial_t p_3\|^2_{L^2(\Omega_3)}+\frac{1}{\varrho_3}\|\nabla p_3\|^2_{L^2(\Omega_3)^3}, \tag{3.34}$$

SE2

and

$$E_2(t) = \|(\rho_1)^{\frac{1}{2}}\partial_t^2 \boldsymbol{u}_1\|^2_{L^2(D_1)^3} + \|(\lambda_1+\mu_1)^{\frac{1}{2}}\nabla\cdot(\partial_t \boldsymbol{u}_1)\|^2_{L^2(D_1)} + \|(\mu_1)^{\frac{1}{2}}\nabla(\partial_t \boldsymbol{u}_1)\|^2_{L^2(D_1)^{3\times 3}}$$
$$+ \|(\rho_2)^{\frac{1}{2}}\partial_t^2 \boldsymbol{u}_2\|^2_{L^2(D_2)^3} + \|(\lambda_2+\mu_2)^{\frac{1}{2}}\nabla\cdot(\partial_t \boldsymbol{u}_2)\|^2_{L^2(D_2)} + \|(\mu_2)^{\frac{1}{2}}\nabla(\partial_t \boldsymbol{u}_2)\|^2_{L^2(D_2)^{3\times 3}}. \tag{3.35}$$
SE3

With initial conditions, it is easy to note that $E(0)=0$ and

$$E(t) = E(t) - E(0) = \int_0^t E'(\tau)\mathrm{d}\tau = \int_0^t \left[E_1'(\tau) + E_2'(\tau)\right]\mathrm{d}\tau. \tag{3.36}$$
SE4

It follows from (3.34) that

$$\begin{aligned}
&\int_0^t E_1'(\tau)\mathrm{d}\tau \\
&= \frac{1}{\varrho_2}\int_0^t\int_{\Omega_1}\left(\frac{1}{c_1^2}\partial_\tau|\partial_\tau p_1|^2 + \partial_\tau|\nabla p_1|^2\right)\mathrm{d}\tau + \frac{1}{\varrho_2}\int_0^t\int_{\Omega_2}\left(\frac{1}{c_2^2}\partial_\tau|\partial_\tau p_2|^2 + \partial_\tau|\nabla p_2|^2\right)\mathrm{d}\tau \\
&\quad + \frac{1}{\varrho_3}\int_0^t\int_{\Omega_3}\left(\frac{1}{c_3^2}\partial_\tau|\partial_\tau p_3|^2 + \partial_\tau|\nabla p_3|^2\right)\mathrm{d}\tau \\
&= \frac{1}{\varrho_2}\int_0^t\int_{\Omega_1}\left(\frac{1}{c_1^2}(\partial_\tau^2 p_1\partial_\tau\bar{p}_1 + \partial_\tau^2\bar{p}_1\partial_\tau p_1) + ((\partial_\tau\nabla p_1)\cdot(\nabla\bar{p}_1) + (\partial_\tau\nabla\bar{p}_1)\cdot(\nabla p_1))\right)\mathrm{d}\tau \\
&\quad + \frac{1}{\varrho_2}\int_0^t\int_{\Omega_2}\left(\frac{1}{c_2^2}(\partial_\tau^2 p_2\partial_\tau\bar{p}_2 + \partial_\tau^2\bar{p}_2\partial_\tau p_2) + ((\partial_\tau\nabla p_2)\cdot(\nabla\bar{p}_2) + (\partial_\tau\nabla\bar{p}_2)\cdot(\nabla p_2))\right)\mathrm{d}\tau \\
&\quad + \frac{1}{\varrho_3}\int_0^t\int_{\Omega_3}\left(\frac{1}{c_3^2}(\partial_\tau^2 p_3\partial_\tau\bar{p}_3 + \partial_\tau^2\bar{p}_3\partial_\tau p_3) + ((\partial_\tau\nabla p_3)\cdot(\nabla\bar{p}_3) + (\partial_\tau\nabla\bar{p}_3)\cdot(\nabla p_3))\right)\mathrm{d}\tau \\
&= \frac{2}{\varrho_2}\Re\int_0^t\int_{\Omega_1}\left(\frac{1}{c_1^2}(\partial_\tau^2 p_1)(\partial_\tau\bar{p}_1) + \nabla p_1\cdot\nabla(\partial_\tau\bar{p}_1)\right)\mathrm{d}\tau \\
&\quad + \frac{2}{\varrho_2}\Re\int_0^t\int_{\Omega_2}\left(\frac{1}{c_2^2}(\partial_\tau^2 p_2)(\partial_\tau\bar{p}_2) + \nabla p_2\cdot\nabla(\partial_\tau\bar{p}_2)\right)\mathrm{d}\tau \\
&\quad + \frac{2}{\varrho_3}\Re\int_0^t\int_{\Omega_3}\left(\frac{1}{c_3^2}(\partial_\tau^2 p_3)(\partial_\tau\bar{p}_3) + \nabla p_3\cdot\nabla(\partial_\tau\bar{p}_3)\right)\mathrm{d}\tau.
\end{aligned} \tag{3.37}$$
SE5

Multiplying $(\Delta - \frac{1}{c_1^2}\partial_t^2 - \frac{\gamma_1}{c_1^2}\partial_t)p_1(\boldsymbol{x},t) = g(\boldsymbol{x},t)$ by a test function $\partial_t\bar{p}_1$, where $p_1 \in L^2(0,T;H^1(\Omega_1))\cap H^1(0,T;L^2(\Omega_1))$. Taking the inner products, using the integration by parts and boundary conditions, which include the TBC condition $\partial_{\boldsymbol{\nu}_{\Gamma+}}p_1 = \mathscr{T}_+ p_1$, we arrive at

$$\begin{aligned}
&\frac{1}{\varrho_2}\int_0^t\int_{\Omega_1}\left[\nabla p_1\cdot\nabla(\partial_\tau\bar{p}_1) + \frac{1}{c_1^2}(\partial_\tau^2 p_1)(\partial_\tau\bar{p}_1) + \frac{\gamma_1}{c_1^2}|\partial_\tau p_1|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= \frac{1}{\varrho_2}\int_0^t\int_{\Gamma_+}(\mathscr{T}_+ p_1)(\partial_\tau\bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau + \frac{1}{\varrho_2}\int_0^t\int_S(\partial_{\boldsymbol{\nu}_S}p_1)(\partial_\tau\bar{p}_1)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau - \frac{1}{\varrho_2}\int_0^t\int_{\Omega_1} g(\partial_\tau\bar{p}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau.
\end{aligned} \tag{3.38}$$
?SE6?

Similarly, we can obtain

$$\begin{aligned}
&\frac{1}{\varrho_2}\int_0^t\int_{\Omega_2}\left[\nabla p_2\cdot\nabla(\partial_\tau\bar{p}_2)+\frac{1}{c_2^2}(\partial_\tau^2 p_2)(\partial_\tau\bar{p}_2)+\frac{\gamma_2}{c_2^2}|\partial_\tau p_2|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau\\
&=\frac{1}{\varrho_2}\int_0^t\int_{\Gamma_-}(\mathscr{T}_- p_2)(\partial_\tau\bar{p}_2)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau-\frac{1}{\varrho_2}\int_0^t\int_S(\partial_{\boldsymbol{\nu}_S}p_2)(\partial_\tau\bar{p}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau\\
&\quad+\int_0^t\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_\tau^2\boldsymbol{u}_1)(\partial_\tau\bar{p}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau,
\end{aligned}\tag{3.39}$$

?SE7?

and

$$\begin{aligned}
&\frac{1}{\varrho_3}\int_0^t\int_{\Omega_3}\left[\nabla p_3\cdot\nabla(\partial_\tau\bar{p}_3)+\frac{1}{c_3^2}(\partial_\tau^2 p_3)(\partial_\tau\bar{p}_3)+\frac{\gamma_3}{c_3^2}|\partial_\tau p_3|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau\\
&=\int_0^t\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_\tau^2\boldsymbol{u}_2)(\partial_\tau\bar{p}_3)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau.
\end{aligned}\tag{3.40}$$

SE8

Then, combining (3.37)-(3.40) and the continuity conditions on S, we find

$$\begin{aligned}
&\int_0^t E_1'(\tau)\mathrm{d}\tau\\
&\le\frac{2}{\varrho_2}\Re\int_0^t\int_{\Gamma_+}(\mathscr{T}_+ p_1)(\partial_\tau\bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau+\frac{2}{\varrho_2}\Re\int_0^t\int_{\Gamma_-}(\mathscr{T}_- p_2)(\partial_\tau\bar{p}_2)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&\quad-\frac{2}{\varrho_2}\Re\int_0^t\int_{\Omega_1}g(\partial_\tau\bar{p}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau\\
&\quad+2\Re\int_0^t\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_\tau^2\boldsymbol{u}_1)(\partial_\tau\bar{p}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau+2\Re\int_0^t\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_\tau^2\boldsymbol{u}_2)(\partial_\tau\bar{p}_3)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau.
\end{aligned}\tag{3.41}$$

SE9

It follows from (3.35) that

$$\begin{aligned}
&\int_0^t E_2'(\tau)\mathrm{d}\tau\\
&=2\Re\int_0^t\int_{D_1}\big[\rho_1(\partial_\tau(\partial_\tau^2\boldsymbol{u}_1)\cdot(\partial_\tau^2\overline{\boldsymbol{u}}_1))+(\lambda_1+\mu_1)(\nabla\cdot(\partial_\tau^2\boldsymbol{u}_1)\nabla\cdot(\partial_\tau\overline{\boldsymbol{u}}_1))\\
&\qquad\qquad+\mu_1(\nabla(\partial_\tau^2\boldsymbol{u}_1):\nabla(\partial_\tau\overline{\boldsymbol{u}}_1))\big]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau\\
&\quad+2\Re\int_0^t\int_{D_2}\big[\rho_2(\partial_\tau(\partial_\tau^2\boldsymbol{u}_2)\cdot(\partial_\tau^2\overline{\boldsymbol{u}}_2))+(\lambda_2+\mu_2)(\nabla\cdot(\partial_\tau^2\boldsymbol{u}_2)\nabla\cdot(\partial_\tau\overline{\boldsymbol{u}}_2))\\
&\qquad\qquad+\mu_2(\nabla(\partial_\tau^2\boldsymbol{u}_2):\nabla(\partial_\tau\overline{\boldsymbol{u}}_2))\big]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau.
\end{aligned}\tag{3.42}$$

SE10

For $j=1,2$, multiplying $(\mu_j\Delta+(\lambda_j+\mu_j)\,\nabla\nabla\cdot-\rho_j\partial_t^2)\partial_t\boldsymbol{u}_j(\boldsymbol{x},t)=0$ by a test function $\partial_t^2\overline{\boldsymbol{u}}_j$, where $\boldsymbol{u}_j\in L^2(0,T;H^1(D_j)^3)\cap H^1(0,T;L^2(D_j)^3)$. Taking the inner products, using the integration by parts and boundary conditions, we arrive at

$$\begin{aligned}
0 =& \int_{D_1} \Big[\mu_1(\nabla(\partial_t \boldsymbol{u}_1) : \nabla(\partial_t^2 \bar{\boldsymbol{u}}_1)) + (\lambda_1 + \mu_1)(\nabla \cdot (\partial_t \boldsymbol{u}_1))(\nabla \cdot (\partial_t^2 \bar{\boldsymbol{u}}_1)) + \rho_1(\partial_t(\partial_t^2 \boldsymbol{u}_1) \cdot ((\partial_t^2 \bar{\boldsymbol{u}}_1))\Big] \mathrm{d}\boldsymbol{x} \\
&- \int_{\Gamma_1} \partial_t^2 \bar{\boldsymbol{u}}_1 \cdot (\boldsymbol{\nu}_{\Gamma_1} \cdot \boldsymbol{\sigma}_1(\partial_t \boldsymbol{u}_1)) \mathrm{d}s_{\boldsymbol{x}} - \int_{\Gamma_2} \partial_t^2 \bar{\boldsymbol{u}}_1 \cdot (\boldsymbol{\nu}_{\Gamma_2} \cdot \boldsymbol{\sigma}_1(\partial_t \boldsymbol{u}_1)) \mathrm{d}s_{\boldsymbol{x}} \\
=& \int_{D_1} \Big[\mu_1(\nabla(\partial_t \boldsymbol{u}_1) : \nabla(\partial_t^2 \bar{\boldsymbol{u}}_1)) + (\lambda_1 + \mu_1)(\nabla \cdot (\partial_t \boldsymbol{u}_1))(\nabla \cdot (\partial_t^2 \bar{\boldsymbol{u}}_1)) + \rho_1(\partial_t(\partial_t^2 \boldsymbol{u}_1) \cdot ((\partial_t^2 \bar{\boldsymbol{u}}_1))\Big] \mathrm{d}\boldsymbol{x} \\
&+ \int_{\Gamma_1} (\partial_t p_2)(\boldsymbol{\nu}_{\Gamma_1} \cdot \partial_t^2 \bar{\boldsymbol{u}}_1) \mathrm{d}s_{\boldsymbol{x}} - \int_{\Gamma_2} \partial_t^2 \bar{\boldsymbol{u}}_1 \cdot (\boldsymbol{\nu}_{\Gamma_2} \cdot \boldsymbol{\sigma}_1(\partial_t \boldsymbol{u}_1)) \mathrm{d}s_{\boldsymbol{x}}, \qquad (3.43)\ ?\underline{\mathtt{SE11}}?
\end{aligned}$$

