A new complex structure-preserving method for QSVD [1]

Cui-E Yu[1], **Xin Liu**[2,*], **Yang Zhang**[3]

[1, 2]Faculty of Innovation Engineering, Macau University of Science and Technology, Avenida Wai Long, TaiPa, Macau, 999078, P. R. China.
[3] Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T 2N2, Canada.

Abstract

In this paper, we propose a new structure-preserving algorithm for computing the singular value decomposition of a quaternion matrix A. We first define a quaternion-type matrix and prove that the multiplication of two quaternion-type matrices still be a quaternion-type matrix. Thus, utilizing this fact, we conduct a sequence of quaternion-type unitary transformations on a half of the elements of the complex adjoint matrix χ_A of A instead of on the whole χ_A. Then, we recover the resulting matrix with the help of the special structures. Compared with direct performing on the complex adjoint matrix, our algorithm needs only half of the computation and storage. This method also provides a novel proof for the existence of the singular value decomposition of a quaternion matrix. Moreover, numerical experiments are given to demonstrate the validity of our approach.

Key words: Quaternion matrix; Singular value decomposition; Householder reflection; Givens rotation.

AMS 2010 Subject Classification: 15A04, 15B33, 39B42.

1 Introduction

It is well-known that the singular value decompositions (SVD) of matrices over real and complex number fields have important applications in many areas such as engineering, data science and signal/image processing. Motivated by increasing applications of quaternion matrices during the past two decades, people have been working on the singular value decompositions of quaternion matrices, and some algorithms for computing the singular value decompositions of quaternion matrices (QSVD) have been proposed.

In general, there are two main ways to compute the QSVD. One way is using quaternionic transformations directly. For instance, Bihan and Sangwine [12] used the quaternionic Householder transformations. In [2], they applied a quaternionic Jacobi method to directly compute the QSVD.

[1]This research is supported by Macao Science and Technology Development Fund (No. 0013/2021/ITP), the grants from the National Natural Science Foundation of China (11571220), and the Natural Sciences and Engineering Research Council of Canada (NSERC) (RGPIN 2020-06746), The joint research and Development fund of Wuyi University, Hong Kong and Macao (2019WGALH20).

* Corresponding author E-mail: xiliu@must.edu.mo

Another way is using real/complex representation methods to get the QSVD. For instance, Li et al. [6] proposed a structure-preserving algorithm for QSVD by using the real counterpart of a quaternion matrix. Bihan and Mars [1] and Pei et al. [10] used a complex adjoint matrix of a quaternion matrix A. They derived the QSVD of $A = U\Sigma V^*$ from the SVD of its complex adjoint matrix χ_A. All of above papers focus on how to find Σ instead of finding U and V from the SVD of the χ_A. Except that, their methods lead to more computational round-off errors and could destroy the particular structure of the complex adjoint matrix (see [2]).

To overcome the disadvantages above, we propose an efficient algorithm to compute QSVD for a given quaternion matrix in this paper. By a successive of structure-preserving unitary transformations on a half of the elements of χ_A, we can derive the QSVD of the quaternion matrix A and produce U, Σ, and V at the same time. Moreover, this new proposed algorithm results in a reduction by half of memory requirements and the computational effort compared with the algorithms proposed in [1] and [10].

This paper is organized as follows. In Section 2, we first recall some basic results. In Section 3, we develop a new method to compute QSVD by utilizing Givens rotations and Householder reflections. Two numerical experiments are presented in Section 4.

Throughout this paper, the symbols $\overline{A}$ and A^* stand for the conjugate and the conjugate transpose of a matrix A, respectively. The notations $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ denote the real number field, the complex number field and the quaternion skew field, respectively. Also, $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$ and $\mathbb{H}^{m\times n}$ represent the set of all $m \times n$ matrices over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, respectively. The $n \times n$ identity matrix is denoted by I_n, and e_i denotes the i-th column of an identity matrix.

