New Tightness Lower and Upper Bounds for the Standard Normal Distribution Function and Related Functions

Omar M. Eidous, Professor of Statistics
Department of Statistics – Yarmouk University
Irbid – Jordan
omarm@yu.edu.jo

Abstract

Most researches interested in finding the bounds of the cumulative standard normal distribution $\Phi(x)$ are not tight for all positive values of the argument x. This paper mainly proposes new simple lower and upper bounds for $\Phi(x)$. Over the whole range of the positive argument x, the maximum absolute difference between the proposed lower bound and $\Phi(x)$ is less than 3×10^{-4}, while it is less than 4.8×10^{-4} between the proposed upper bound and $\Phi(x)$. Numerical comparisons have been made between the proposed bounds and some of the other existing bounds, which showed that the proposed bounds are more compact than most alternative bounds found in the literature.

Keywords: Bounds for standard normal distribution function; Error function; Q-function; Maximum absolute error.

Mathematics Subject Classifications (2000) 65D20; 26D15

1. Introduction

The probability that a standard normal random variable X is less than a real value x is known as the cumulative standard normal distribution, which is mathematically given by,

$$\Phi(x) = \int_{-\infty}^{x} \phi_X(t) dt,$$

where $\phi_X(t) = \frac{\exp(-t^2/2)}{\sqrt{2\pi}}$, $-\infty < t < \infty$ is the standard normal probability density function of a random variable X. The function $\Phi(x)$ cannot be expressed in a closed form, therefore, there are numerous proposed approximations for $\Phi(x)$ that developed in the literature (see Eidous and Abu-Shareefa, 2020). Also, many works are interested to find upper bounds for $\Phi(x)$ (Eidous, 2022) and lower bounds for $\Phi(x)$ (Peric *et al.*, 2019). There are two main interesting functions related to $\Phi(x)$. The first one is known as Q-function and the other one is the error function, which are, respectively, defined as follows,

$$\begin{aligned} Q(x) &= \int_{x}^{\infty} \phi_X(t) dt\,, \\ &= 1 - \Phi(x) \end{aligned}$$

and

$$\begin{aligned} erf(x) &= \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-t^2)\, dt \\ &= 2\Phi\left(\sqrt{2}x\right) - 1. \end{aligned}$$

Several works have been discussed the importance of the functions $\Phi(x)$, $Q(x)$ and $erf(x)$ and their broad scope of application for science and engineering (See, Simon, 2006 and Sandoval-Hernandez et al., 2019).

In this paper, we will focus our attention on providing new lower and upper bounds for the function $\Phi(x)$ and consequently for the two functions $Q(x)$ and $erf(x)$. The proposed bounds are very tight for their corresponding functions for all values of the argument $x \geq 0$. More specifically, the maximum absolute difference between the proposed lower bound and $\Phi(x)$ is less than 3×10^{-4}, and it is less than 4.8×10^{-4} between the proposed upper bound and $\Phi(x)$. In addition, the mathematical and numerical results show that the absolute differences between the proposed bounds and the actual function converge to zero for large x. It is worthwhile to mention here that the relation $\Phi(-x) = 1 - \Phi(x)$ can be used to address the case of a negative argument x.

2. Overview of Some Bounds for $\boldsymbol{\Phi}(x)$

This section summarizes some existing lower and upper bounds for $\Phi(x)$. The symbol $\Phi_{L-}(x)$ is used to represent the lower bound of $\Phi(x)$, while $\Phi_{U-}(x)$ denotes its upper bound.

- Boyd (1959) gave the following upper and lower bound for $\Phi(x)$ (for $x > 0$),

$$1 - \frac{\pi\phi(x)}{2x + \sqrt{(\pi-1)^2x^2 + 2\pi}} \leq \Phi(x) \leq 1 - \frac{\pi\phi(x)}{(\pi-1)x + \sqrt{x^2 + 2\pi}}$$

  Ruskai and Werner (2000) improved the lower bound of Boyd (1959). They gave the following lower and upper bounds for $\Phi(x)$,

$$\Phi_{L-RW}(x) \leq \Phi(x) \leq \Phi_{U-RW}(x)$$

where $\Phi_{L-RW}(x) = 1 - \frac{4\phi(x)}{3x+\sqrt{x^2+8}}$ and $\Phi_{U-RW}(x) = 1 - \frac{\pi\phi(x)}{(\pi-1)x+\sqrt{x^2+2\pi}}$.