and

$$\begin{aligned}
0 =& \int_{D_2} \Big[\mu_2(\nabla(\partial_t \boldsymbol{u}_2) : \nabla(\partial_t^2 \bar{\boldsymbol{u}}_2)) + (\lambda_2 + \mu_2)(\nabla \cdot (\partial_t \boldsymbol{u}_2))(\nabla \cdot (\partial_t^2 \bar{\boldsymbol{u}}_2)) + \rho_2(\partial_t(\partial_t^2 \boldsymbol{u}_2) \cdot ((\partial_t^2 \bar{\boldsymbol{u}}_2))\Big] \mathrm{d}\boldsymbol{x} \\
&- \int_{\Gamma_3} \partial_t^2 \bar{\boldsymbol{u}}_2 \cdot (\boldsymbol{\nu}_{\Gamma_3} \cdot \boldsymbol{\sigma}_2(\partial_t \boldsymbol{u}_2)) \mathrm{d}s_{\boldsymbol{x}} + \int_{\Gamma_2} \partial_t^2 \bar{\boldsymbol{u}}_2 \cdot (\boldsymbol{\nu}_{\Gamma_2} \cdot \boldsymbol{\sigma}_2(\partial_t \boldsymbol{u}_2)) \mathrm{d}s_{\boldsymbol{x}} \\
=& \int_{D_2} \Big[\mu_2(\nabla(\partial_t \boldsymbol{u}_2) : \nabla(\partial_t^2 \bar{\boldsymbol{u}}_2)) + (\lambda_2 + \mu_2)(\nabla \cdot (\partial_t \boldsymbol{u}_2))(\nabla \cdot (\partial_t^2 \bar{\boldsymbol{u}}_2)) + \rho_2(\partial_t(\partial_t^2 \boldsymbol{u}_2) \cdot ((\partial_t^2 \bar{\boldsymbol{u}}_2))\Big] \mathrm{d}\boldsymbol{x} \\
&+ \int_{\Gamma_3} (\partial_t p_3)(\boldsymbol{\nu}_{\Gamma_3} \cdot \partial_t^2 \bar{\boldsymbol{u}}_2) \mathrm{d}s_{\boldsymbol{x}} + \int_{\Gamma_2} \partial_t^2 \bar{\boldsymbol{u}}_2 \cdot (\boldsymbol{\nu}_{\Gamma_2} \cdot \boldsymbol{\sigma}_2(\partial_t \boldsymbol{u}_2)) \mathrm{d}s_{\boldsymbol{x}}. \qquad (3.44)\ \boxed{\mathtt{SE12}}
\end{aligned}$$

Then, combining (3.42)-(3.44) and the continuity conditions on Γ_2, we find

$$\begin{aligned}
&\int_0^t E_2'(\tau) \mathrm{d}\tau \\
&= -2\Re \int_0^t \int_{\Gamma_1} (\partial_t p_2)(\boldsymbol{\nu}_{\Gamma_1} \cdot \partial_t^2 \bar{\boldsymbol{u}}_1) \mathrm{d}s_{\boldsymbol{x}} \mathrm{d}\tau - 2\Re \int_0^t \int_{\Gamma_3} (\partial_t p_3)(\boldsymbol{\nu}_{\Gamma_3} \cdot \partial_t^2 \bar{\boldsymbol{u}}_2) \mathrm{d}s_{\boldsymbol{x}} \mathrm{d}\tau. \qquad (3.45)\ \boxed{\mathtt{SE13}}
\end{aligned}$$

From lemma 2.6, combining (3.36), (3.41) and (3.45) gives

$$\begin{aligned}
E(t) =& \int_0^t E'(\tau) \mathrm{d}\tau = \int_0^t \big[E_1'(\tau) + E_2'(\tau)\big] \mathrm{d}\tau. \\
\le& \frac{2}{\varrho_2} \Re \int_0^t \int_{\Gamma_+} (\mathscr{T}_+ p_1)(\partial_\tau \bar{p}_1) \mathrm{d}\boldsymbol{r} \mathrm{d}\tau + \frac{2}{\varrho_2} \Re \int_0^t \int_{\Gamma_-} (\mathscr{T}_- p_2)(\partial_\tau \bar{p}_2) \mathrm{d}\boldsymbol{r} \mathrm{d}\tau \\
&- \frac{2}{\varrho_2} \Re \int_0^t \int_{\Omega_1} g(\partial_\tau \bar{p}_1) \mathrm{d}\boldsymbol{x} \mathrm{d}\tau \\
\le& -\frac{2}{\varrho_2} \Re \int_0^t \int_{\Omega_1} g(\partial_\tau \bar{p}_1) \mathrm{d}\boldsymbol{x} \mathrm{d}\tau \le \frac{2}{\varrho_2} \int_0^t \int_{\Omega_1} \big|g(\partial_\tau \bar{p}_1)\big| \mathrm{d}\boldsymbol{x} \mathrm{d}\tau \\
\le& \frac{2}{\varrho_2} (\max_{t \in [0,T]} \|\partial_t p_1\|_{L^2(\Omega_1)}) \|g\|_{L^1(0,T;L^2(\Omega_1))} \\
\le& \frac{2}{\varrho_2} \big(\|\partial_t p_1\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\nabla(\partial_t p_1)\|_{L^\infty(0,T;L^2(\Omega_1))}\big) \|g\|_{L^1(0,T;L^2(\Omega_1))}. \qquad (3.46)\ \boxed{\mathtt{SE14}}
\end{aligned}$$

Taking the first partial derivative of (3.24) with respect to t, we get

$$\begin{cases}
(\Delta - \frac{1}{c_1^2}\partial_t^2 - \frac{\gamma_1}{c_1^2}\partial_t)\partial_t p_1(\boldsymbol{x},t) = \partial_t g(\boldsymbol{x},t) & \text{in} \quad \Omega_1,\ t>0,\\
(\Delta - \frac{1}{c_l^2}\partial_t^2 - \frac{\gamma_l}{c_l^2}\partial_t)\partial_t p_l(\boldsymbol{x},t) = 0 & \text{in} \quad \Omega_l,\ t>0,\ l=2,3,\\
(\mu_j\Delta + (\lambda_j+\mu_j)\,\nabla\nabla\cdot - \rho_j\partial_t^2)\partial_t^2\boldsymbol{u}_j(\boldsymbol{x},t) = 0 & \text{in} \quad D_j,\ t>0,\ j=1,2,\\
\partial_t p_1 = \partial_t p_2, \qquad \partial_{\boldsymbol{\nu}_S}(\partial_t p_1) = \partial_{\boldsymbol{\nu}_S}(\partial_t p_2) & \text{on} \quad S, \quad t>0,\\
\partial_{\boldsymbol{\nu}_{\Gamma_1}}(\partial_t p_2) = -\varrho_2\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t^2(\partial_t\boldsymbol{u}_1), \quad -(\partial_t^2 p_2)\ \boldsymbol{\nu}_{\Gamma_1} = \boldsymbol{\nu}_{\Gamma_1}\cdot\boldsymbol{\sigma}_1(\partial_t^2\boldsymbol{u}_1) & \text{on} \quad \Gamma_1,\ t>0,\\
\partial_t^2\boldsymbol{u}_1 = \partial_t^2\boldsymbol{u}_2, \quad \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\partial_t^2\boldsymbol{u}_1) = \boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\partial_t^2\boldsymbol{u}_2) & \text{on} \quad \Gamma_2,\ t>0,\\
\partial_{\boldsymbol{\nu}_{\Gamma_3}}(\partial_t p_3) = -\varrho_3\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t^2(\partial_t\boldsymbol{u}_2), \quad -(\partial_t^2 p_3)\ \boldsymbol{\nu}_{\Gamma_3} = \boldsymbol{\nu}_{\Gamma_3}\cdot\boldsymbol{\sigma}_2(\partial_t^2\boldsymbol{u}_2) & \text{on} \quad \Gamma_3,\ t>0,\\
\partial_t p_l|_{t=0} = \partial_t^2 p_l|_{t=0} = 0 & \text{in} \quad \Omega_l,\ l=1,2,3,\\
\partial_t^2\boldsymbol{u}_j|_{t=0} = \partial_t(\partial_t^2\boldsymbol{u}_j)|_{t=0} = 0 & \text{in} \quad D_j,\ j=1,2,\\
\partial_{\boldsymbol{\nu}_{\Gamma+}}(\partial_t p_1) = \mathscr{T}_+(\partial_t p_1) & \text{on} \quad \Gamma_+,\ t>0,\\
\partial_{\boldsymbol{\nu}_{\Gamma-}}(\partial_t p_2) = \mathscr{T}_-(\partial_t p_2) & \text{on} \quad \Gamma_-,\ t>0.
\end{cases} \tag{3.47}$$

TEUT1-1

For $\forall t \in [0,T]$, we define the energy function

$$\widetilde{E}(t) = \widetilde{E}_1(t) + \widetilde{E}_2(t), \tag{3.48}$$

?SE1-1?

where

$$\begin{aligned}
\widetilde{E}_1(t) = {} & \frac{1}{\varrho_2 c_1^2}\|\partial_t^2 p_1\|^2_{L^2(\Omega_1)} + \frac{1}{\varrho_2}\|\nabla(\partial_t p_1)\|^2_{L^2(\Omega_1)^3} + \frac{1}{\varrho_2 c_2^2}\|\partial_t^2 p_2\|^2_{L^2(\Omega_2)} + \frac{1}{\varrho_2}\|\nabla(\partial_t p_2)\|^2_{L^2(\Omega_2)^3}\\
& + \frac{1}{\varrho_3 c_3^2}\|\partial_t^2 p_3\|^2_{L^2(\Omega_3)} + \frac{1}{\varrho_3}\|\nabla(\partial_t p_3)\|^2_{L^2(\Omega_3)^3},
\end{aligned} \tag{3.49}$$

SE2-1

and

$$\begin{aligned}
\widetilde{E}_2(t) = {} & \|(\rho_1)^{\frac{1}{2}}\partial_t^2(\partial_t\boldsymbol{u}_1)\|^2_{L^2(D_1)^3} + \|(\lambda_1+\mu_1)^{\frac{1}{2}}\nabla\cdot(\partial_t^2\boldsymbol{u}_1)\|^2_{L^2(D_1)} + \|(\mu_1)^{\frac{1}{2}}\nabla(\partial_t^2\boldsymbol{u}_1)\|^2_{L^2(D_1)^{3\times 3}}\\
& + \|(\rho_2)^{\frac{1}{2}}\partial_t^2(\partial_t\boldsymbol{u}_2)\|^2_{L^2(D_2)^3} + \|(\lambda_2+\mu_2)^{\frac{1}{2}}\nabla\cdot(\partial_t^2\boldsymbol{u}_2)\|^2_{L^2(D_2)} + \|(\mu_2)^{\frac{1}{2}}\nabla(\partial_t^2\boldsymbol{u}_2)\|^2_{L^2(D_2)^{3\times 3}}.
\end{aligned} \tag{3.50}$$

SE3-1

With initial conditions, it is easy to note that $\widetilde{E}(0) = 0$ and

$$\widetilde{E}(t) = \widetilde{E}(t) - \widetilde{E}(0) = \int_0^t \widetilde{E}'(\tau)\mathrm{d}\tau = \int_0^t \left[\widetilde{E}_1'(\tau) + \widetilde{E}_2'(\tau)\right]\mathrm{d}\tau. \tag{3.51}$$