2 Preliminaries

For any quaternion $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$, it can be rewritten as $q = c + sj$, where $c = q_0 + q_1 i,\ s = q_2 + q_3 i \in \mathbb{C}$, and its 2×2 matrix representation is $\begin{bmatrix} c & s \\ -\overline{s} & \overline{c} \end{bmatrix}$. The conjugate of q is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k = \overline{c} - sj$, and the norm of q is defined as

$$|q| = \sqrt{\overline{q}q} = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = \sqrt{|c|^2 + |s|^2}.$$

Therefore,

$$\begin{bmatrix} \overline{c} & -s \\ \overline{s} & c \end{bmatrix} \begin{bmatrix} c & s \\ -\overline{s} & \overline{c} \end{bmatrix} = \begin{bmatrix} c & s \\ -\overline{s} & \overline{c} \end{bmatrix} \begin{bmatrix} \overline{c} & -s \\ \overline{s} & c \end{bmatrix} = \begin{bmatrix} |c|^2 + |s|^2 & 0 \\ 0 & |c|^2 + |s|^2 \end{bmatrix}.$$

Let $F = \frac{1}{\sqrt{|c|^2+|s|^2}} \begin{bmatrix} \overline{c} & -s \\ \overline{s} & c \end{bmatrix}$. Then $FF^* = F^*F = I_2$, and thus, F is unitary.

Definition 2.1 *Given two complex matrices $A, B \in \mathbb{C}^{m\times n}$, we can define the quaternion-type matrix as following:*

$$\begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix} \in \mathbb{C}^{2m\times 2n}. \tag{1}$$

Clearly, there is a one-to-one correspondence between quaternion-type matrices and quaternion matrices with compatible sizes. Furthermore, for any $C, D \in \mathbb{C}^{n\times l}$,

$$\begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix} \begin{bmatrix} C & D \\ -\overline{D} & \overline{C} \end{bmatrix} = \begin{bmatrix} AC - B\overline{D} & AD + B\overline{C} \\ -\overline{A}\,\overline{D} - \overline{B}C & \overline{A}\,\overline{C} - \overline{B}D \end{bmatrix}.$$

Thus, we have the following simple result. But it will be critical for the design of the new algorithm for QSVD in Section 3.

Lemma 2.2 *The multiplication of two quaternion-type matrices with compatible sizes is also a quaternion-type matrix.*

Given a quaternion matrix $A = A_1 + A_2 j \in \mathbb{H}^{m\times n}$ with $A_1, A_2 \in \mathbb{C}^{m\times n}$. A well-known complex adjoint matrix of A is given by

$$\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}. \tag{2}$$

Obviously, by Definition 2.1, χ_A is a quaternion-type matrix which is corresponding to the quaternion matrix $A = A_1 + A_2 j$.

Next, we summarize some important properties of the complex adjoint matrix of a quaternion matrix. We refer to the readers to [11, 13] for more details.

Lemma 2.3 *Let* $A, B \in \mathbb{H}^{m\times n}$, $C \in \mathbb{H}^{n\times l}$, $D \in \mathbb{H}^{m\times m}$. *Then*

(1) $A = B \Longleftrightarrow \chi_A = \chi_B$;

(2) $\chi_{A+B} = \chi_A + \chi_B$;

(3) $\chi_{AC} = \chi_A \chi_C$;

(4) $\chi_{I_n} = I_{2n}$;

(5) $\chi_{A^*} = \chi_A^*$;

(6) χ_D *is unitary, Hermitian, or normal if and only if* D *is unitary, Hermitian, or normal, respectively;*

By Lemma 2.2, if some quaternion-type transformations $T_1, \ldots, T_k$ act on the complex adjoint matrix 2 of $A \in \mathbb{H}^{m\times n}$, then the resulting matrix $(T_k \cdots T_1)\chi_A$ is still a quaternion-type matrix.

3 QSVD based on structure-preserving unitary transformations

Recall that the Givens rotation can zero out a particular entry in a vector, and Householder reflection can be used to simultaneously zero out up to $n-1$ elements in an n-vectors. In this section, we will mix the Givens rotations and Householder transformations to derive our QSVD algorithms.

For any quaternion matrix $A = A_1 + A_2 j \in \mathbb{H}^{m\times n}$, without loss of generality, we assume $m \geq n$. Our method can be divided into two steps. In this first step, we will conduct a series of quaternion-type unitary matrices on a half of χ_A to derive a real upper-bidiagonal matrix $\hat{A}$, that is,

$$A = \begin{bmatrix} * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ * & * & * & \cdots & * & * \end{bmatrix} \longrightarrow \hat{A} = \begin{bmatrix} \times & \times & 0 & \cdots & 0 & 0 \\ 0 & \times & \times & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & \times & \times \\ 0 & 0 & 0 & \cdots & 0 & \times \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