- The more recent bounds of Alzer (2010) depend on the function $\tanh(x)$. The lower and upper bounds were given as follows (for $x > 0$),

$$\Phi_{L-AL}(x) \leq \Phi(x) \leq \Phi_{U-AL}(x)$$

where $\Phi_{L-AL}(x) = 0.5 + \frac{Tanh(\sqrt{2/\pi}\,x)}{2}$ and $\Phi_{U-AL}(x) = 0.5 + \frac{1.0407\ Tanh(\sqrt{2/\pi}\,x)}{2}$.

- Abreu (2012) gave the following bounds for $\Phi(x)$ (for $x > 0$),

$$\Phi_{L-AB}(x) \leq \Phi(x) \leq \Phi_{U-AB}(x)$$

where $\Phi_{L-AB}(x) = 1 - \frac{\exp(-x^2)}{50} - \frac{\exp(-x^2/2)}{2(1+x)}$ and $\Phi_{U-AB}(x) = 1 - \frac{\exp(-x^2)}{12} - \frac{\phi(x)}{1+x}$.

- The Mastin and Jaillet (2013)'s lower and upper bounds for $\Phi(x)$ are (for $x > 0$),

$$\Phi_{L-MJ}(x) \leq \Phi(x) \leq \Phi_{U-MJ}(x)$$

where $\Phi_{L-MJ}(x) = 1 - \frac{1}{2}\exp\left(-\sqrt{2/\pi}\,x - x^2/\pi\right)$ and $\Phi_{U-MJ}(x) = 1 - \frac{1}{2}\exp\left(-\sqrt{2/\pi}\,x - x^2/2\right)$.

- Peric *et al.* (2019) suggested the following lower and upper bounds for $\Phi(x)$, for $x > 0$,

$$\Phi_{L-PE}(x) \leq \Phi(x) \leq \Phi_{U-PE}(x)$$

where $\Phi_{L-PE}(x) = 1 - \frac{(x^2+2)\phi(x)}{x(x^2+3)}$ and $\Phi_{U-PE}(x) = 1 - \frac{x\phi(x)}{x^2+1}$.

- Bercu (2020) gave the following lower and upper bounds for $\Phi(x)$ when $0 \leq x \leq 6.248$,

$$\Phi_{L-BE}(x) \leq \Phi(x) \leq \Phi_{U-BE}(x)$$

where $\Phi_{L-BE}(x) = \frac{1}{2} + \frac{\beta_1(x/\sqrt{2})}{\sqrt{\pi}}$ and $\Phi_{U-BE}(x) = \frac{1}{2} + \frac{\beta_2(x/\sqrt{2})}{\sqrt{\pi}}$. Also $\beta_1(\mathrm{t}) = \frac{110\mathrm{t}^3+210t}{39\mathrm{t}^4+180\mathrm{t}^2+210}$ and $\beta_2(\mathrm{t}) = \frac{113400\, t}{29t^8-660t^6+1260t^4+37800t^2+113400}$. As noted by Bercu (2020), the upper bound $\Phi_{U-BE}(x)$ is valid only for $0 \leq x \leq 6.248$.

Concerning the above lower and upper bounds of $\Phi(x)$, a limitation shared by almost all of them is that they do not exhibit a tight lower and upper bound for all values of the argument $x > 0$. For instance, the numerical results in Section (4) of this paper showed that Bercu (2020)'s bounds (i.e. $\Phi_{L-BE}(x)$ and $\Phi_{U-BE}(x)$) are very tight for small values of x but it is less sharp for large values of x. The converse is true for the upper and lower bounds $\Phi_{L-PE}(x)$ and $\Phi_{U-PE}(x)$ of that derived by Peric *et al.* (2019).