SE4-1

It follows from (3.49) that

$$
\begin{aligned}
&\int_0^t \widetilde{E}_1'(\tau)\mathrm{d}\tau \\
&= \frac{1}{\varrho_2}\int_0^t\int_{\Omega_1}\left(\frac{1}{c_1^2}\partial_\tau|\partial_\tau^2 p_1|^2 + \partial_\tau|\nabla(\partial_\tau p_1)|^2\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau + \frac{1}{\varrho_2}\int_0^t\int_{\Omega_2}\left(\frac{1}{c_2^2}\partial_\tau|\partial_\tau^2 p_2|^2 + \partial_\tau|\nabla(\partial_\tau p_2)|^2\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\quad + \frac{1}{\varrho_3}\int_0^t\int_{\Omega_3}\left(\frac{1}{c_3^2}\partial_\tau|\partial_\tau^2 p_3|^2 + \partial_\tau|\nabla(\partial_\tau p_3)|^2\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= \frac{2}{\varrho_2}\Re\int_0^t\int_{\Omega_1}\left(\frac{1}{c_1^2}(\partial_\tau^3 p_1)(\partial_\tau^2\bar{p}_1) + \nabla(\partial_\tau^2 p_1)\cdot\nabla(\partial_\tau\bar{p}_1)\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\quad + \frac{2}{\varrho_2}\Re\int_0^t\int_{\Omega_2}\left(\frac{1}{c_2^2}(\partial_\tau^3 p_2)(\partial_\tau^2\bar{p}_2) + \nabla(\partial_\tau^2 p_2)\cdot\nabla(\partial_\tau\bar{p}_2)\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\quad + \frac{2}{\varrho_3}\Re\int_0^t\int_{\Omega_3}\left(\frac{1}{c_3^2}(\partial_\tau^3 p_3)(\partial_\tau^2\bar{p}_3) + \nabla(\partial_\tau^2 p_3)\cdot\nabla(\partial_\tau\bar{p}_3)\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau. \qquad (3.52)\ \boxed{\texttt{SE5-1}}
\end{aligned}
$$

Multiplying $(\Delta - \frac{1}{c_1^2}\partial_t^2 - \frac{\gamma_1}{c_1^2}\partial_t)\partial_t p_1(\boldsymbol{x},t) = \partial_t g(\boldsymbol{x},t)$ by a test function $\partial_t^2\bar{p}_1$, where $p_1 \in L^2(0,T;H^1(\Omega_1)) \cap H^1(0,T;L^2(\Omega_1))$. Taking the inner products, using the integration by parts and boundary conditions, which include the Boundary condition $\partial_{\boldsymbol{\nu}_{\Gamma+}}(\partial_t p_1) = \mathscr{T}_+(\partial_t p_1)$, we arrive at

$$
\begin{aligned}
&\frac{1}{\varrho_2}\int_0^t\int_{\Omega_1}\left[\nabla(\partial_\tau p_1)\cdot\nabla(\partial_\tau^2\bar{p}_1) + \frac{1}{c_1^2}(\partial_\tau^3 p_1)(\partial_\tau^2\bar{p}_1) + \frac{\gamma_1}{c_1^2}|\partial_\tau^2 p_1|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= \frac{1}{\varrho_2}\int_0^t\int_{\Gamma_+}(\mathscr{T}_+(\partial_\tau p_1))(\partial_\tau^2\bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau + \frac{1}{\varrho_2}\int_0^t\int_S(\partial_{\boldsymbol{\nu}_S}(\partial_\tau p_1))(\partial_\tau^2\bar{p}_1)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau \\
&\quad - \frac{1}{\varrho_2}\int_0^t\int_{\Omega_1}(\partial_\tau g)(\partial_\tau^2\bar{p}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau. \qquad (3.53)\ ?\underline{\texttt{SE6-1}}?
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
&\frac{1}{\varrho_2}\int_0^t\int_{\Omega_2}\left[\nabla(\partial_\tau^2 p_2)\cdot\nabla(\partial_\tau\bar{p}_2) + \frac{1}{c_2^2}(\partial_\tau^3 p_2)(\partial_\tau^2\bar{p}_2) + \frac{\gamma_2}{c_2^2}|\partial_\tau^2 p_2|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= \frac{1}{\varrho_2}\int_0^t\int_{\Gamma_-}(\mathscr{T}_-(\partial_\tau p_2))(\partial_\tau^2\bar{p}_2)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau - \frac{1}{\varrho_2}\int_0^t\int_S(\partial_{\boldsymbol{\nu}_S}(\partial_\tau p_2))(\partial_\tau^2\bar{p}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau \\
&\quad + \int_0^t\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_\tau^2(\partial_\tau\boldsymbol{u}_1))(\partial_\tau^2\bar{p}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau, \qquad (3.54)\ ?\underline{\texttt{SE7-1}}?
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\varrho_3}\int_0^t\int_{\Omega_3}\left[\nabla(\partial_\tau p_3)\cdot\nabla(\partial_\tau^2\bar{p}_3) + \frac{1}{c_3^2}(\partial_\tau^3 p_3)(\partial_\tau^2\bar{p}_3) + \frac{\gamma_3}{c_3^2}|\partial_\tau^2 p_3|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= \int_0^t\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_\tau^2(\partial_\tau\boldsymbol{u}_2))(\partial_\tau^2\bar{p}_3)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau. \qquad (3.55)\ \boxed{\texttt{SE8-1}}
\end{aligned}
$$

Then, combining (3.52)-(3.55) and the continuity conditions on S to obtain

$$
\begin{aligned}
&\int_0^t \widetilde{E}_1'(\tau)\mathrm{d}\tau \\
&\le \frac{2}{\varrho_2}\Re\int_0^t\int_{\Gamma_+}(\mathscr{T}_+(\partial_\tau p_1))(\partial_\tau^2\bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau + \frac{2}{\varrho_2}\Re\int_0^t\int_{\Gamma_-}(\mathscr{T}_-(\partial_\tau p_2))(\partial_\tau^2\bar{p}_2)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau \\
&\quad - \frac{2}{\varrho_2}\Re\int_0^t\int_{\Omega_1}(\partial_\tau g)(\partial_\tau^2\bar{p}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\quad + 2\Re\int_0^t\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_\tau^2(\partial_\tau\boldsymbol{u}_1))(\partial_\tau^2\bar{p}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau + 2\Re\int_0^t\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_\tau^2(\partial_\tau\boldsymbol{u}_2))(\partial_\tau^2\bar{p}_3)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau. \qquad (3.56)\ \boxed{\texttt{SE9-1}}
\end{aligned}
$$

It follows from (3.50) that

$$
\begin{aligned}
\int_0^t \widetilde{E}_2'(\tau)\mathrm{d}\tau = 2\Re\int_0^t\int_{D_1}&\big[\rho_1((\partial_\tau^4\boldsymbol{u}_1)\cdot(\partial_\tau^3\overline{\boldsymbol{u}}_1)) + (\lambda_1+\mu_1)(\nabla\cdot(\partial_\tau^3\boldsymbol{u}_1)\nabla\cdot(\partial_\tau^2\overline{\boldsymbol{u}}_1)) \\
&+ \mu_1(\nabla(\partial_\tau^3\boldsymbol{u}_1):\nabla(\partial_\tau^2\overline{\boldsymbol{u}}_1))\big]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
+ 2\Re\int_0^t\int_{D_2}&\big[\rho_2((\partial_\tau^4\boldsymbol{u}_2)\cdot(\partial_\tau^3\overline{\boldsymbol{u}}_2)) + (\lambda_2+\mu_2)(\nabla\cdot(\partial_\tau^3\boldsymbol{u}_2)\nabla\cdot(\partial_\tau^2\overline{\boldsymbol{u}}_2)) \\
&+ \mu_2(\nabla(\partial_\tau^3\boldsymbol{u}_2):\nabla(\partial_\tau^2\overline{\boldsymbol{u}}_2))\big]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau. \qquad (3.57)\ \boxed{\texttt{SE10-1}}
\end{aligned}
$$

For $j=1,2$, multiplying $(\mu_j\Delta + (\lambda_j+\mu_j)\nabla\nabla\cdot - \rho_j\partial_t^2)\partial_t^2\boldsymbol{u}_j(\boldsymbol{x},t) = 0$ by a test function $\partial_t^3\overline{\boldsymbol{u}}_j$, where $\boldsymbol{u}_j\in L^2(0,T;H^1(D_j)^3)\cap H^1(0,T;L^2(D_j)^3)$. Taking the inner products, using the integration by parts and boundary conditions, we arrive at

$$
\begin{aligned}
0 &= \int_{D_1}\Big[\mu_1(\nabla(\partial_t^2\boldsymbol{u}_1):\nabla(\partial_t^3\overline{\boldsymbol{u}}_1)) + (\lambda_1+\mu_1)(\nabla\cdot(\partial_t^2\boldsymbol{u}_1))(\nabla\cdot(\partial_t^3\overline{\boldsymbol{u}}_1)) + \rho_1((\partial_t^4\boldsymbol{u}_1)\cdot(\partial_t^3\overline{\boldsymbol{u}}_1))\Big]\mathrm{d}\boldsymbol{x} \\
&\quad - \int_{\Gamma_1}\partial_t^3\overline{\boldsymbol{u}}_1\cdot(\boldsymbol{\nu}_{\Gamma_1}\cdot\boldsymbol{\sigma}_1(\partial_t^2\boldsymbol{u}_1))\mathrm{d}s_{\boldsymbol{x}} - \int_{\Gamma_2}\partial_t^3\overline{\boldsymbol{u}}_1\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\partial_t^2\boldsymbol{u}_1))\mathrm{d}s_{\boldsymbol{x}} \\
&= \int_{D_1}\Big[\mu_1(\nabla(\partial_t^2\boldsymbol{u}_1):\nabla(\partial_t^3\overline{\boldsymbol{u}}_1)) + (\lambda_1+\mu_1)(\nabla\cdot(\partial_t^2\boldsymbol{u}_1))(\nabla\cdot(\partial_t^3\overline{\boldsymbol{u}}_1)) + \rho_1((\partial_t^4\boldsymbol{u}_1)\cdot(\partial_t^3\overline{\boldsymbol{u}}_1))\Big]\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\Gamma_1}(\partial_t^2 p_2)(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t^3\overline{\boldsymbol{u}}_1)\mathrm{d}s_{\boldsymbol{x}} - \int_{\Gamma_2}\partial_t^3\overline{\boldsymbol{u}}_1\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\partial_t^2\boldsymbol{u}_1))\mathrm{d}s_{\boldsymbol{x}}, \qquad (3.58)\ ?\underline{\texttt{SE11-1}}?
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \int_{D_2}\Big[\mu_2(\nabla(\partial_t^2\boldsymbol{u}_2):\nabla(\partial_t^3\overline{\boldsymbol{u}}_2)) + (\lambda_2+\mu_2)(\nabla\cdot(\partial_t^2\boldsymbol{u}_2))(\nabla\cdot(\partial_t^3\overline{\boldsymbol{u}}_2)) + \rho_2((\partial_t^4\boldsymbol{u}_2)\cdot(\partial_t^3\overline{\boldsymbol{u}}_2))\Big]\mathrm{d}\boldsymbol{x} \\
&\quad - \int_{\Gamma_3}\partial_t^3\overline{\boldsymbol{u}}_2\cdot(\boldsymbol{\nu}_{\Gamma_3}\cdot\boldsymbol{\sigma}_2(\partial_t^2\boldsymbol{u}_2))\mathrm{d}s_{\boldsymbol{x}} + \int_{\Gamma_2}\partial_t^3\overline{\boldsymbol{u}}_2\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\partial_t^2\boldsymbol{u}_2))\mathrm{d}s_{\boldsymbol{x}} \\
&= \int_{D_2}\Big[\mu_2(\nabla(\partial_t^2\boldsymbol{u}_2):\nabla(\partial_t^3\overline{\boldsymbol{u}}_2)) + (\lambda_2+\mu_2)(\nabla\cdot(\partial_t^2\boldsymbol{u}_2))(\nabla\cdot(\partial_t^3\overline{\boldsymbol{u}}_2)) + \rho_2((\partial_t^4\boldsymbol{u}_2)\cdot(\partial_t^3\overline{\boldsymbol{u}}_2))\Big]\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\Gamma_3}(\partial_t^2 p_3)(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t^3\overline{\boldsymbol{u}}_2)\mathrm{d}s_{\boldsymbol{x}} + \int_{\Gamma_2}\partial_t^3\overline{\boldsymbol{u}}_2\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\partial_t^2\boldsymbol{u}_2))\mathrm{d}s_{\boldsymbol{x}}. \qquad (3.59)\ \boxed{\texttt{SE12-1}}
\end{aligned}
$$