In the second step, we derive the QSVD of A from the SVD of $\hat{A} = \hat{U}\Sigma\hat{V}^T$, that is,

$$\hat{A} = \begin{bmatrix} \times & \times & 0 & \cdots & 0 & 0 \\ 0 & \times & \times & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & \times & \times \\ 0 & 0 & 0 & \cdots & 0 & \times \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \longrightarrow \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & \sigma_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \sigma_n \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Now we show how step (1) works correctly. For $A = A_1 + A_2 j = (q_{ts})_{m\times n} = (a_{ts} + b_{ts} j)_{m\times n} \in \mathbb{H}^{m\times n}$, where $A_1, A_2 \in \mathbb{C}^{m\times n}$, $q_{ts} \in \mathbb{H}$ and $a_{ts}, b_{ts} \in \mathbb{C}, 1 \le t \le m, 1 \le s \le n$. We define a complex Givens rotation matrix for nonzero q_{ts} as follows:

$$G_{ts} = \begin{bmatrix} I_{t-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\bar{a}_{ts}}{|q_{ts}|} & 0 & 0 & -\frac{b_{ts}}{|q_{ts}|} & 0 \\ 0 & 0 & I_{m-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{t-1} & 0 & 0 \\ 0 & \frac{\bar{b}_{ts}}{|q_{ts}|} & 0 & 0 & \frac{a_{ts}}{|q_{ts}|} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{m-t} \end{bmatrix} \in \mathbb{C}^{2m\times 2m}.$$

When $q_{ts} = 0$, we just define $G_{ts} = I_{2m}$. It is easy to verify that G_{ts} is a unitary quaternion-type matrix by $|q_{ts}| = \sqrt{a_{ts}\bar{a}_{ts} + b_{ts}\bar{b}_{ts}}$. Let

$$B = \begin{bmatrix} A_1 \\ -\overline{A_2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ -\bar{b}_{11} & -\bar{b}_{12} & \cdots & -\bar{b}_{1n} \\ -\bar{b}_{21} & -\bar{b}_{22} & \cdots & -\bar{b}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{b}_{m1} & -\bar{b}_{m2} & \cdots & -\bar{b}_{mn} \end{bmatrix}.$$

Then multiplying G_{11} on the left-hand side of B, we obtain

$$G_{11}B = \begin{bmatrix} |q_{11}| & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ 0 & -\bar{b}_{12}^{(1)} & \cdots & -\bar{b}_{1n}^{(1)} \\ -\bar{b}_{21} & -\bar{b}_{22} & \cdots & -\bar{b}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{b}_{m1} & -\bar{b}_{m2} & \cdots & -\bar{b}_{mn} \end{bmatrix}.$$

Next multiplying G_{21} on the left side of $G_{11}B$,

$$G_{21}G_{11}B = \begin{bmatrix} |q_{11}| & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ |q_{21}| & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ 0 & -\bar{b}_{12}^{(1)} & \cdots & -\bar{b}_{1n}^{(1)} \\ 0 & -\bar{b}_{22}^{(1)} & \cdots & -\bar{b}_{2n}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{b}_{m1} & -\bar{b}_{m2} & \cdots & -\bar{b}_{mn} \end{bmatrix}.$$

In an analogous way, we have

$$G_{m1}\ldots G_{21}G_{11}B = \begin{bmatrix} |q_{11}| & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ |q_{21}| & a_{22}^{(1)} & \ldots & a_{2n}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ |q_{m1}| & a_{m2}^{(1)} & \cdots & a_{mn}^{(1)} \\ 0 & -\bar{b}_{12}^{(1)} & \cdots & -\bar{b}_{1n}^{(1)} \\ 0 & -\bar{b}_{22}^{(1)} & \cdots & -\bar{b}_{2n}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & -\bar{b}_{m2}^{(1)} & \cdots & -\bar{b}_{mn}^{(1)} \end{bmatrix}. \tag{3}$$

Denoting $G_{l1} = G_{m1} \cdots G_{21}G_{11}$. By Lemma 2.2, G_{l1} is still a $2m \times 2m$ unitary quaternion-type matrix.