The main contribution of this work is to provide new lower and upper bounds for $\Phi(x)$, which are very close to the true value of $\Phi(x)$ for all values of the argument $x \geq 0$.

3. Proposed New Bounds for $\boldsymbol{\Phi(x)}$

The following lemma gives the proposed lower and upper bounds for $\Phi(x)$.
Lemma 1: Let

$$\Phi_L(x) = \frac{1}{2}\sqrt{1 - \exp\left(-x^2\left(\frac{203}{320} - \frac{x^2}{125} + \frac{x^6}{1000000}\right)\right)}$$

and

$$\Phi_U(x) = \frac{1}{2}\sqrt{1 - \exp\left(-x^2\left(\frac{1500}{2351} - \frac{4x^2}{485} + \frac{x^6}{40000}\right)\right)}$$

then, for $x \geq 0$, $\Phi_L(x)$ and $\Phi_{U-EI}(x)$ are a lower and an upper bounds for $\Phi(x)$ respectively. That is,

$$\Phi_L(x) \leq \Phi(x) \leq \Phi_U(x).$$

Also, the maximum absolute difference between $\Phi_L(x)$ and $\Phi(x)$ is 3.06×10^{-4} occurs at $x \cong 2.462$ and it is 4.746×10^{-4} between $\Phi_U(x)$ and $\Phi(x)$, which occurs at $x \cong 1.17709$.

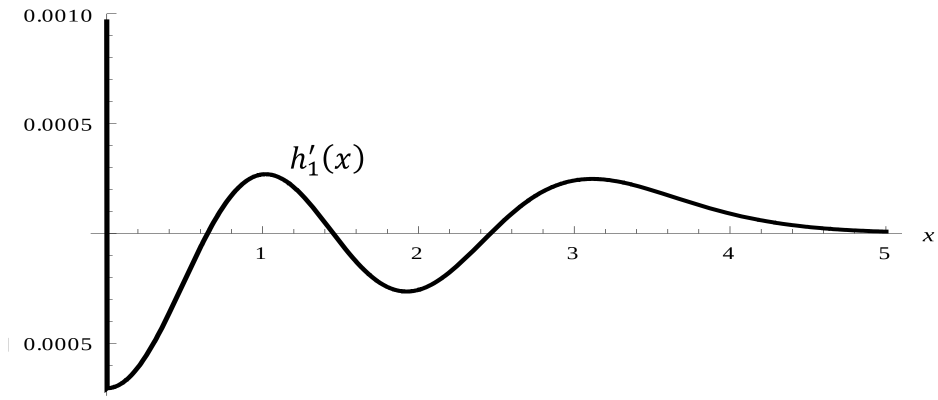
Proof: To verify the above inequality, it is enough to show that $h_1(x) = \Phi_L(x) - \Phi(x) \leq 0$ and $h_2(x) = \Phi_U(x) - \Phi(x) \geq 0$ for all $x \geq 0$. The two bounds $\Phi_L(x)$ and $\Phi_U(x)$ take the same form, which can be written as

$$\frac{1}{2}\sqrt{1 - \exp\left(-x^2(a - bx^2 + cx^6)\right)},$$

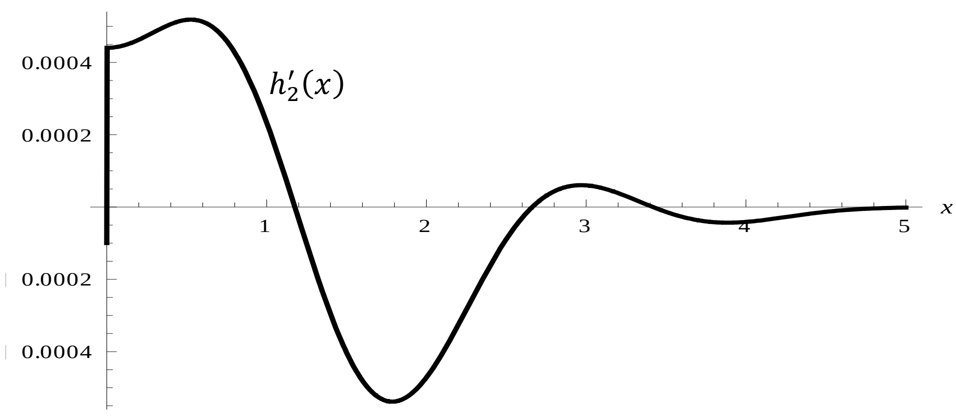
where the constants a, b and c equal $\frac{203}{320}$, $\frac{1}{125}$ and $\frac{1}{1000000}$ for $\Phi_L(x)$ and their values are $\frac{1500}{2351}$, $\frac{4}{485}$ and $\frac{1}{40000}$ for $\Phi_U(x)$. Accordingly, the first derivative of $h_i(x), i = 1, 2$ with respect to x is of the form,