Then, we can combine (3.57)-(3.59) and the continuity conditions on Γ_2 to obtain

$$
\begin{aligned}
&\int_0^t \widetilde{E}_2'(\tau)\mathrm{d}\tau \\
&= 2\Re \int_0^t \int_{D_1} \big[\rho_1((\partial_\tau^4 \boldsymbol{u}_1)\cdot(\partial_\tau^3 \overline{\boldsymbol{u}}_1)) + (\lambda_1+\mu_1)(\nabla\cdot(\partial_\tau^3 \boldsymbol{u}_1)\nabla\cdot(\partial_\tau^2 \overline{\boldsymbol{u}}_1)) \\
&\qquad\qquad + \mu_1(\nabla(\partial_\tau^3 \boldsymbol{u}_1) : \nabla(\partial_\tau^2 \overline{\boldsymbol{u}}_1))\big]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\quad + 2\Re \int_0^t \int_{D_2} \big[\rho_2((\partial_\tau^4 \boldsymbol{u}_2)\cdot(\partial_\tau^3 \overline{\boldsymbol{u}}_2)) + (\lambda_2+\mu_2)(\nabla\cdot(\partial_\tau^3 \boldsymbol{u}_2)\nabla\cdot(\partial_\tau^2 \overline{\boldsymbol{u}}_2)) \\
&\qquad\qquad + \mu_2(\nabla(\partial_\tau^3 \boldsymbol{u}_2) : \nabla(\partial_\tau^2 \overline{\boldsymbol{u}}_2))\big]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= -2\Re \int_0^t \int_{\Gamma_1} (\partial_\tau^2 p_2)(\boldsymbol{\nu}_{\Gamma_1}\cdot \partial_\tau^3 \overline{\boldsymbol{u}}_1)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau - 2\Re \int_0^t \int_{\Gamma_3} (\partial_\tau^2 p_3)(\boldsymbol{\nu}_{\Gamma_3}\cdot \partial_\tau^3 \overline{\boldsymbol{u}}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}\tau. \qquad (3.60)\ \boxed{\texttt{SE13-1}}
\end{aligned}
$$

From lemma 2.7, combining (3.51), (3.56) and (3.60) gives

$$
\begin{aligned}
\widetilde{E}(t) &= \int_0^t \widetilde{E}'(\tau)\mathrm{d}\tau = \int_0^t \big[\widetilde{E}_1'(\tau) + \widetilde{E}_2'(\tau)\big]\mathrm{d}\tau. \\
&\le \frac{2}{\varrho_2}\Re \int_0^t \int_{\Gamma_+} (\mathscr{T}_+(\partial_\tau p_1))(\partial_\tau^2 \bar{p}_1)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau + \frac{2}{\varrho_2}\Re \int_0^t \int_{\Gamma_-} (\mathscr{T}_-(\partial_\tau p_2))(\partial_\tau^2 \bar{p}_2)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau \\
&\quad - \frac{2}{\varrho_2}\Re \int_0^t \int_{\Omega_1} (\partial_\tau g)(\partial_\tau^2 \bar{p}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\le -\frac{2}{\varrho_2}\Re \int_0^t \int_{\Omega_1} (\partial_\tau g)(\partial_\tau^2 \bar{p}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau = -\frac{2}{\varrho_2}\Re \int_{\Omega_1} \int_0^t (\partial_\tau g)(\partial_\tau^2 \bar{p}_1)\mathrm{d}\tau\mathrm{d}\boldsymbol{x} \\
&= -\frac{2}{\varrho_2}\Re \int_{\Omega_1} \left[(\partial_\tau g)(\partial_\tau \bar{p}_1)|_0^t - \int_0^t (\partial_\tau^2 g)(\partial_\tau \bar{p}_1)\mathrm{d}\tau\right]\mathrm{d}\boldsymbol{r} \\
&= -\frac{2}{\varrho_2}\Re \int_{\Omega_1} (\partial_t g(\cdot,t))(\partial_t \bar{p}_1(\cdot,t))\mathrm{d}\boldsymbol{x} + \frac{2}{\varrho_2}\Re \int_{\Omega_1} \int_0^t (\partial_\tau^2 g)(\partial_\tau \bar{p}_1)\mathrm{d}\tau\mathrm{d}\boldsymbol{x} \\
&\le \frac{2}{\varrho_2}\|\partial_t g(\cdot,t)\|_{L^2(\Omega_1)}\|\partial_t p_1(\cdot,t)\|_{L^2(\Omega_1)} \\
&\quad + \frac{2}{\varrho_2}\int_0^t \|\partial_t^2 g(\cdot,\tau)\|_{L^2(\Omega_1)}\|\partial_t p_1(\cdot,\tau)\|_{L^2(\Omega_1)}\mathrm{d}\tau \\
&\le \frac{2}{\varrho_2}[\max_{t\in[0,T]}\|\partial_t g(\cdot,t)\|_{L^2(\Omega_1)}][\max_{t\in[0,T]}\|\partial_t p_1(\cdot,t)\|_{L^2(\Omega_1)}] \\
&\quad + \frac{2}{\varrho_2}[\max_{t\in[0,T]}\|\partial_t p_1(\cdot,t)\|_{L^2(\Omega_1)}]\int_0^t \|\partial_t^2 g(\cdot,\tau)\|_{L^2(\Omega_1)}\mathrm{d}\tau \\
&= \frac{2}{\varrho_2}[\max_{t\in[0,T]}\|\partial_t p_1\|_{H^1(\Omega_1)}]\left[\max_{t\in[0,T]}\|\partial_t g\|_{L^2(\Omega_1)} + \int_0^t \|\partial_t^2 g(\cdot,\tau)\|_{L^2(\Omega_1)}\mathrm{d}\tau\right] \\
&= \frac{2}{\varrho_2}[\|\partial_t p_1\|_{L^\infty(0,T;H^1(\Omega_1))}]\left[\|\partial_t g\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\partial_t^2 g\|_{L^1(0,T;L^2(\Omega_1))}\right] \\
&= \frac{2}{\varrho_2}\big(\|\partial_t p_1\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\nabla(\partial_t p_1)\|_{L^\infty(0,T;L^2(\Omega_1))}\big)\times \\
&\qquad \big(\|\partial_t g\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\partial_t^2 g\|_{L^1(0,T;L^2(\Omega_1))}\big). \qquad (3.61)\ \boxed{\texttt{SE14-1}}
\end{aligned}
$$

Combining (3.46) and (3.61) gives

$$
\begin{aligned}
&E(t)+\widetilde{E}(t)\\
&\le \frac{2}{\varrho_2}\left(\|\partial_t p_1\|_{L^\infty(0,T;L^2(\Omega_1))}+\|\nabla(\partial_t p_1)\|_{L^\infty(0,T;L^2(\Omega_1))}\right)\times\\
&\qquad\left(\|g\|_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|_{L^\infty(0,T;L^2(\Omega_1))}+\|\partial_t^2 g\|_{L^1(0,T;L^2(\Omega_1))}\right).
\end{aligned}
\tag{3.62}
$$ SE15

It follows from (3.62) and Young's inequality that

$$
\begin{aligned}
&\max_{t\in[0,T]}\sum_{l=1}^{3}\left[\|\partial_t p_l\|^2_{L^2(\Omega_l)}+\|\nabla p_l\|^2_{L^2(\Omega_l)^3}\right]+\max_{t\in[0,T]}\sum_{l=1}^{3}\left[\|\partial_t^2 p_l\|^2_{L^2(\Omega_l)}+\|\nabla(\partial_t p_l)\|^2_{L^2(\Omega_l)^3}\right]\\
&+\max_{t\in[0,T]}\sum_{j=1}^{2}\left[\|\partial_t^2\boldsymbol{u}_j\|^2_{L^2(D_j)^3}+\|\nabla\cdot(\partial_t\boldsymbol{u}_j)\|^2_{L^2(D_j)}+\|\nabla(\partial_t\boldsymbol{u}_j)\|^2_{L^2(D_j)^{3\times3}}\right]\\
&+\max_{t\in[0,T]}\sum_{j=1}^{2}\left[\|\partial_t^3\boldsymbol{u}_j\|^2_{L^2(D_j)^3}+\|\nabla\cdot(\partial_t^2\boldsymbol{u}_j)\|^2_{L^2(D_j)}+\|\nabla(\partial_t^2\boldsymbol{u}_j)\|^2_{L^2(D_j)^{3\times3}}\right]\\
&\le M_3\left[\|g\|^2_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|^2_{L^\infty(0,T;L^2(\Omega_1))}+\|\partial_t^2 g\|^2_{L^1(0,T;L^2(\Omega_1))}\right],
\end{aligned}
\tag{3.63}
$$?SE15-1?

which shows the stability estimate (3.31) and

$$
\begin{aligned}
&\max_{t\in[0,T]}\sum_{j=1}^{2}\left[\|\partial_t^2\boldsymbol{u}_j\|^2_{L^2(D_j)^3}+\|\nabla\cdot(\partial_t\boldsymbol{u}_j)\|^2_{L^2(D_j)}+\|\nabla(\partial_t\boldsymbol{u}_j)\|^2_{L^2(D_j)^{3\times3}}\right]\\
&\le M_3\left[\|g\|^2_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|^2_{L^\infty(0,T;L^2(\Omega_1))}+\|\partial_t^2 g\|^2_{L^1(0,T;L^2(\Omega_1))}\right].
\end{aligned}
\tag{3.64}
$$ SE16

Note that for any function $\boldsymbol{v}$ with $\boldsymbol{v}(\boldsymbol{x},0)=0$, we have

$$
\|\boldsymbol{v}(\cdot,t)\|\le\int_0^t\|\partial_t\boldsymbol{v}(\cdot,\tau)\|\mathrm{d}\tau\le T\max_{t\in[0,T]}\|\partial_t\boldsymbol{v}\|,
\tag{3.65}
$$ SE17

which holds for any norm. Hence, it follows from (3.64) and (3.65) that

$$
\begin{aligned}
&\max_{t\in[0,T]}\sum_{j=1}^{2}\left[\|\partial_t\boldsymbol{u}_j\|^2_{L^2(D_j)^3}+\|\nabla\cdot\boldsymbol{u}_j\|^2_{L^2(D_j)}+\|\nabla\boldsymbol{u}_j\|^2_{L^2(D_j)^{3\times3}}\right]\\
&\le M_4\left[\|g\|^2_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|^2_{L^\infty(0,T;L^2(\Omega_1))}+\|\partial_t^2 g\|^2_{L^1(0,T;L^2(\Omega_1))}\right],
\end{aligned}
\tag{3.66}
$$?SE18?

which shows the stability estimate (3.32). □

3.3 A priori estimates

?⟨RE⟩? **Theorem 3.5.** *We assume that* $g,\partial_t g\in L^1(0,T;L^2(\Omega_1))$. *Let* $p_l\in H^1(\Omega_l)$ $(l=1,2,3)$ *and* $\boldsymbol{u}_j\in H^1(D_j)^3$ $(j=1,2)$ *be the solution of* (2.16), *then, for any* $T>0$, *we have*

$$
\begin{aligned}
&\sum_{l=1}^{3}\left[\|p_l\|_{L^\infty(0,T;L^2(\Omega_l))}+\|\nabla p_l\|_{L^\infty(0,T;L^2(\Omega_l))}\right]\\
&\le M_3\left(\|g\|_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right),
\end{aligned}
\tag{3.67}
$$?RE01?

$$\sum_{j=1}^{2}\left[\|\nabla \boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j)^{3\times 3})}+\|\nabla\cdot\boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j))}+\|\partial_t\boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j))}\right]$$
$$\leq M_3\left(T\|g\|_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right), \tag{3.68}$$?RE02?

and

$$\sum_{l=1}^{3}\left[\|p_l\|_{L^2(0,T;L^2(\Omega_l))}+\|\nabla p_l\|_{L^2(0,T;L^2(\Omega_l))}\right]$$
$$\leq M_3\left(T\|g\|_{L^1(0,T;L^2(\Omega_1))}+\|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right), \tag{3.69}$$?RE03?

$$\sum_{j=1}^{2}\left[\|\nabla \boldsymbol{u}_j\|_{L^2(0,T;L^2(D_j)^{3\times 3})}+\|\nabla\cdot\boldsymbol{u}_j\|_{L^2(0,T;L^2(D_j))}+\|\partial_t\boldsymbol{u}_j\|_{L^2(0,T;L^2(D_j))}\right]$$
$$\leq M_3\left(T^{\frac{3}{2}}\|g\|_{L^1(0,T;L^2(\Omega_1))}+T^{\frac{1}{2}}\|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right), \tag{3.70}$$?RE04?

where $M_3>0$ is a constant.