For the real vector $v_1 = \begin{bmatrix} |q_{11}|, & |q_{21}|, & \cdots, & |q_{m1}| \end{bmatrix}^T$, we define a real Householder reflection H_1 such that $H_1 v_1 = \|v_1\| e_1$, where $\|v_1\| = \sqrt{|q_{11}|^2 + |q_{21}|^2 + \cdots + |q_{m1}|^2}$. We denote

$$H_{l1} = \begin{bmatrix} H_1 & 0 \\ 0 & H_1 \end{bmatrix}. \tag{4}$$

Clearly, H_{l1} is a $2m \times 2m$ quaternion-type unitary matrix, and

$$H_{l1}G_{l1}B = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1n}^{(2)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & a_{m2}^{(2)} & \cdots & a_{mn}^{(2)} \\ 0 & -\bar{b}_{12}^{(2)} & \cdots & -\bar{b}_{1n}^{(2)} \\ 0 & -\bar{b}_{22}^{(2)} & \cdots & -\bar{b}_{2n}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & -\bar{b}_{m2}^{(2)} & \cdots & -\bar{b}_{mn}^{(2)} \end{bmatrix} \triangleq \begin{bmatrix} A_1^{(1)} \\ \hline -\overline{A_2^{(1)}} \end{bmatrix}, \tag{5}$$

where $a_{11}^{(2)} = \|v_1\|$, $A_1^{(1)}, A_2^{(1)} \in \mathbb{C}^{m\times n}$.

By Lemma 2.2, $H_{l1}G_{l1}\chi_A$ should be a quaternion-type matrix. Note that $H_{l1}G_{l1}B$ is the first block column of $H_{l1}G_{l1}\chi_A$. Thus, we can recover from (5) that

$$H_{l1}G_{l1}\chi_A = \begin{bmatrix} A_1^{(1)} & A_2^{(1)} \\ \hline -\overline{A_2^{(1)}} & \overline{A_1^{(1)}} \end{bmatrix}, \tag{6}$$

Next, we take a half of $H_{l1}G_{l1}\chi_A$ as follows:

$$C \triangleq \begin{bmatrix} A_1^{(1)}, & A_2^{(1)} \end{bmatrix} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1n}^{(2)} & 0 & b_{12}^{(2)} & \cdots & b_{1n}^{(2)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & 0 & b_{22}^{(2)} & \cdots & b_{2n}^{(2)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a_{m2}^{(2)} & \cdots & a_{mn}^{(2)} & 0 & b_{m2}^{(2)} & \cdots & b_{mn}^{(2)} \end{bmatrix}. \tag{7}$$

We define a complex Givens rotation matrix

$$G_{ts}^{(2)} = \begin{bmatrix} I_{s-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\bar{a}_{ts}^{(2)}}{|q_{ts}^{(2)}|} & 0 & 0 & -\frac{b_{ts}^{(2)}}{|q_{ts}^{(2)}|} & 0 \\ 0 & 0 & I_{n-s} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{s-1} & 0 & 0 \\ 0 & \frac{\bar{b}_{ts}^{(2)}}{|q_{ts}^{(2)}|} & 0 & 0 & \frac{a_{ts}^{(2)}}{|q_{ts}^{(2)}|} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n-s} \end{bmatrix}$$

when $|q_{ts}^{(2)}| = \sqrt{|a_{ts}^{(2)}|^2 + |b_{ts}^{(2)}|^2} \neq 0$. In case, $q_{ts}^{(2)} = 0$, we just simply set $G_{ts}^{(2)} = I_{2n}$. Obviously, the $2n \times 2n$ complex matrix $G_{ts}^{(2)}$ is a unitary quaternion-type matrix.

Similarly, we conduct a series of quaternion-type unitary matrices on C by $G_{12}^{(2)}, G_{13}^{(2)}, \cdots, G_{1n}^{(2)}$ such that

$$CG_{12}^{(2)}G_{13}^{(2)}\cdots G_{1n}^{(2)} = \begin{bmatrix} a_{11}^{(2)} & |q_{12}^{(2)}| & \cdots & |q_{1n}^{(2)}| & 0 & 0 & \cdots & 0 \\ 0 & a_{22}^{(3)} & \cdots & a_{2n}^{(3)} & 0 & b_{22}^{(3)} & \cdots & b_{2n}^{(3)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a_{m2}^{(3)} & \cdots & a_{mn}^{(3)} & 0 & b_{m2}^{(3)} & \cdots & b_{mn}^{(3)} \end{bmatrix}. \tag{8}$$

Denoting $G_{r1} = G_{12}^{(2)} G_{13}^{(2)} \cdots G_{1n}^{(2)}$. Then it is still a $2n \times 2n$ quaternion-type unitary matrix. For a nonzero real vector $u_1^T = \begin{bmatrix} |q_{12}^{(2)}|, & |q_{13}^{(2)}|, & \cdots, & |q_{1n}^{(2)}| \end{bmatrix}^T$, we define a real Householder reflection $\tilde{H}_1$ such that $u_1 \tilde{H}_1 = \|u_1\| e_1^T$. For a zero vector, we just choose I_n. Let

$$H_{r1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{H}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \tilde{H}_1 \end{bmatrix}. \tag{9}$$