$$h_i{}'(x) = \frac{x(a - 2bx^2 + 4cx^6)\exp\left(-x^2(a - bx^2 + cx^6)\right)}{2\sqrt{1 - \exp\left(-x^2(a - bx^2 + cx^6)\right)}} - \frac{\exp(-x^2/2)}{\sqrt{2\pi}} .$$

Using Mathematica, Ver 11, it is found that there are three roots for each equation $h_1{}'(x) = 0$ and $h_2{}'(x) = 0$, which are approximately equal to 0.64620, 1.45355 and 2.46200 for $h_1'(x) = 0$ and 1.17709, 2.66243 and 3.41754 for $h_2{}'(x) = 0$. This is illustrated in Graph (1) and Graph (2), which give the plots of $h_1'(x)$ and $h_2'(x)$ respectively.



Graph (1). Plot of $h_1'(x)$.



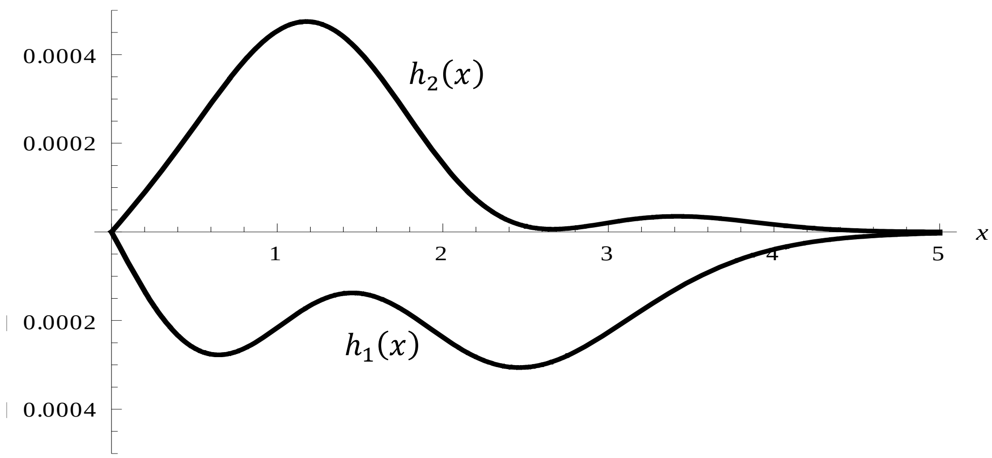
Graph (2). Plot of $h_2'(x)$.

By using the derivative sign (see also, Graph 3 below), it is found that $h_1(x)$ decreasing for $x \in [0, 0.64620]$ and $x \in [1.45355, 2.462]$, while it increases for $x \in [0.6462, 1.45355]$ and $x \in [2.462, \infty)$. It is a simple technique to show that the minimum absolute value of $h_1(x)$ occurs at $x = 1.45355$ with $h_1(1.45355) =$

$-1.3762 \times 10^{-4} < 0$ and $h_1(x) \to 0$ as $x \to 0$ or ∞. Therefore, $h_1(x) \le 0, \forall\, x \ge 0$. In addition, the two points $x = 0.6462$ and 2.462 give local maximum absolute values for $h_1(x)$. Since $|h_1(0.6462)| = 2.77 \times 10^{-4} < |h_1(2.6462)| = 3.06 \times 10^{-4}$ then the maximum absolute difference between $\Phi_L(x)$ and $\Phi(x)$ is 3.06×10^{-4}.