Proof. For a given constant $\eta\in(0,T)$, define some functions

$$\psi_l(\boldsymbol{x},t)=\int_t^\eta p_l(\boldsymbol{x},\tau)\mathrm{d}\tau, \qquad \text{in } \Omega_l\times[0,\eta], \quad l=1,2,3, \tag{3.71}$$ RE1

which satisfy $\psi_1=\psi_2$ on S and

$$\psi_l(\boldsymbol{x},\eta)=0, \qquad \partial_t\psi_l(\boldsymbol{x},t)=-p_l(\boldsymbol{x},t) \qquad l=1,2,3. \tag{3.72}$$ RE2

For any $\phi_l(\boldsymbol{x},t)\in L^2(0,\eta;L^2(\Omega_l))$ $(l=1,2,3)$, we have from the integration by parts and (3.72) that

$$\begin{aligned}
&\int_0^\eta \phi_l(\boldsymbol{x},t)\overline{\psi}_l(\boldsymbol{x},t)\mathrm{d}t=\int_0^\eta\phi_l(\boldsymbol{x},t)\left(\int_t^\eta \bar{p}_l(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\mathrm{d}t\\
&=\int_0^\eta\left(\int_t^\eta \bar{p}_l(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\partial_t\left(\int_0^t\phi_l(\boldsymbol{x},\zeta)\mathrm{d}\zeta\right)\mathrm{d}t\\
&=\left(\int_t^\eta \bar{p}_l(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\left(\int_0^t\phi_l(\boldsymbol{x},\zeta)\mathrm{d}\zeta\right)\Big|_0^\eta-\int_0^\eta\left(\int_0^t\phi_l(\boldsymbol{x},\zeta)\mathrm{d}\zeta\right)\partial_t\left(\int_t^\eta \bar{p}_l(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\mathrm{d}t\\
&=0-\int_0^\eta\left[\left(\int_0^t\phi_l(\boldsymbol{x},\zeta)\mathrm{d}\zeta\right)(-\bar{p}_l(\boldsymbol{x},t))\right]\mathrm{d}t=\int_0^\eta\left[\left(\int_0^t\phi_l(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\bar{p}_l(\boldsymbol{x},t)\right]\mathrm{d}t.
\end{aligned} \tag{3.73}$$ RE3

Multiplying $(\Delta-\frac{1}{c_1^2}\partial_t^2-\frac{\gamma_1}{c_1^2}\partial_t)p_1(\boldsymbol{x},t)=g(\boldsymbol{x},t)$ by a test function $\bar{\psi}_1$. Taking the inner products, using the integration by parts and boundary conditions, which include the TBC condition $\partial_{\boldsymbol{\nu}_{\Gamma+}}p_1=\mathscr{T}_+p_1$, we arrive at

$$\begin{aligned}
&\frac{1}{\varrho_2}\int_0^\eta\int_{\Omega_1}\left[\nabla p_1\cdot\nabla\bar{\psi}_1+\frac{1}{c_1^2}(\partial_t^2p_1)\bar{\psi}_1+\frac{\gamma_1}{c_1^2}(\partial_tp_1)\bar{\psi}_1\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\frac{1}{\varrho_2}\int_0^\eta\int_{\Gamma_+}(\mathscr{T}_+p_1)\bar{\psi}_1\mathrm{d}\boldsymbol{r}\mathrm{d}t+\frac{1}{\varrho_2}\int_0^\eta\int_S(\partial_{\boldsymbol{\nu}_S}p_1)\bar{\psi}_1\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t-\frac{1}{\varrho_2}\int_0^\eta\int_{\Omega_1}g\bar{\psi}_1\mathrm{d}\boldsymbol{x}\mathrm{d}t.
\end{aligned} \tag{3.74}$$ RE5

Similarly, we get

$$\begin{aligned}
&\frac{1}{\varrho_2}\int_0^\eta\int_{\Omega_2}\left[\nabla p_2\cdot\nabla\bar\psi_2+\frac{1}{c_2^2}(\partial_t^2 p_2)\bar\psi_2+\frac{\gamma_2}{c_2^2}(\partial_t p_2)\bar\psi_2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\frac{1}{\varrho_2}\int_0^\eta\int_{\Gamma_-}(\mathscr{T}_- p_2)\bar\psi_2\mathrm{d}\boldsymbol{r}\mathrm{d}t-\frac{1}{\varrho_2}\int_0^\eta\int_S(\partial_{\boldsymbol{\nu}_S}p_2)\bar\psi_2\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t+\int_0^\eta\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t^2\boldsymbol{u}_1)\bar\psi_2\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t, \quad (3.75)\ \text{?RE6?}
\end{aligned}$$

and

$$\frac{1}{\varrho_3}\int_0^\eta\int_{\Omega_3}\left[\nabla p_3\cdot\nabla\bar\psi_3+\frac{1}{c_3^2}(\partial_t^2 p_3)\bar\psi_3+\frac{\gamma_3}{c_3^2}(\partial_t p_3)\bar\psi_3\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t=\int_0^\eta\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t^2\boldsymbol{u}_2)\bar\psi_3\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t. \quad (3.76)\ \text{?RE7?}$$

It follows from (3.72) and the initial conditions in (3.24) that

$$\begin{aligned}
&\Re\int_0^\eta\int_{\Omega_l}\frac{1}{c_l^2}(\partial_t^2 p_l)\bar\psi_l\mathrm{d}\boldsymbol{x}\mathrm{d}t=\Re\int_{\Omega_l}\int_0^\eta\frac{1}{c_l^2}\big[\partial_t(\partial_t p_l\bar\psi_1)-\partial_t p_l\partial_t\bar\psi_l\big]\mathrm{d}t\mathrm{d}\boldsymbol{x}\\
&=\Re\int_{\Omega_l}\int_0^\eta\frac{1}{c_l^2}\big[\partial_t(\partial_t p_l\bar\psi_l)-\partial_t p_l(-\bar p_l)\big]\mathrm{d}t\mathrm{d}\boldsymbol{x}=\Re\int_{\Omega_l}\frac{1}{c_l^2}\left[(\partial_t p_l\bar\psi_l)\big|_0^\eta+\frac{1}{2}|p_l|^2\big|_0^\eta\right]\mathrm{d}t\mathrm{d}\boldsymbol{x}\\
&=0+\frac{1}{2c_l^2}\int_{\Omega_l}|p_l(\boldsymbol{x},\eta)|^2\mathrm{d}\boldsymbol{x}=\frac{1}{2c_l^2}\|p_l(\cdot,\eta)\|^2_{L^2(\Omega_l)},\qquad l=1,2,3, \quad (3.77)\ \text{?RE8?}
\end{aligned}$$

$$\begin{aligned}
&\Re\int_0^\eta\int_{\Omega_l}\frac{\gamma_l}{c_l^2}(\partial_t p_l)\bar\psi_l\mathrm{d}\boldsymbol{x}\mathrm{d}t=\frac{\gamma_l}{c_l^2}\Re\int_{\Omega_l}\int_0^\eta(\partial_t p_l)\bar\psi_l\mathrm{d}t\mathrm{d}\boldsymbol{x}\\
&=\frac{\gamma_l}{c_l^2}\Re\int_{\Omega_l}\left[(p_l\bar\psi_l)\big|_0^\eta-\int_0^\eta p_l(\partial_t\bar\psi_l)\mathrm{d}t\right]\mathrm{d}t\mathrm{d}\boldsymbol{x}=\frac{\gamma_l}{c_l^2}\int_{\Omega_l}\int_0^\eta|p_l|^2\mathrm{d}t\mathrm{d}\boldsymbol{x},\qquad l=1,2,3, \quad (3.78)\ \text{?RE8-1?}
\end{aligned}$$

and

$$\Re\int_0^\eta\int_{\Omega_l}(\nabla p_l\cdot\nabla\bar\psi_l)\mathrm{d}\boldsymbol{x}\mathrm{d}t=\frac{1}{2}\int_{\Omega_l}\left|\int_0^\eta\nabla p_l(\cdot,t)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x},\qquad l=1,2,3. \quad (3.79)\ \text{?RE11?}$$

Similarly, it can be shown that

$$\begin{aligned}
&\Re\int_0^\eta\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t^2\boldsymbol{u}_1)\bar\psi_2\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t=\Re\int_{\Gamma_1}\int_0^\eta\big[\partial_t((\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\boldsymbol{u}_1)\bar\psi_2)-(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\boldsymbol{u}_1)(\partial_t\bar\psi_2)\big]\mathrm{d}t\mathrm{d}s_{\boldsymbol{x}}\\
&=\Re\int_{\Gamma_1}\int_0^\eta\big[\partial_t((\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\boldsymbol{u}_1)\bar\psi_2)-(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\boldsymbol{u}_1)(-\bar p_2)\big]\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t\\
&=\Re\int_{\Gamma_1}((\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\boldsymbol{u}_1)\bar\psi_2)\big|_0^\eta\mathrm{d}s_{\boldsymbol{x}}+\Re\int_0^\eta\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\boldsymbol{u}_1)\bar p_2\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t\\
&=\Re\int_0^\eta\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\boldsymbol{u}_1)\bar p_2\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t, \quad (3.80)\ \text{?RE9?}
\end{aligned}$$

and

$$\Re\int_0^\eta\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t^2\boldsymbol{u}_2)\bar\psi_3\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t=\Re\int_0^\eta\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t\boldsymbol{u}_2)\bar p_3\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t. \quad (3.81)\ \boxed{\text{RE10}}$$

Combining (3.74)-(3.81) and the boundary condition on S, we find

$$\begin{aligned}
&\frac{1}{2\varrho_2 c_1^2}\|p_1(\cdot,\eta)\|^2_{L^2(\Omega_1)} + \frac{1}{2\varrho_2 c_2^2}\|p_2(\cdot,\eta)\|^2_{L^2(\Omega_2)} + \frac{1}{2\varrho_3 c_3^2}\|p_1(\cdot,\eta)\|^3_{L^2(\Omega_3)} \\
&\quad + \frac{\gamma_1}{\varrho_2 c_1^2}\int_{\Omega_1}\int_0^\eta |p_1|^2 \mathrm{d}t\mathrm{d}\boldsymbol{x} + \frac{\gamma_2}{\varrho_2 c_2^2}\int_{\Omega_2}\int_0^\eta |p_2|^2 \mathrm{d}t\mathrm{d}\boldsymbol{x} + \frac{\gamma_3}{\varrho_3 c_3^2}\int_{\Omega_1}\int_0^\eta |p_3|^2 \mathrm{d}t\mathrm{d}\boldsymbol{x} \\
&\quad + \frac{1}{2\varrho_2}\int_{\Omega_1}\left|\int_0^\eta \nabla p_1 \mathrm{d}t\right|^2 \mathrm{d}\boldsymbol{x} + \frac{1}{2\varrho_2}\int_{\Omega_2}\left|\int_0^\eta \nabla p_2 \mathrm{d}t\right|^2 \mathrm{d}\boldsymbol{x} + \frac{1}{2\varrho_3}\int_{\Omega_3}\left|\int_0^\eta \nabla p_3 \mathrm{d}t\right|^2 \mathrm{d}\boldsymbol{x} \\
&= \frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Gamma_+}(\mathscr{T}_+ p_1)\bar{\psi}_1 \mathrm{d}\boldsymbol{r}\mathrm{d}\tau + \frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Gamma_-}(\mathscr{T}_- p_2)\bar{\psi}_2 \mathrm{d}\boldsymbol{r}\mathrm{d}\tau - \frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Omega_1} g\bar{\psi}_1 \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\quad + \Re\int_0^\eta\int_{\Gamma_1}(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\boldsymbol{u}_1)\bar{p}_2 \mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t + \Re\int_0^\eta\int_{\Gamma_3}(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t\boldsymbol{u}_2)\bar{p}_3 \mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t. \qquad (3.82)\ \boxed{\mathrm{RE12}}
\end{aligned}$$

For $j = 1, 2$, we define

$$\boldsymbol{\varphi}_j(\boldsymbol{x},t) = \int_t^\eta \partial_\tau \boldsymbol{u}_j(\boldsymbol{x},\tau)\mathrm{d}\tau, \qquad \boldsymbol{x}\in D_j,\ 0\le t\le\eta, \qquad (3.83)\ ?\underline{\mathrm{RE13}}?$$

which satisfy $\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_2$ on Γ_2 and

$$\boldsymbol{\varphi}_j(\boldsymbol{x},\eta) = 0, \qquad \partial_t\boldsymbol{\varphi}_j(\boldsymbol{x},t) = -\partial_t\boldsymbol{u}_j(\boldsymbol{x},t). \qquad (3.84)\ \boxed{\mathrm{RE14}}$$