Then H_{r1} is a $2n \times 2n$ quaternion-type unitary matrix, and

$$CG_{r1}H_{r1} = \begin{bmatrix} a_{11}^{(4)} & a_{12}^{(4)} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{22}^{(4)} & \cdots & a_{2n}^{(4)} & 0 & b_{22}^{(4)} & \cdots & b_{2n}^{(4)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a_{m2}^{(4)} & \cdots & a_{mn}^{(4)} & 0 & b_{m2}^{(4)} & \cdots & b_{mn}^{(4)} \end{bmatrix} \triangleq \begin{bmatrix} A_1^{(2)}, & A_2^{(2)} \end{bmatrix}, \tag{10}$$

where $a_{11}^{(4)} = a_{11}^{(2)}, a_{12}^{(4)} = \sqrt{|q_{12}^{(2)}|^2 + \cdots + |q_{1n}^{(2)}|^2}$, $A_1^{(2)}, A_2^{(2)} \in \mathbb{C}^{m \times n}$. Since $H_{l1} G_{l1} \chi_A G_{r1} H_{r1}$ is a quaternion-type unitary matrix and $CG_{r1}H_{r1}$ is the first block row of $H_{l1} G_{l1} \chi_A G_{r1} H_{r1}$, we can recover the matrix

$$H_{l1} G_{l1} \chi_A G_{r1} H_{r1} = \begin{bmatrix} A_1^{(2)} & A_2^{(2)} \\ -\overline{A_2^{(2)}} & \overline{A_1^{(2)}} \end{bmatrix} \tag{11}$$

from (6), (7) and (10).

Since $H_{l1}G_{l1}$ and $G_{r1}H_{r1}$ are unitary quaternion-type matrices, by Definition 2.1, equality (2), and (6) of Lemma 2.3, we can read two unitary quaternion matrices U_1, V_1 from $H_{l1}G_{l1}$ and $G_{r1}H_{r1}$ such that $H_{l1}G_{l1} = \chi_{U_1}, G_{r1}H_{r1} = \chi_{V_1}$. Thus (11) is equivalent to $\chi_{U_1} \chi_A \chi_{V_1} = \chi_{A_1^{(2)} + A_2^{(2)} j}$. By (3) of Lemma 2.3,

$$\chi_{U_1 A V_1} = \chi_{A_1^{(2)} + A_2^{(2)} j}. \tag{12}$$

Hence (12) is equivalent to $U_1 A V_1 = A_1^{(2)} + A_2^{(2)} j$. As the process described above, there exist unitary matrices $U_1, U_2, \cdots, U_{n-1}, U_n \in \mathbb{H}^{m \times m}$ and $V_1, V_2, \cdots, V_{n-1} \in \mathbb{H}^{n \times n}$ such that

$$U_n U_{n-1} \cdots U_1 A V_1 \cdots V_{n-1} = \hat{A}_1 + \hat{A}_2 j,$$

where

$$\hat{A}_1 = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & 0 & \cdots & 0 & 0 \\ 0 & \hat{a}_{22} & \hat{a}_{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{a}_{(n-1)(n-1)} & \hat{a}_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & \hat{a}_{nn} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{m \times n}, \hat{A}_2 = 0,$$

i.e., there exist unitary matrices $\tilde{U} \triangleq U_n U_{n-1} \cdots U_2 U_1 \in \mathbb{H}^{m \times m}$ and $\tilde{V} \triangleq V_1 V_2 \cdots V_{n-1} \in \mathbb{H}^{n \times n}$ such that

$$\tilde{U} A \tilde{V} = \hat{A}_1. \tag{13}$$

Since $\hat{A}_1$ is a real upper-bidiagonal matrix, thus there exist orthogonal matrices $\hat{U} \in \mathbb{R}^{m\times m}, \hat{V} \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ such that

$$\hat{A}_1 = \hat{U}\Sigma\hat{V}^T, \tag{14}$$

where $\Sigma = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, $r = \mathrm{rank}(\hat{A}_1)$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Thus, up to now, we have already proved that any quaternion matrix has a QSVD decomposition.