For the second function, $h_2(x)$ increases for $x \in [0, 1.7709]$ and $x \in [2.66243, 3.41754]$, while it is decreases for $x \in [1.17709, 2.66243]$ and $x \in [3.41754, \infty)$. The local minimum value of $h_2(x)$ is $h_2(2.66243) = 6.2 \times 10^{-6} > 0$. Also, it is clear that $h_2(x) \to 0$ as $x \to 0$ or ∞. Therefore, $h_2(x) \ge 0, \forall\, x \ge 0$. Also and in the same manner followed previously for $h_1(x)$, it can easily be proven that the maximum absolute difference between $\Phi_U(x)$ and $\Phi(x)$ is 4.746×10^{-4}.

This completes the proof.



Graph (3). Plots of $h_1(x) = \Phi_L(x) - \Phi(x)$ and $h_2(x) = \Phi_U(x) - \Phi(x)$.

The following two observations should be noted here:

1. Since the Q-function, $Q(x) = 1 - \Phi(x)$ then the lower and upper bounds for $Q(x)$ based on the proposed $\Phi_L(x)$ and $\Phi_U(x)$ are given by the following inequality,

$$1 - \Phi_U(x) \le Q(x) \le 1 - \Phi_L(x)\,.$$

2. The lower and upper bounds for $erf(x)$ based on the proposed $\Phi_L(x)$ and $\Phi_U(x)$ are given by,

$$2\Phi_L(\sqrt{2}x) - 1 \le erf(x) \le 2\Phi_U(\sqrt{2}x) - 1.$$

4. Numerical Comparisons and Conclusion

In this section, we have provided some numerical results to check the tightness of our proposed bounds of $\Phi(x)$ and to compare their performances with that of some current bounds mentioned in Section 2. Let $h_{U-t}(x)$ be the absolute error function between the upper bound of $\Phi(x)$ and the exact $\Phi(x)$. Also, let $h_{L-t}(x)$ be the absolute error function between the lower bound of $\Phi(x)$ and the exact $\Phi(x)$. That is, $h_{s-t}(x) = |\Phi_{s-t}(x) - \Phi(x)|$, where s stands for U or L and t stands for RW (Roskai and Werner, 2000), AL (Alzer, 2010), AB (Abreu, 2012), MJ (Mastin and Jaillet, 2013), PE (Peric *et al.,* 2019), BE (Bercu, 2020) and EI (Proposed bounds as given in Section 3).

Table (1) shows the values of the error function $h_{s-t}(x)$ for some values of x, varying from 0.1 to 8.7, although all the bounds considered in this paper (except the upper bound $\Phi_{U-BE}(x)$) can be applied to any value of $x \geq 0$. We focused our analysis on the small and moderate values of x since our suggested bounds are too precise for large values of x. By examining the numerical results of Table (1), the following points can be drawn:

1. The proposed bounds are very accurate for moderate to large values of the argument x. Compared with other considered bounds, the accuracy of the proposed bounds is very satisfactory for small values of x.
2. For large x, the proposed bounds as well as those of Abreu (AB), Mastin and Jaillet (MJ) and Peric *et al*. (PE) are very tight. The absolute error associated with these bounds becomes very small as the size of x increases. However, the results clearly showed that the proposed bounds are much tighter than these bounds for $0 < x < 2.1$.
3. For $0 < x < 1.7$, the absolute error associated with Bercu (BE) bounds is very small compared to the other bounds. However, the absolute error becomes large when x gets large. The same conclusion can be said for the bounds of Alzer (AL). In addition, taking into account all the considered values of the argument x, in this numerical study, one can conclude that the tightness of the bounds BE, AL and those given by Ruskai and Werner (RW) are not as good as the other bounds.
4. In general and for all values of x, the numerical results of Table (1) clearly show that the proposed bounds are tighter than the other ones considered in this study. Because of their tightness, one can use the lower bound or upper bound as an approximation for $\Phi(x)$, with some preference for the lower bound over the upper bound. Its absolute maximum error is 3.06×10^{-4} instead of 3.06×10^{-4} for the upper bound.