Using a similar proof as that for (3.73), for any $\boldsymbol{\phi}_j(\boldsymbol{x},t)\in L^2(0,\eta;L^2(D_j)^3)$, we may show that

$$\int_0^\eta \boldsymbol{\phi}_j(\boldsymbol{x},t)\cdot\overline{\boldsymbol{\varphi}}_j(\boldsymbol{x},t)\mathrm{d}t = -\int_0^\eta\left[\left(\int_0^t \boldsymbol{\phi}_j(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\partial_t\overline{\boldsymbol{\varphi}}_j(\boldsymbol{x},t)\right]\mathrm{d}t, \qquad j=1,2. \qquad (3.85)\ ?\underline{\mathrm{RE15}}?$$

For $j = 1, 2$, multiplying $(\mu_j\Delta + (\lambda_j+\mu_j)\nabla\nabla\cdot - \rho_j\partial_t^2)\partial_t\boldsymbol{u}_j(\boldsymbol{x},t) = 0$ by a test function $\overline{\boldsymbol{\varphi}}_j$. Taking the inner products, using the integration by parts and boundary conditions, we arrive at

$$\begin{aligned}
0 &= \int_{D_1}\left[\mu_1(\nabla(\partial_t\boldsymbol{u}_1):\nabla\overline{\boldsymbol{\varphi}}_1) + (\lambda_1+\mu_1)(\nabla\cdot(\partial_t\boldsymbol{u}_1))(\nabla\cdot\overline{\boldsymbol{\varphi}}_1) + \rho_1(\partial_t^2(\partial_t\boldsymbol{u}_1))\cdot\overline{\boldsymbol{\varphi}}_1\right]\mathrm{d}\boldsymbol{x} \\
&\quad - \int_{\Gamma_1}\overline{\boldsymbol{\varphi}}_1\cdot(\boldsymbol{\nu}_{\Gamma_1}\cdot\boldsymbol{\sigma}_1(\partial_t\boldsymbol{u}_1))\mathrm{d}s_{\boldsymbol{x}} - \int_{\Gamma_2}\overline{\boldsymbol{\varphi}}_1\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\partial_t\boldsymbol{u}_1))\mathrm{d}s_{\boldsymbol{x}} \\
&= \int_{D_1}\left[\mu_1(\nabla(\partial_t\boldsymbol{u}_1):\nabla\overline{\boldsymbol{\varphi}}_1) + (\lambda_1+\mu_1)(\nabla\cdot(\partial_t\boldsymbol{u}_1))(\nabla\cdot\overline{\boldsymbol{\varphi}}_1) + \rho_1(\partial_t^2(\partial_t\boldsymbol{u}_1))\cdot\overline{\boldsymbol{\varphi}}_1\right]\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\Gamma_1}(\partial_t p_2)(\boldsymbol{\nu}_{\Gamma_1}\cdot\overline{\boldsymbol{\varphi}}_1)\mathrm{d}s_{\boldsymbol{x}} - \int_{\Gamma_2}\overline{\boldsymbol{\varphi}}_1\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_1(\partial_t\boldsymbol{u}_1))\mathrm{d}s_{\boldsymbol{x}}, \qquad (3.86)\ \boxed{\mathrm{RE16}}
\end{aligned}$$

and

$$\begin{aligned}
0 &= \int_{D_2}\left[\mu_2(\nabla(\partial_t\boldsymbol{u}_2):\nabla\overline{\boldsymbol{\varphi}}_2) + (\lambda_2+\mu_2)(\nabla\cdot(\partial_t\boldsymbol{u}_2))(\nabla\cdot\overline{\boldsymbol{\varphi}}_2) + \rho_2(\partial_t^2(\partial_t\boldsymbol{u}_2))\cdot\overline{\boldsymbol{\varphi}}_2\right]\mathrm{d}\boldsymbol{x} \\
&\quad - \int_{\Gamma_3}\overline{\boldsymbol{\varphi}}_2\cdot(\boldsymbol{\nu}_{\Gamma_3}\cdot\boldsymbol{\sigma}_2(\partial_t\boldsymbol{u}_2))\mathrm{d}s_{\boldsymbol{x}} + \int_{\Gamma_2}\overline{\boldsymbol{\varphi}}_2\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\partial_t\boldsymbol{u}_2))\mathrm{d}s_{\boldsymbol{x}} \\
&= \int_{D_2}\left[\mu_2(\nabla(\partial_t\boldsymbol{u}_2):\nabla\overline{\boldsymbol{\varphi}}_2) + (\lambda_2+\mu_2)(\nabla\cdot(\partial_t\boldsymbol{u}_2))(\nabla\cdot\overline{\boldsymbol{\varphi}}_2) + \rho_2(\partial_t^2(\partial_t\boldsymbol{u}_2))\cdot\overline{\boldsymbol{\varphi}}_2\right]\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\Gamma_3}(\partial_t p_3)(\boldsymbol{\nu}_{\Gamma_3}\cdot\overline{\boldsymbol{\varphi}}_2)\mathrm{d}s_{\boldsymbol{x}} + \int_{\Gamma_2}\overline{\boldsymbol{\varphi}}_2\cdot(\boldsymbol{\nu}_{\Gamma_2}\cdot\boldsymbol{\sigma}_2(\partial_t\boldsymbol{u}_2))\mathrm{d}s_{\boldsymbol{x}}. \qquad (3.87)\ \boxed{\mathrm{RE17}}
\end{aligned}$$

Combining (3.86)-(3.87) and the boundary condition on Γ_j ($j = 1, 2, 3$) we find

$$\begin{aligned}
&\int_{D_1} \Big[\mu_1(\nabla(\partial_t \boldsymbol{u}_1) : \nabla\overline{\boldsymbol{\varphi}}_1) + (\lambda_1 + \mu_1)(\nabla \cdot (\partial_t \boldsymbol{u}_1))(\nabla \cdot \overline{\boldsymbol{\varphi}}_1) + \rho_1(\partial_t^2(\partial_t \boldsymbol{u}_1)) \cdot \overline{\boldsymbol{\varphi}}_1\Big]\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{D_2} \Big[\mu_2(\nabla(\partial_t \boldsymbol{u}_2) : \nabla\overline{\boldsymbol{\varphi}}_2) + (\lambda_2 + \mu_2)(\nabla \cdot (\partial_t \boldsymbol{u}_2))(\nabla \cdot \overline{\boldsymbol{\varphi}}_2) + \rho_2(\partial_t^2(\partial_t \boldsymbol{u}_2)) \cdot \overline{\boldsymbol{\varphi}}_2\Big]\mathrm{d}\boldsymbol{x} \\
&= -\int_{\Gamma_1} (\partial_t p_2)(\boldsymbol{\nu}_{\Gamma_1} \cdot \overline{\boldsymbol{\varphi}}_1)\mathrm{d}s_{\boldsymbol{x}} - \int_{\Gamma_3} (\partial_t p_3)(\boldsymbol{\nu}_{\Gamma_3} \cdot \overline{\boldsymbol{\varphi}}_2)\mathrm{d}s_{\boldsymbol{x}},
\end{aligned} \tag{3.88}$$
?RE18?

which satisfies

$$\begin{aligned}
&\int_0^\eta \int_{D_1} \Big[\mu_1(\nabla(\partial_t \boldsymbol{u}_1) : \nabla\overline{\boldsymbol{\varphi}}_1) + (\lambda_1 + \mu_1)(\nabla \cdot (\partial_t \boldsymbol{u}_1))(\nabla \cdot \overline{\boldsymbol{\varphi}}_1) + \rho_1(\partial_t^2(\partial_t \boldsymbol{u}_1)) \cdot \overline{\boldsymbol{\varphi}}_1\Big]\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\quad + \int_0^\eta \int_{D_2} \Big[\mu_2(\nabla(\partial_t \boldsymbol{u}_2) : \nabla\overline{\boldsymbol{\varphi}}_2) + (\lambda_2 + \mu_2)(\nabla \cdot (\partial_t \boldsymbol{u}_2))(\nabla \cdot \overline{\boldsymbol{\varphi}}_2) + \rho_2(\partial_t^2(\partial_t \boldsymbol{u}_2)) \cdot \overline{\boldsymbol{\varphi}}_2\Big]\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&= -\int_0^\eta \int_{\Gamma_1} (\partial_t p_2)(\boldsymbol{\nu}_{\Gamma_1} \cdot \overline{\boldsymbol{\varphi}}_1)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t - \int_0^\eta \int_{\Gamma_3} (\partial_t p_3)(\boldsymbol{\nu}_{\Gamma_3} \cdot \overline{\boldsymbol{\varphi}}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t.
\end{aligned} \tag{3.89}$$
RE19

It follows from (3.84) and the initial conditions in (3.24) that

$$\begin{aligned}
&\Re \int_0^\eta \int_{D_j} \rho_j(\partial_t^2(\partial_t \boldsymbol{u}_j)) \cdot \overline{\boldsymbol{\varphi}}_j \mathrm{d}\boldsymbol{x}\mathrm{d}t = \rho_j \Re \int_{D_j} \int_0^\eta [\partial_t(\partial_t^2 \boldsymbol{u}_j \cdot \overline{\boldsymbol{\varphi}}_j) - (\partial_t^2 \boldsymbol{u}_j \cdot \partial_t \overline{\boldsymbol{\varphi}}_j)]\mathrm{d}t\mathrm{d}\boldsymbol{x} \\
&= \rho_j \Re \int_{D_j} \int_0^\eta [\partial_t(\partial_t^2 \boldsymbol{u}_j \cdot \overline{\boldsymbol{\varphi}}_j) + (\partial_t^2 \boldsymbol{u}_j \cdot \partial_t \overline{\boldsymbol{u}}_j)]\mathrm{d}t\mathrm{d}\boldsymbol{x} \\
&= \rho_j \Re \int_{D_j} \Big[(\partial_t^2 \boldsymbol{u}_j \cdot \overline{\boldsymbol{\varphi}}_j)\big|_0^\eta + \frac{1}{2}|\partial_t \boldsymbol{u}_j|^2\big|_0^\eta\Big]\mathrm{d}\boldsymbol{x} = 0 + \frac{\rho_j}{2}\int_{D_j} |\partial_t \boldsymbol{u}_j(\cdot, \eta)|^2 \mathrm{d}\boldsymbol{x} \\
&= \frac{\rho_j}{2}\|\partial_t \boldsymbol{u}_j(\cdot, \eta)\|^2_{L^2(D_j)^3}, \qquad\qquad j = 1, 2,
\end{aligned} \tag{3.90}$$
?RE20?

and

$$\begin{aligned}
&\Re \int_0^\eta \int_{\Gamma_1} (\partial_t p_2)(\boldsymbol{\nu}_{\Gamma_1} \cdot \overline{\boldsymbol{\varphi}}_1)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t = \Re \int_{\Gamma_1} \int_0^\eta [\partial_t(p_2 \boldsymbol{\nu}_{\Gamma_1} \cdot \overline{\boldsymbol{\varphi}}_1) - (p_2 \boldsymbol{\nu}_{\Gamma_1} \cdot \partial_t \overline{\boldsymbol{\varphi}}_1)]\mathrm{d}t\mathrm{d}\boldsymbol{x} \\
&= \Re \int_{\Gamma_1} \int_0^\eta [\partial_t(p_2 \boldsymbol{\nu}_{\Gamma_1} \cdot \overline{\boldsymbol{\varphi}}_1) + (p_2 \boldsymbol{\nu}_{\Gamma_1} \cdot \partial_t \overline{\boldsymbol{u}}_1)]\mathrm{d}t\mathrm{d}\boldsymbol{x} \\
&= \Re \int_{\Gamma_1} (p_2 \boldsymbol{\nu}_{\Gamma_1} \cdot \overline{\boldsymbol{\varphi}}_1)\big|_0^\eta \mathrm{d}\boldsymbol{x} + \Re \int_0^\eta \int_{\Gamma_1} p_2(\boldsymbol{\nu}_{\Gamma_1} \cdot \partial_t \overline{\boldsymbol{u}}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&= \Re \int_0^\eta \int_{\Gamma_1} p_2(\boldsymbol{\nu}_{\Gamma_1} \cdot \partial_t \overline{\boldsymbol{u}}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}t,
\end{aligned} \tag{3.91}$$
?RE21?

and

$$\Re \int_0^\eta \int_{\Gamma_3} (\partial_t p_3)(\boldsymbol{\nu}_{\Gamma_3} \cdot \overline{\boldsymbol{\varphi}}_2)\mathrm{d}s_{\boldsymbol{x}}\mathrm{d}t = \Re \int_0^\eta \int_{\Gamma_3} p_3(\boldsymbol{\nu}_{\Gamma_3} \cdot \partial_t \overline{\boldsymbol{u}}_2)\mathrm{d}\boldsymbol{x}\mathrm{d}t. \tag{3.92}$$
?RE22?