Theorem 3.1 *For any matrix $A \in \mathbb{H}^{m\times n}$, there exist unitary matrices $U \in \mathbb{H}^{m\times m}, V \in \mathbb{H}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ such that*

$$A = U\Sigma V^*, \tag{15}$$

where $U = \widetilde{U}^*\hat{U} \in \mathbb{H}^{m\times m}$ and $V = \widetilde{V}\hat{V} \in \mathbb{H}^{n\times n}$ are unitary, $\Sigma \in \mathbb{R}^{m\times n}$ is diagonal, and $\widetilde{U}, \widetilde{V}, \hat{U}, \hat{V}$ and Σ are given as equations (13) and (14).

Remark 3.2 *For $A \in \mathbb{H}^{m\times n}$, if $m < n$, to find the QSVD for A , we just need to take*

$$A^* = U\Sigma V^*,$$

and then form

$$A = V\Sigma U^*.$$

4 Algorithms and Numerical experiments

In this section, we first propose two algorithms for computing QSVDs which are based on two steps discussed in Section 3.

Algorithm 1. Computing the upper-bidiagonalization form of a quaternion matrix
Input: $A = A_1 + A_2 j \in \mathbb{H}^{m\times n}$ with $A_1, A_2 \in \mathbb{C}^{m\times n}$.
Output: $\tilde{U} \in \mathbb{H}^{m\times m}, \tilde{V} \in \mathbb{H}^{n\times n}, \hat{A} \in \mathbb{R}^{m\times n}$ satisfying $\tilde{U}A\tilde{V} = \hat{A}$ in the upper-bidiagonalization form and $\chi_{\tilde{U}} = H_{ln}G_{ln}\cdots H_{l1}G_{l1}, \chi_{\tilde{V}} = G_{r1}H_{r1}\cdots G_{r(n-2)}H_{r(n-2)}G_{r(n-1)}$.

(i) Calculate $G_{l1}, \cdots, G_{ln}$ (similar to (3));
(ii) Calculate $H_{l1}, \cdots, H_{ln}$ (similar to (4));
(iii) Calculate $G_{r1}, \cdots, G_{r(n-1)}$ (similar to (8));
(iv) Calculate $H_{r1}, \cdots, H_{r(n-2)}$ (similar to (9));

Algorithm 2. Computing the QSVD of a quaternion matrix
Input: $A = A_1 + A_2 j \in \mathbb{H}^{m\times n}, A_1, A_2 \in \mathbb{C}^{m\times n}$.
Output: U, Σ, V satisfying $A = U\Sigma V^*$, where $U = \widetilde{U}^*\hat{U}$ and $V = \widetilde{V}\hat{V}$.

(i) Calculate $\tilde{U}, \tilde{V}, \hat{A}$ based on Algorithm 1;
(ii) Compute the SVD for the real diagonal matrix $\hat{A}$ by (14) and Theorem 3.1.

Finally, we give two numerical examples.

Example 4.1 *Find the QSVD of the quaternion matrix $A = A_1 + A_2 j$ with*

$$A_1 = \begin{bmatrix} 2-i & -3+i & 4-i & 1+i \\ -2 & 5i & -4i & 3i \\ 1+2i & 3-4i & 2i & 3-i \\ -4i & -4 & 1-3i & 4+2i \\ 9-2i & 3 & 2-2i & 5i \end{bmatrix}, A_2 = \begin{bmatrix} 3-5i & -3i & 8-5i & 6 \\ 2+i & -2+4i & -4i & 3i \\ 2-5i & 5i & -4-4i & 7-2i \\ 2-i & 6-2i & -1+i & 5+5i \\ 3-3i & -2 & 3-2i & -6i \end{bmatrix}.$$

Using Algorithm 2, we obtain the following result (which quoted in three decimal places):

$$\Sigma = \begin{bmatrix} 19.681 & 0 & 0 & 0 \\ 0 & 16.266 & 0 & 0 \\ 0 & 0 & 13.109 & 0 \\ 0 & 0 & 0 & 4.717 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ U = U_1 + U_2 j, \ and\ V = V_1 + V_2 j \ with$$

$$U_1 = \begin{bmatrix} -0.178+0.131i & 0.154+0.059i & 0.122+0.160i & -0.188+0.185i & 0.060+0.201i \\ -0.011+0.001i & -0.260-0.200i & -0.017-0.207i & 0.088+0.320i & -0.010-0.010i \\ -0.209-0.063i & -0.105+0.175i & -0.194+0.109i & -0.368+0.126i & 0.075+0.118i \\ -0.249+0.296i & -0.330+0.097i & -0.012+0.479i & 0.261+0.108i & 0.236-0.329i \\ -0.083+0.038i & 0.580-0.018i & -0.425+0.126i & 0.272-0.329i & -0.056-0.113i \end{bmatrix},$$