Table (1). The values of error function $h_{s-t}(x) = |\Phi_{s-t}(x) - \Phi(x)|$ for some values of x.

x		$h_{s-RW}(x)$	$h_{s-AL}(x)$	$h_{s-AB}(x)$	$h_{s-MJ}(x)$	$h_{s-PE}(x)$	$h_{s-BE}(x)$	$h_{s-EI}(x)$
0.1	L	7.95×10^{-2}	1.81×10^{-5}	1.19×10^{-2}	1.65×10^{-5}	2.19	6.92×10^{-14}	6.96×10^{-5}
	U	2.16×10^{-3}	1.60×10^{-3}	1.68×10^{-2}	8.19×10^{-4}	4.21×10^{-1}	0.00	4.43×10^{-5}
0.3	L	2.35×10^{-1}	4.63×10^{-4}	3.88×10^{-3}	3.60×10^{-4}	4.78×10^{-1}	1.29×10^{-9}	1.90×10^{-4}
	U	3.76×10^{-3}	4.32×10^{-3}	1.26×10^{-2}	5.84×10^{-3}	2.77×10^{-1}	3.11×10^{-12}	1.37×10^{-4}
0.5	L	3.82×10^{-1}	1.94×10^{-3}	1.20×10^{-3}	1.31×10^{-3}	1.79×10^{-1}	1.16×10^{-7}	2.62×10^{-4}
	U	3.57×10^{-3}	5.77×10^{-3}	8.93×10^{-3}	1.24×10^{-2}	1.68×10^{-1}	8.05×10^{-10}	2.38×10^{-4}
0.7	L	5.15×10^{-1}	4.60×10^{-3}	4.96×10^{-4}	2.76×10^{-3}	7.63×10^{-2}	2.07×10^{-6}	2.75×10^{-4}
	U	2.80×10^{-3}	5.72×10^{-3}	7.23×10^{-3}	1.81×10^{-2}	9.53×10^{-2}	2.96×10^{-8}	3.40×10^{-4}
0.9	L	6.31×10^{-1}	8.08×10^{-3}	3.57×10^{-4}	4.36×10^{-3}	3.40×10^{-2}	1.65×10^{-5}	2.42×10^{-4}
	U	1.97×10^{-3}	4.46×10^{-3}	6.94×10^{-3}	2.14×10^{-2}	5.18×10^{-2}	4.16×10^{-7}	4.24×10^{-4}
1.1	L	7.28×10^{-1}	1.17×10^{-2}	3.16×10^{-4}	5.76×10^{-3}	1.53×10^{-2}	8.02×10^{-5}	1.90×10^{-4}
	U	1.29×10^{-3}	2.64×10^{-3}	7.08×10^{-3}	2.22×10^{-2}	2.72×10^{-2}	3.25×10^{-6}	4.70×10^{-4}
1.3	L	8.06×10^{-1}	1.48×10^{-2}	2.72×10^{-4}	6.68×10^{-3}	6.91×10^{-3}	2.80×10^{-4}	1.48×10^{-4}
	U	7.96×10^{-4}	1.01×10^{-3}	6.92×10^{-3}	2.07×10^{-2}	1.40×10^{-2}	1.71×10^{-5}	4.64×10^{-4}
1.5	L	8.66×10^{-1}	1.69×10^{-2}	2.31×10^{-4}	7.01×10^{-3}	3.09×10^{-3}	7.66×10^{-4}	1.39×10^{-4}
	U	4.69×10^{-4}	9.57×10^{-5}	6.22×10^{-3}	1.78×10^{-2}	7.03×10^{-3}	6.80×10^{-5}	4.06×10^{-4}
1.7	L	9.10×10^{-1}	1.77×10^{-2}	2.03×10^{-4}	6.77×10^{-3}	1.36×10^{-3}	1.75×10^{-3}	1.64×10^{-4}
	U	2.64×10^{-4}	1.63×10^{-4}	5.10×10^{-3}	1.42×10^{-2}	3.46×10^{-3}	2.17×10^{-4}	3.1×10^{-4}
1.9	L	9.42×10^{-1}	1.73×10^{-2}	1.82×10^{-4}	6.08×10^{-3}	5.93×10^{-4}	3.47×10^{-3}	2.11×10^{-4}
	U	1.43×10^{-4}	1.19×10^{-3}	3.84×10^{-3}	1.07×10^{-2}	1.67×10^{-3}	5.86×10^{-4}	2.04×10^{-4}
2.1	L	9.64×10^{-1}	1.60×10^{-2}	1.61×10^{-4}	5.13×10^{-3}	2.54×10^{-4}	6.14×10^{-3}	2.62×10^{-4}