It is easy to verify that

$$\begin{aligned}
&\Re\int_0^\eta\int_{D_j}\mu_j[\nabla(\partial_t\boldsymbol{u}_j):\nabla\overline{\boldsymbol{\varphi}}_j]\mathrm{d}\boldsymbol{x}\mathrm{d}t=\frac{1}{2}\int_{D_j}\mu_j\left|\int_0^\eta\nabla(\partial_t\boldsymbol{u}_j)\mathrm{d}t\right|^2_{L^2(D_j)^{3\times3}}\mathrm{d}\boldsymbol{x}\\
&=\frac{\mu_j}{2}\int_{D_j}\left[\left(\int_0^\eta\nabla(\partial_t\boldsymbol{u}_j)\mathrm{d}t\right):\left(\int_0^\eta\nabla(\partial_t\overline{\boldsymbol{u}}_j)\mathrm{d}t\right)\right]\mathrm{d}\boldsymbol{x}\\
&=\frac{\mu_j}{2}\|\nabla\boldsymbol{u}_j(\cdot,\eta)\|^2_{L^2(D_j)^{3\times3}},\qquad j=1,2,
\end{aligned}\tag{3.93}$$?RE23?

and

$$\begin{aligned}
&\Re\int_0^\eta\int_{D_j}(\lambda_j+\mu_j)[\nabla\cdot(\partial_t\boldsymbol{u}_j)(\nabla\cdot\overline{\boldsymbol{\varphi}}_j)]\mathrm{d}\boldsymbol{x}\mathrm{d}t=\frac{\lambda_j+\mu_j}{2}\int_{D_j}\left|\int_0^\eta\nabla\cdot(\partial_t\boldsymbol{u}_j)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}\\
&=\frac{\lambda_j+\mu_j}{2}\int_{D_j}|\nabla\cdot\boldsymbol{u}_j(\cdot,\eta)|^2\mathrm{d}\boldsymbol{x}=\frac{\lambda_j+\mu_j}{2}\|\nabla\cdot\boldsymbol{u}_j(\cdot,\eta)\|^2_{L^2(D_j)},\qquad j=1,2.
\end{aligned}\tag{3.94}$$ RE24

It follows from (3.89)-(3.94) that

$$\begin{aligned}
&\frac{\mu_1}{2}\|\nabla\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)^{3\times3}}+\frac{\lambda_1+\mu_1}{2}\|\nabla\cdot\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)}+\frac{\rho_1}{2}\|\partial_t\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)^3}\\
&\quad+\frac{\mu_2}{2}\|\nabla\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)^{3\times3}}+\frac{\lambda_2+\mu_2}{2}\|\nabla\cdot\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)}+\frac{\rho_2}{2}\|\partial_t\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)^3}\\
&=-\Re\int_0^\eta\int_{\Gamma_1}p_2(\boldsymbol{\nu}_{\Gamma_1}\cdot\partial_t\overline{\boldsymbol{u}}_1)\mathrm{d}\boldsymbol{x}\mathrm{d}t-\Re\int_0^\eta\int_{\Gamma_3}p_3(\boldsymbol{\nu}_{\Gamma_3}\cdot\partial_t\overline{\boldsymbol{u}}_2)\mathrm{d}\boldsymbol{x}\mathrm{d}t,
\end{aligned}\tag{3.95}$$ RE25

Combining (3.82) and (3.95), we find

$$\begin{aligned}
&\frac{1}{2\varrho_2c_1^2}\|p_1(\cdot,\eta)\|^2_{L^2(\Omega_1)}+\frac{1}{2\varrho_2c_2^2}\|p_2(\cdot,\eta)\|^2_{L^2(\Omega_2)}+\frac{1}{2\varrho_3c_3^2}\|p_1(\cdot,\eta)\|^2_{L^2(\Omega_3)}\\
&\quad+\frac{\gamma_1}{\varrho_2c_1^2}\int_{\Omega_1}\int_0^\eta|p_1|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}+\frac{\gamma_2}{\varrho_2c_2^2}\int_{\Omega_2}\int_0^\eta|p_2|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}+\frac{\gamma_3}{\varrho_3c_3^2}\int_{\Omega_1}\int_0^\eta|p_3|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{1}{2\varrho_2}\int_{\Omega_1}\left|\int_0^\eta\nabla p_1\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}+\frac{1}{2\varrho_2}\int_{\Omega_2}\left|\int_0^\eta\nabla p_2\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}+\frac{1}{2\varrho_3}\int_{\Omega_3}\left|\int_0^\eta\nabla p_3\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{\mu_1}{2}\|\nabla\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)^{3\times3}}+\frac{\lambda_1+\mu_1}{2}\|\nabla\cdot\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)}+\frac{\rho_1}{2}\|\partial_t\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)^3}\\
&\quad+\frac{\mu_2}{2}\|\nabla\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)^{3\times3}}+\frac{\lambda_2+\mu_2}{2}\|\nabla\cdot\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)}+\frac{\rho_2}{2}\|\partial_t\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)^3}\\
&=\frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Gamma_+}(\mathscr{T}_+p_1)\bar{\psi}_1\mathrm{d}\boldsymbol{r}\mathrm{d}\tau+\frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Gamma_-}(\mathscr{T}_-p_2)\bar{\psi}_2\mathrm{d}\boldsymbol{r}\mathrm{d}\tau-\frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Omega_1}g\bar{\psi}_1\mathrm{d}\boldsymbol{x}\mathrm{d}t.
\end{aligned}\tag{3.96}$$ RE26

Using Lemma 2.5 and (3.73), we can obtain

$$\begin{aligned}
&\Re\int_0^\eta\int_{\Gamma_+}(\mathscr{T}_+p_1)\bar{\psi}_1\mathrm{d}\boldsymbol{r}\mathrm{d}\tau=\Re\int_{\Gamma_+}\int_0^\eta\left(\int_0^t(\mathscr{T}_+p_1)(\cdot,\tau)\mathrm{d}\tau\right)\bar{p}_1(\cdot,t)\mathrm{d}t\mathrm{d}\boldsymbol{r}\le0,\\
&\Re\int_0^\eta\int_{\Gamma_-}(\mathscr{T}_-p_2)\bar{\psi}_2\mathrm{d}\boldsymbol{r}\mathrm{d}\tau=\Re\int_{\Gamma_-}\int_0^\eta\left(\int_0^t(\mathscr{T}_-p_2)(\cdot,\tau)\mathrm{d}\tau\right)\bar{p}_2(\cdot,t)\mathrm{d}t\mathrm{d}\boldsymbol{r}\le0.
\end{aligned}\tag{3.97}$$?RE27?

Noting that from (3.71), we have

$$
\begin{aligned}
&\Re\int_0^\eta\int_{\Omega_1} g\bar{\psi}_1 \mathrm{d}\boldsymbol{x}\mathrm{d}t = \Re\int_0^\eta\int_{\Omega_1} g\Big(\int_t^\eta \bar{p}_1(\boldsymbol{x},\tau)\mathrm{d}\tau\Big)\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\le \int_0^\eta\left(\int_0^\eta \|g(\cdot,t)\|_{L^2(\Omega_1)}\mathrm{d}t\right)\|p_1(\cdot,t)\|_{L^2(\Omega_1)}\mathrm{d}t \\
&= \left(\int_0^\eta \|g(\cdot,t)\|_{L^2(\Omega_1)}\mathrm{d}t\right)\left(\int_0^\eta \|p_1(\cdot,t)\|_{L^2(\Omega_1)}\mathrm{d}t\right).
\end{aligned} \tag{3.98}
$$

RE029

Then, from (3.96)-(3.98), we have

$$
\begin{aligned}
&\frac{1}{2\varrho_2 c_1^2}\|p_1(\cdot,\eta)\|^2_{L^2(\Omega_1)} + \frac{1}{2\varrho_2 c_2^2}\|p_2(\cdot,\eta)\|^2_{L^2(\Omega_2)} + \frac{1}{2\varrho_3 c_3^2}\|p_1(\cdot,\eta)\|^2_{L^2(\Omega_3)} \\
&\quad + \frac{\gamma_1}{\varrho_2 c_1^2}\int_{\Omega_1}\int_0^\eta |p_1|^2\mathrm{d}t\mathrm{d}\boldsymbol{x} + \frac{\gamma_2}{\varrho_2 c_2^2}\int_{\Omega_2}\int_0^\eta |p_2|^2\mathrm{d}t\mathrm{d}\boldsymbol{x} + \frac{\gamma_3}{\varrho_3 c_3^2}\int_{\Omega_1}\int_0^\eta |p_3|^2\mathrm{d}t\mathrm{d}\boldsymbol{x} \\
&\quad + \frac{1}{2\varrho_2}\int_{\Omega_1}\left|\int_0^\eta \nabla p_1\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x} + \frac{1}{2\varrho_2}\int_{\Omega_2}\left|\int_0^\eta \nabla p_2\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x} + \frac{1}{2\varrho_3}\int_{\Omega_3}\left|\int_0^\eta \nabla p_3\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x} \\
&\quad + \frac{\mu_1}{2}\|\nabla\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)^{3\times 3}} + \frac{\lambda_1+\mu_1}{2}\|\nabla\cdot\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)} + \frac{\rho_1}{2}\|\partial_t\boldsymbol{u}_1(\cdot,\eta)\|^2_{L^2(D_1)^3} \\
&\quad + \frac{\mu_2}{2}\|\nabla\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)^{3\times 3}} + \frac{\lambda_2+\mu_2}{2}\|\nabla\cdot\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)} + \frac{\rho_2}{2}\|\partial_t\boldsymbol{u}_2(\cdot,\eta)\|^2_{L^2(D_2)^3} \\
&\le \frac{1}{\varrho_2}\left(\int_0^\eta \|g(\cdot,t)\|_{L^2(\Omega_1)}\mathrm{d}t\right)\left(\int_0^\eta \|p_1(\cdot,t)\|_{L^2(\Omega_1)}\mathrm{d}t\right) \\
&\le \frac{1}{\varrho_2}\left(\int_0^T \|g(\cdot,t)\|_{L^2(\Omega_1)}\mathrm{d}t\right)\left(\int_0^T \|p_1(\cdot,t)\|_{L^2(\Omega_1)}\mathrm{d}t\right) \\
&\le \frac{1}{\varrho_2}T\|g\|_{L^1(0,T;L^2(\Omega_1))}\|p_1\|_{L^\infty(0,T;H^1(\Omega_1))}.
\end{aligned} \tag{3.99}
$$

RE30

We define another auxiliary functions

$$
\widetilde{\psi}_l(\boldsymbol{x},t) = \int_t^\eta \partial_t p_l(\boldsymbol{x},\tau)\mathrm{d}\tau, \qquad \boldsymbol{x}\in\Omega_l,\ 0\le t\le\eta,\ l=1,2,3. \tag{3.100}
$$

?RE1-1?

$$
\widetilde{\boldsymbol{\varphi}}_j(\boldsymbol{x},t) = \int_t^\eta \partial_\tau^2\boldsymbol{u}_j(\boldsymbol{x},\tau)\mathrm{d}\tau, \qquad \boldsymbol{x}\in D_j,\ 0\le t\le\eta,\ j=1,2. \tag{3.101}
$$

?RE13-1?