$$U_2 = \begin{bmatrix} -0.527+0.175i & 0.009-0.310i & 0.241+0.229i & -0.312+0.256i & 0.069+0.296i \\ -0.255+0.179i & -0.091+0.303i & 0.000-0.049i & 0.179-0.154i & -0.567+0.403i \\ 0.065+0.419i & 0.314-0.242i & 0.104-0.390i & 0.080-0.305i & -0.153-0.259i \\ -0.112-0.037i & 0.070+0.049i & 0.069+0.331i & 0.252-0.052i & -0.002-0.239i \\ -0.246+0.293i & -0.089-0.022i & -0.225-0.002i & 0.152-0.027i & -0.180-0.012i \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -0.476+0.000i & 0.572+0.000i & -0.383+0.000i & 0.548+0.000i \\ 0.190+0.001i & 0.139+0.120i & -0.467-0.377i & -0.306-0.387i \\ -0.576+0.169i & 0.132+0.045i & 0.042-0.247i & -0.609-0.072i \\ -0.322-0.088i & 0.140+0.156i & 0.406+0.418i & -0.142+0.053i \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.000+0.000i & 0.000+0.000i & 0.000+0.000i & 0.000+0.000i \\ 0.177+0.284i & 0.335+0.248i & 0.083+0.107i & -0.138+0.062i \\ -0.125-0.093i & -0.321-0.109i & -0.116+0.054i & 0.145+0.070i \\ 0.168+0.320i & 0.239+0.480i & 0.140+0.193i & -0.006-0.089i \end{bmatrix}.$$

It can be verified that $UU^* = U^*U = I_5, VV^* = V^*V = I_4$, and $U\Sigma V^* - A = 1.0e^{-14}(W_1 + W_2 j)$, where

$$W_1 = \begin{bmatrix} -0.311+0.100i & 0.355-0.344i & -0.355+0.111i & 0.133-0.089i \\ 0.178+0.217i & 0.000-0.178i & -0.150+0.089i & 0.222-0.222i \\ -0.289-0.222i & -0.178-0.355i & 0.044+0.133i & -0.711-0.222i \\ -0.071+0.178i & 0.044-0.411i & -0.511+0.044i & 0.267-0.089i \\ -0.178+0.000i & 0.355-0.666i & -0.022-0.044i & -0.427+0.000i \end{bmatrix},$$

$$W_2 = \begin{bmatrix} -0.622+0.355i & 0.233-0.267i & -0.444+0.355i & -0.533+0.022i \\ -0.289+0.044i & 0.022+0.178i & -0.178+0.267i & -0.067-0.089i \\ -0.200+0.799i & 0.644-0.533i & 0.089+0.622i & 0.000+0.489i \\ -0.089-0.133i & 0.178-0.267i & 0.067-0.011i & -0.355-0.355i \\ -0.133+0.311i & -0.178+0.000i & 0.267+0.311i & -0.511-0.267i \end{bmatrix}.$$

That is, $A = U\Sigma V^*$.

Example 4.2 *Find the QSVD of $A = A_1 + A_2 j$ with*

$$A_1 = rand(3) + rand(3)i = \begin{bmatrix} 0.0975+0.8003i & 0.9575+0.9157i & 0.9706+0.6557i \\ 0.2785+0.1419i & 0.9649+0.7922i & 0.9572+0.0357i \\ 0.5469+0.4218i & 0.1576+0.9595i & 0.4854+0.8491i \end{bmatrix},$$

$$A_2 = rand(3) + rand(3)i = \begin{bmatrix} 0.9340+0.2769i & 0.7431+0.8235i & 0.1712+0.9502i \\ 0.6787+0.0462i & 0.3922+0.6948i & 0.7060+0.0344i \\ 0.7577+0.0971i & 0.6555+0.3171i & 0.0318+0.4387i \end{bmatrix}.$$