	U	7.48×10^{-5}	2.98×10^{-3}	2.66×10^{-3}	7.54×10^{-3}	7.91×10^{-4}	1.38×10^{-3}	1.10×10^{-4}
2.3	L	9.78×10^{-1}	1.41×10^{-2}	1.35×10^{-4}	4.09×10^{-3}	1.06×10^{-4}	9.94×10^{-3}	2.97×10^{-4}
	U	3.77×10^{-5}	5.23×10^{-3}	1.72×10^{-3}	5.06×10^{-3}	3.66×10^{-4}	2.92×10^{-3}	4.54×10^{-5}
2.5	L	9.88×10^{-1}	1.20×10^{-2}	1.06×10^{-4}	3.09×10^{-3}	4.37×10^{-5}	1.49×10^{-2}	3.05×10^{-4}
	U	1.84×10^{-5}	7.65×10^{-3}	1.04×10^{-3}	3.22×10^{-3}	1.65×10^{-4}	5.65×10^{-3}	1.29×10^{-5}
2.7	L	9.93×10^{-1}	9.81×10^{-3}	7.66×10^{-5}	2.23×10^{-3}	1.75×10^{-5}	2.11×10^{-2}	2.88×10^{-4}
	U	8.63×10^{-6}	1.00×10^{-2}	5.94×10^{-4}	1.95×10^{-3}	7.29×10^{-5}	1.01×10^{-2}	6.49×10^{-6}
3.1	L	9.98×10^{-1}	6.09×10^{-3}	3.24×10^{-5}	1.01×10^{-3}	2.64×10^{-6}	3.66×10^{-2}	2.02×10^{-4}
	U	1.72×10^{-6}	1.40×10^{-2}	1.65×10^{-4}	6.22×10^{-4}	1.31×10^{-5}	2.75×10^{-2}	2.64×10^{-5}
3.5	L	1. 00	3.51×10^{-3}	1.05×10^{-5}	3.88×10^{-4}	3.59×10^{-7}	5.52×10^{-2}	1.10×10^{-4}
	U	3.01×10^{-7}	1.67×10^{-2}	3.83×10^{-5}	1.66×10^{-4}	2.11×10^{-6}	6.30×10^{-2}	3.46×10^{-5}
3.9	L	1.00	1.93×10^{-3}	2.72×10^{-6}	1.28×10^{-4}	4.37×10^{-8}	7.53×10^{-2}	4.92×10^{-5}
	U	4.58×10^{-8}	1.83×10^{-2}	7.53×10^{-6}	3.70×10^{-5}	3.01×10^{-7}	1.29×10^{-1}	2.11×10^{-5}
4.5	L	1.00	7.57×10^{-4}	2.45×10^{-7}	1.85×10^{-5}	1.50×10^{-9}	1.06×10^{-1}	1.14×10^{-5}
	U	2.10×10^{-9}	1.96×10^{-2}	4.91×10^{-7}	2.85×10^{-6}	1.29×10^{-8}	3.31×10^{-1}	3.13×10^{-6}
5.1	L	1.00	2.92×10^{-4}	1.45×10^{-8}	2.00×10^{-6}	3.89×10^{-11}	1.35×10^{-1}	2.25×10^{-6}
	U	7.05×10^{-11}	2.00×10^{-2}	2.27×10^{-8}	1.51×10^{-7}	4.1×10^{-10}	6.78×10^{-1}	1.70×10^{-7}
5.9	L	1.00	8.15×10^{-5}	1.83×10^{-10}	6.77×10^{-8}	1.9×10^{-13}	1.70×10^{-1}	2.38×10^{-7}
	U	4.64×10^{-13}	2.03×10^{-2}	2.21×10^{-10}	1.69×10^{-9}	2.57×10^{-12}	2.98×10^{-1}	1.82×10^{-9}
6.7	L	1.00	2.27×10^{-5}	1.19×10^{-12}	1.47×10^{-9}	5.55×10^{-16}	---	1.85×10^{-8}
	U	1.78×10^{-15}	2.03×10^{-2}	1.16×10^{-12}	9.99×10^{-12}	9.21×10^{-15}	2.97×10^{-1}	1.04×10^{-11}
7.5	L	1.00	6.34×10^{-6}	3.89×10^{-15}	2.11×10^{-11}	1.11×10^{-16}	---	3.52×10^{-10}
	U	1.11×10^{-16}	2.03×10^{-2}	3.33×10^{-15}	3.12×10^{-14}	1.11×10^{-16}	4.40×10^{-1}	3.20×10^{-14}
8.7	L	1.00	9.35×10^{-7}	0.00	1.67×10^{-14}	0.00	---	2.22×10^{-16}
	U	0.00	2.04×10^{-2}	0.00	0.00	0.00	4.85×10^{-1}	0.00