In (3.47), proceeding as in the proof of (3.96), we can derive that

$$
\begin{aligned}
&\frac{1}{2\varrho_2 c_1^2}\|\partial_t p_1(\cdot,\eta)\|_{L^2(\Omega_1)}^2+\frac{1}{2\varrho_2 c_2^2}\|\partial_t p_2(\cdot,\eta)\|_{L^2(\Omega_2)}^2+\frac{1}{2\varrho_3 c_3^2}\|\partial_t p_1(\cdot,\eta)\|_{L^2(\Omega_3)}^2\\
&\quad+\frac{\gamma_1}{\varrho_2 c_1^2}\int_{\Omega_1}\int_0^\eta|\partial_t p_1|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}+\frac{\gamma_2}{\varrho_2 c_2^2}\int_{\Omega_2}\int_0^\eta|\partial_t p_2|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}+\frac{\gamma_3}{\varrho_3 c_3^2}\int_{\Omega_3}\int_0^\eta|\partial_t p_3|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{1}{2\varrho_2}\int_{\Omega_1}\left|\int_0^\eta\nabla(\partial_t p_1)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}+\frac{1}{2\varrho_2}\int_{\Omega_2}\left|\int_0^\eta\nabla(\partial_t p_2)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}+\frac{1}{2\varrho_3}\int_{\Omega_3}\left|\int_0^\eta\nabla(\partial_t p_3)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{\mu_1}{2}\int_{D_1}\left|\int_0^\eta\nabla(\partial_t^2\boldsymbol{u}_1)\mathrm{d}t\right|^2_{L^2(D_1)^{3\times3}}\mathrm{d}\boldsymbol{x}+\frac{\lambda_1+\mu_1}{2}\int_{D_1}\left|\int_0^\eta\nabla\cdot(\partial_t^2\boldsymbol{u}_1)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}+\frac{\rho_1}{2}\|\partial_t^2\boldsymbol{u}_1(\cdot,\eta)\|_{L^2(D_1)^3}^2\\
&\quad+\frac{\mu_2}{2}\int_{D_2}\left|\int_0^\eta\nabla(\partial_t^2\boldsymbol{u}_2)\mathrm{d}t\right|^2_{L^2(D_2)^{3\times3}}\mathrm{d}\boldsymbol{x}+\frac{\lambda_2+\mu_2}{2}\int_{D_2}\left|\int_0^\eta\nabla\cdot(\partial_t^2\boldsymbol{u}_2)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{x}+\frac{\rho_2}{2}\|\partial_t^2\boldsymbol{u}_2(\cdot,\eta)\|_{L^2(D_2)^3}^2\\
&=\frac{1}{2\varrho_2 c_1^2}\|\partial_t p_1(\cdot,\eta)\|_{L^2(\Omega_1)}^2+\frac{1}{2\varrho_2 c_2^2}\|\partial_t p_2(\cdot,\eta)\|_{L^2(\Omega_2)}^2+\frac{1}{2\varrho_3 c_3^2}\|\partial_t p_1(\cdot,\eta)\|_{L^2(\Omega_3)}^2\\
&\quad+\frac{\gamma_1}{\varrho_2 c_1^2}\int_{\Omega_1}\int_0^\eta|\partial_t p_1|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}+\frac{\gamma_2}{\varrho_2 c_2^2}\int_{\Omega_2}\int_0^\eta|\partial_t p_2|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}+\frac{\gamma_3}{\varrho_3 c_3^2}\int_{\Omega_3}\int_0^\eta|\partial_t p_3|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{1}{2\varrho_2}\int_{\Omega_1}|\nabla p_1(\cdot,\eta)|^2\mathrm{d}\boldsymbol{x}+\frac{1}{2\varrho_2}\int_{\Omega_2}|\nabla p_2(\cdot,\eta)|^2\mathrm{d}\boldsymbol{x}+\frac{1}{2\varrho_3}\int_{\Omega_3}|\nabla p_3(\cdot,\eta)|^2\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{\mu_1}{2}\int_{D_1}|\nabla(\partial_t\boldsymbol{u}_1(\cdot,\eta))|^2_{L^2(D_1)^{3\times3}}\mathrm{d}\boldsymbol{x}+\frac{\lambda_1+\mu_1}{2}\int_{D_1}|\nabla\cdot(\partial_t\boldsymbol{u}_1(\cdot,\eta))|^2\mathrm{d}\boldsymbol{x}+\frac{\rho_1}{2}\|\partial_t^2\boldsymbol{u}_1(\cdot,\eta)\|_{L^2(D_1)^3}^2\\
&\quad+\frac{\mu_2}{2}\int_{D_2}|\nabla(\partial_t\boldsymbol{u}_2(\cdot,\eta))|^2_{L^2(D_2)^{3\times3}}\mathrm{d}\boldsymbol{x}+\frac{\lambda_2+\mu_2}{2}\int_{D_2}|\nabla\cdot(\partial_t\boldsymbol{u}_2(\cdot,\eta))|^2\mathrm{d}\boldsymbol{x}+\frac{\rho_2}{2}\|\partial_t^2\boldsymbol{u}_2(\cdot,\eta)\|_{L^2(D_2)^3}^2\\
&=\frac{1}{2\varrho_2 c_1^2}\|\partial_t p_1(\cdot,\eta)\|_{L^2(\Omega_1)}^2+\frac{1}{2\varrho_2 c_2^2}\|\partial_t p_2(\cdot,\eta)\|_{L^2(\Omega_2)}^2+\frac{1}{2\varrho_3 c_3^2}\|\partial_t p_1(\cdot,\eta)\|_{L^2(\Omega_3)}^2\\
&\quad+\frac{\gamma_1}{\varrho_2 c_1^2}\int_{\Omega_1}\int_0^\eta|\partial_t p_1|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}+\frac{\gamma_2}{\varrho_2 c_2^2}\int_{\Omega_2}\int_0^\eta|\partial_t p_2|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}+\frac{\gamma_3}{\varrho_3 c_3^2}\int_{\Omega_3}\int_0^\eta|\partial_t p_3|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{1}{2\varrho_2}\|\nabla p_1(\cdot,\eta)\|_{L^2(\Omega_1)}^2+\frac{1}{2\varrho_2}\|\nabla p_2(\cdot,\eta)\|_{L^2(\Omega_1)}^2+\frac{1}{2\varrho_3}\|\nabla p_3(\cdot,\eta)\|_{L^2(\Omega_1)}^2\\
&\quad+\frac{\mu_1}{2}\|\nabla\partial_t\boldsymbol{u}_1(\cdot,\eta)\|_{L^2(D_1)^{3\times3}}^2+\frac{\lambda_1+\mu_1}{2}\|\nabla\cdot\partial_t\boldsymbol{u}_1(\cdot,\eta)\|_{L^2(D_1)}^2+\frac{\rho_1}{2}\|\partial_t^2\boldsymbol{u}_1(\cdot,\eta)\|_{L^2(D_1)^3}^2\\
&\quad+\frac{\mu_2}{2}\|\nabla\partial_t\boldsymbol{u}_2(\cdot,\eta)\|_{L^2(D_2)^{3\times3}}^2+\frac{\lambda_2+\mu_2}{2}\|\nabla\cdot\partial_t\boldsymbol{u}_2(\cdot,\eta)\|_{L^2(D_2)}^2+\frac{\rho_2}{2}\|\partial_t^2\boldsymbol{u}_2(\cdot,\eta)\|_{L^2(D_2)^3}^2\\
&=\frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Gamma_+}(\mathscr{T}_+(\partial_t p_1))\overline{\overline{\psi}}_1\mathrm{d}\boldsymbol{r}\mathrm{d}t+\frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Gamma_-}(\mathscr{T}_-(\partial_t p_2))\overline{\overline{\psi}}_2\mathrm{d}\boldsymbol{r}\mathrm{d}t-\frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Omega_1}(\partial_t g)\overline{\psi}_1\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&\le-\frac{1}{\varrho_2}\Re\int_0^\eta\int_{\Omega_1}(\partial_t g)\overline{\psi}_1\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&\le\frac{1}{\varrho_2}\|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\|p_1\|_{L^\infty(0,T;H^1(\Omega_1))}.
\end{aligned}
\tag{3.102}
$$

RE30-1

Taking the L^∞ norm with respect to η on both sides of (3.99) and (3.102) yields

$$\begin{aligned}
&\|p_1\|^2_{L^\infty(0,T;L^2(\Omega_1))} + \|p_2\|^2_{L^\infty(0,T;L^2(\Omega_2))} + \|p_3\|^2_{L^\infty(0,T;L^2(\Omega_3))} \\
&\quad + \|\nabla p_1\|^2_{L^\infty(0,T;L^2(\Omega_1))} + \|\nabla p_2\|^2_{L^\infty(0,T;L^2(\Omega_2))} + \|\nabla p_3\|^2_{L^\infty(0,T;L^2(\Omega_3))} \\
&\quad + \|\nabla \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2(D_1)^{3\times 3})} + \|\nabla\cdot \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2(D_1))} + \|\partial_t \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2(D_1))} \\
&\quad + \|\nabla \boldsymbol{u}_2\|^2_{L^\infty(0,T;L^2(D_2)^{3\times 3})} + \|\nabla\cdot \boldsymbol{u}_2\|^2_{L^\infty(0,T;L^2(D_2))} + \|\partial_t \boldsymbol{u}_2\|^2_{L^\infty(0,T;L^2(D_2))} \\
&\le \tilde{M}_3 \left(T\|g\|_{L^1(0,T;L^2(\Omega_1))} + \|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right) \|p_1\|_{L^\infty(0,T;H^1(\Omega_1))}.
\end{aligned} \tag{3.103} \text{ ?RE31?}$$

With the help of the Cauchy-Schwarz inequality, we yields

$$\begin{aligned}
&\|p_1\|^2_{L^\infty(0,T;L^2(\Omega_1))} + \|p_2\|^2_{L^\infty(0,T;L^2(\Omega_2))} + \|p_3\|^2_{L^\infty(0,T;L^2(\Omega_3))} \\
&\quad + \|\nabla p_1\|^2_{L^\infty(0,T;L^2(\Omega_1))} + \|\nabla p_2\|^2_{L^\infty(0,T;L^2(\Omega_2))} + \|\nabla p_3\|^2_{L^\infty(0,T;L^2(\Omega_3))} \\
&\quad + \|\nabla \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2(D_1)^{3\times 3})} + \|\nabla\cdot \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2(D_1))} + \|\partial_t \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2(D_1))} \\
&\quad + \|\nabla \boldsymbol{u}_2\|^2_{L^\infty(0,T;L^2(D_2)^{3\times 3})} + \|\nabla\cdot \boldsymbol{u}_2\|^2_{L^\infty(0,T;L^2(D_2))} + \|\partial_t \boldsymbol{u}_2\|^2_{L^\infty(0,T;L^2(D_2))} \\
&\le \tilde{M}_3^2 \left(T\|g\|_{L^1(0,T;L^2(\Omega_1))} + \|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right)^2,
\end{aligned} \tag{3.104} \text{ ?RE32?}$$

which implies that

$$\begin{aligned}
&\sum_{l=1}^{3} \left[\|p_l\|_{L^\infty(0,T;L^2(\Omega_l))} + \|\nabla p_l\|_{L^\infty(0,T;L^2(\Omega_l))} \right] \\
&\le M_3 \left(T\|g\|_{L^1(0,T;L^2(\Omega_1))} + \|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right),
\end{aligned} \tag{3.105} \text{ ?RE33?}$$

and

$$\begin{aligned}
&\sum_{j=1}^{2} \left[\|\nabla \boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j)^{3\times 3})} + \|\nabla\cdot \boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j))} + \|\partial_t \boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j))} \right] \\
&\le M_3 \left(T\|g\|_{L^1(0,T;L^2(\Omega_1))} + \|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right),
\end{aligned} \tag{3.106} \text{ ?RE34?}$$

where $\tilde{M}_3$ and M_3 are positive constants. Furthermore, we can get

$$\begin{aligned}
&\sum_{l=1}^{3} \left[\|p_l\|_{L^2(0,T;L^2(\Omega_l))} + \|\nabla p_l\|_{L^2(0,T;L^2(\Omega_l))} \right] \\
&\le T^{\frac{1}{2}} \sum_{l=1}^{3} \left[\|p_l\|_{L^\infty(0,T;L^2(\Omega_l))} + \|\nabla p_l\|_{L^\infty(0,T;L^2(\Omega_l))} \right] \\
&\le M_3 \left(T^{\frac{3}{2}}\|g\|_{L^1(0,T;L^2(\Omega_1))} + T^{\frac{1}{2}}\|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right),
\end{aligned} \tag{3.107} \text{ ?RE35?}$$

and

$$\begin{aligned}
&\sum_{j=1}^{2} \left[\|\nabla \boldsymbol{u}_j\|_{L^2(0,T;L^2(D_j)^{3\times 3})} + \|\nabla\cdot \boldsymbol{u}_j\|_{L^2(0,T;L^2(D_j))} + \|\partial_t \boldsymbol{u}_j\|_{L^2(0,T;L^2(D_j))} \right] \\
&\le T^{\frac{1}{2}} \sum_{j=1}^{2} \left[\|\nabla \boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j)^{3\times 3})} + \|\nabla\cdot \boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j))} + \|\partial_t \boldsymbol{u}_j\|_{L^\infty(0,T;L^2(D_j))} \right] \\
&\le M_3 \left(T^{\frac{3}{2}}\|g\|_{L^1(0,T;L^2(\Omega_1))} + T^{\frac{1}{2}}\|\partial_t g\|_{L^1(0,T;L^2(\Omega_1))}\right).
\end{aligned} \tag{3.108} \text{ ?RE36?}$$

□

Acknowledgments

The work of L. Zhang was supported by NSFC (No.11871198, No.11801116) and the Scientific Research Starting Foundation (No. 2022109001429).

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