By using Algorithm 2, we obtain the following result (which quoted in four decimal places):

$$
\begin{aligned}
U &= \begin{bmatrix} -0.2330-0.3360i & -0.0884+0.3398i & 0.6046-0.1289i \\ -0.2546-0.2114i & -0.5383-0.4193i & -0.4140-0.0768i \\ -0.0384-0.2801i & 0.4957+0.1586i & -0.5117+0.3357i \end{bmatrix} + \begin{bmatrix} -0.5439-0.1699i & 0.0297-0.0085i & -0.0420-0.0012i \\ -0.3967+0.0421i & 0.1283+0.2314i & 0.1299+0.0399i \\ -0.3845-0.1085i & 0.1017-0.2428i & -0.1607-0.1411i \end{bmatrix} j, \\
V &= \begin{bmatrix} -0.4506+0.0000i & 0.6038+0.0000i & -0.6575+0.0000i \\ -0.5920-0.2129i & -0.1899+0.2103i & 0.2313+0.3391i \\ -0.4210-0.3174i & -0.0272-0.4411i & 0.2635-0.1876i \end{bmatrix} + \begin{bmatrix} 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \\ -0.1034+0.2468i & 0.0255+0.4589i & 0.0943+0.2523i \\ -0.1703+0.1500i & -0.0012-0.3854i & 0.1157-0.4568i \end{bmatrix} j, \\
\Sigma &= \begin{bmatrix} 3.6747 & 0 & 0 \\ 0 & 0.9415 & 0 \\ 0 & 0 & 0.5987 \end{bmatrix}.
\end{aligned}
$$

It can be verified that $UU^* = U^*U = I_3,\ VV^* = V^*V = I_3,$

$$
U\Sigma V^* - A = \\
1.0e^{-15} \cdot \left\{ \begin{bmatrix} 0.2220+0.3331i & 0.3331-0.4441i & 0.1110-0.2220i \\ 0.1665+0.1110i & -0.3331+0.2220i & -0.2220-0.3886i \\ 0.0000+0.0000i & -0.4718+0.1110i & -0.2220+0.1110i \end{bmatrix} + \begin{bmatrix} 0.1110+0.0555i & -0.1110-0.2220i & -0.2220+0.2220i \\ 0.0000-0.0139i & -0.7772+0.3331i & 0.1110-0.7216i \\ 0.1110+0.0416i & 0.1110-0.3331i & -0.2220+0.0000i \end{bmatrix} j \right\}.
$$

That is, $A = U\Sigma V^*$.

References

[1] N. L. Bihan and J. Mars, Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing, Signal Processing, 84 (7) (2004) 1177-1199.

[2] N. L. Bihan and S. J. Sangwine, Jacobi method for quaternion matrix singular value decomposition, Appl. Math. Comput. 187 (2) (2007) 1265-1271.

[3] E. Doukhnitch, E. Ozen, Hardware-oriented algorithm for quaternion-valued matrix decomposition, IEEE Trans. Circuits Syst. II, Express Briefs 58 (2011) 225-229.

[4] Z. G. Jia, M. K. Ng, G. J. Song, Lanczos method for large-scale quaternion singular value decomposition, Numer. Algorithms (2018) 1-19.

[5] Z. G. Jia, M. S. Wei, S. T. Ling, A new structure-preserving method for quaternion Hermitian eigenvalue problems, J. Comput. Appl. Math. 239 (2013) 12-24.

[6] Y. Li, M. Wei, F. Zhang, and J. Zhao, A fast structure-preserving method for computing the singular value decomposition of quaternion matrix, Appl. Math Comput. 235 (2014) 157-167.

[7] Y. Li, M. Wei, F. Zhang, J. Zhao, Real structure-preserving algorithms of Householder based transformations for quaternion matrices, J. Comput. Appl. Math. 305 (2016) 82-91.

[8] R. R. Ma, Z. G. Jia, Z. J. Bai, A structure-preserving Jacobi algorithm for quaternion Hermitian eigenvalue problems, Comput. Math. Appl. 75 (2018) 809-820.

[9] R. R. Ma, Z. J. Bai, A structure-preserving one-sided Jacobi method for computing the SVD of a quaternion matrix, Applied Numerical Mathematics, 147 (2020) 101-117.

[10] S. C. Pei, J. H. Chang, and J. J. Ding, Quaternion matrix singular value decomposition and its applications for color image processing, Image Processing, 2003. ICIP 2003. Proceedings. 2003 International Conference on IEEE Xplore, 2003.

[11] L. Rodman, Topics in Quaternion Linear Algebra, Princeton Series in Applied Mathematics, Princeton University Press, 2014.

[12] S. J. Sangwine, and N. L. Bihan, Quaternion singular value decomposition based on bidiagonalization to a real or complex matrix using quaternion householder transformations, Applied Mathematics and Computation, 182 (1) (2006) 727-738.

[13] F. Z. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21-57.