References

Abreu, G. (2012). Very simple tight bounds on the Q-function. *IEEE Trans Commun.* 60(9): 2415-2420.

Alzer, H. (2010). Error function inequalities. *Adv. Comput. Math*., 33: 349–379. doi: 10.1007/s10444-009-9139-2.

Bercu, G. (2020). New refinements for the error function with applications in diffusion theory. *Symmetry*, 12. doi:10.3390/sym12122017.

Boyd, A. V. (1959). Inequalities for Mills' ratio. *Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs.* 6 1959, 44–46.

Eidous, O. (2022). Improvements of Polya upper bound for cumulative standard normal distribution and related functions. https://doi.org/10.48550/arXiv.2205.03485

Eidous, O. and Abu Shareefa, R. (2020). New approximations for standard normal distribution function. *Communications in Statistics - Simulation and computation,* 49 (6),1357-1374.

Lipoth, J., Tereda, Y., Papalexiou, S. and Spiteri, R. (2022). A new very simply explicitly invertible approximation for the standard normal cumulative distribution function. *AIMS Mathematics*, 7(7): 11635–11646. DOI:10.3934/math.2022648.

Mastin, A. and Jaillet, P. (2013).Log-Quadratic Bounds for the Gaussian Q-function. https://arxiv.org/abs/1304.2488.

Peric, Z. H., Nikolic, J. R. and Petkovic, M. D. (2019). Class of tight bounds on the Q - function with closed-form upper bound on relative error. *Math. Meth. Appl. Sci.,* 42, 1786-1794.

Ruskai, M. B. and Werner, E. (2000). Study of a class of regularizations of $1/|X|$ using Gaussian integrals. *SIAM Journal of Mathematical Analysis*, 32 (2), 435-463.

Sandoval-Hernandez, M., Vazquez-Leal, H., Filobello-Nino, U. and Hernandez-Martinez, L. (2019). New handy and accurate approximation for the Gaussian integrals with applications to science and engineering. *Open Mathematics*, 17 (1), 1774-1793. https://doi.org/10.1515/math-2019-0131

Simon, M. (2006). Probability Distributions Involving Gaussian Random Variables: *A Handbook for Engineers and Scientists*, 2nd ed. Springer.

Trigui, I., Agbogla, E. Benjillali, M., Ajib, W. and Zhu, W. (2021). Bit Error Rate Analysis for Reconfigurable Intelligent Surfaces with Phase Errors. *IEEE Communications Letters*, 25 (7): 2176 -2180.