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RESEARCH ARTICLE

The Cauchy problem for the non-isentropic compressible MHD fluids: optimal time-decay rates

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Abstract

This paper is concerned with the time-decay rates of the strong solutions of the three-dimensional non-isentropic compressible magnetohydrodynamic (MHD) system. First, motivated by Pu–Guo's result [Z. Angew. Math. Phys. 64 (2013) 519–538], we establish the existence result of a unique local-in-time strong solution for the MHD system. Then, we derive *a priori* estimates and use the continuity argument to obtain the global-in-time solution, where the initial data should be bounded in L^1-norm and is small in H^2-norm. Finally, based on Fourier theory and the idea of cancellation of a low-medium frequent part as in [Sci. China Math. 65 (2022) 1199–1228], we get the optimal time-decay rates (including highest-order derivatives) of strong solutions for non-isentropic MHD fluids. Our result is the first one concerning with the optimal decay estimates of the highest-order derivatives of the non-isentropic MHD system.

KEYWORDS:
non-isentropic MHD fluids, global well-posedness, Fourier theory, optimal time-decay rates.

1 | INTRODUCTION

In this paper, we are interested in the optimal time-decay rates of the strong solutions to the Cauchy problem of the non-isentropic compressible MHD fluids equations, which are formulated as follows:

$$\begin{cases} \rho_t + \mathrm{div}\,(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) - \mu_1\Delta\mathbf{u} - \left(\mu_1+\mu_2\right)\nabla\mathrm{div}\mathbf{u} + \nabla P\,(\rho,\theta) = \mathrm{curl}\mathbf{H}\times\mathbf{H}, \\ \rho c_v\left[\theta_t + (\mathbf{u}\cdot\nabla)\,\theta\right] + \theta P_\theta\,(\rho,\theta)\,\mathrm{div}\mathbf{u} = \kappa\Delta\theta + \Psi\,(\mathbf{u}) + \nu(\mathrm{curl}\mathbf{H})^2, \\ \mathbf{H}_t - \mathrm{curl}\,(\mathbf{u}\times\mathbf{H}) + \mathrm{curlcurl}\mathbf{H} = 0, \\ \mathrm{div}\mathbf{H} = 0, \end{cases} \tag{1}$$

where the unknown functions $\rho = \rho\,(t,\mathbf{x})$, $\mathbf{u} = \mathbf{u}\,(t,\mathbf{x})$, $\theta = \theta\,(t,\mathbf{x})$ and $\mathbf{H} = \mathbf{H}\,(t,\mathbf{x})$ denote the density, velocity, temperature and magnetic field of the MHD fluids, respectively. Here $(t,\mathbf{x}) \in \mathbb{R}_+\times\mathbb{R}^3$. In addition, $P = P\,(\rho,\theta)$ represents the pressure, μ_1 and μ_2 stand for the viscosity coefficients which satisfy $\mu_1 > 0$ and $2\mu_1 + 3\mu_2 \geqslant 0$. c_v, κ and ν represent the specific heat at constant volume, the coefficient of heat conduction and the magnetic diffusivity, respectively. In addition, the classical dissipation function $\Psi\,(\mathbf{u})$ is expressed as follows

$$\Psi\,(\mathbf{u}) = \frac{\mu_1}{2}\sum_{i,j=1}^{3}\left(\partial_i\mathbf{u}^j + \partial_j\mathbf{u}^i\right)^2 + \mu_2\sum_{i=1}^{3}\left(\partial_i\mathbf{u}^i\right)^2.$$

To investigate the well-posedness of Cauchy problem of the system of equations (1), we shall pose the initial data

$$(\rho, \mathbf{u}, \theta, \mathbf{H})(0, \mathbf{x}) = \left(\rho^0(\mathbf{x}), \mathbf{u}^0(\mathbf{x}), \theta^0(\mathbf{x}), \mathbf{H}^0(\mathbf{x})\right). \tag{2}$$

Let $\tilde{\rho}$ and $\tilde{\theta}$ be constants, then $(\rho, \mathbf{u}, \theta, \mathbf{H}) = \left(\tilde{\rho}, 0, \tilde{\theta}, 0\right)$ is an equilibrium state solution of the system (1). In the rest paper, we assume that $\tilde{\rho} = 1$ and $\tilde{\theta} = 1$ for the sake of simplicity.

At present, the incompressible/compressible MHD equations have been widely investigated, see [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16] and the references cited therein. In particular, for the compressible MHD system, there are many mathematical progress in the existence, stability and convergence rates of the solutions. Here we only list some of them, which are related to our study for the global well-posedness and decay estimates of the solutions of Cauchy problem. The interested readers can also refer to [17,18,1,19,20,21,22,23,24,25,26,27,28] for details.

- *Well-posedness for the MHD system*. For the isentropic case, Chen–Tan [2] studied the Cauchy problem of compressible MHD equations with the initial data being close to a constant equilibrium state and proved the global existence of the smooth solutions. Hu–Wang [29] established the global existence and large-time behavior of the solutions. In addition, the global existence of weak solutions for isentropic case can be also extended to the non-isentropic case [30]. Later on, Xu et al. [31] studied the Cauchy problem for the multi-dimensional ($N \geqslant 3$) non-isentropic full compressible magnetohydrodynamic equations. Besides, they proved the existence and uniqueness of a global strong solution when the initial data was close to a stable equilibrium state in critical Besov spaces. For the stability of MHD fluids, Jiang–Jiang [32] investigated the nonlinear stability and instability in the Rayleigh-Taylor problem of stratified compressible MHD systems.

- *Time-decay rates for the MHD system*. Matsumura and Nishida [33,34,35] have done some outstanding early work on the global existence and uniqueness of the initial value problem of compressible Navier-Stokes equations when the solution is perturbed near the equilibrium state. In particular, when the initial perturbation is sufficiently small in $H^3 \cap L^1$-norm, Matsumura and Nishida [35] gave the following decay estimates
$$\|(\rho - 1, \mathbf{u})(t)\|_{H^2} \lesssim (1+t)^{-\frac{3}{4}}.$$
Moreover, the compressible MHD fluid model, as the relevant promotion of the classical Navier-Stokes system, is also one of the goals widely concerned by mathematical experts at home and abroad. In the case of isentropic, Tan–Wang [36] concluded that when the initial perturbation belongs to $H^l \cap H^{-s}$ with $l \geqslant 3$ and $s \in \left[0, \frac{3}{2}\right)$, the higher-order derivatives for the solutions to the MHD flows enjoys the following optimal decay estimate
$$\left\|\nabla^k (\rho - 1, \mathbf{u}, \mathbf{H})(t)\right\|_{H^{l-k}} \lesssim (1+t)^{-\frac{k+s}{2}}, \ 0 \leqslant k \leqslant l-1.$$
The initial perturbation is bounded in L^q-norm with $q \in [1, 6/5)$ and is small sufficiently in H^3-norm, Pu–Guo [37] obtained the decay estimate of classical solutions for non-isentropic case in $\mathbb{R}^3$:
$$\left\|\nabla^k (\rho - 1, \mathbf{u}, \theta - 1, \mathbf{H})(t)\right\|_{H^{3-k}} \leqslant C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{2}}, \quad k = 0,\ 1.$$
Gao et al. [38] further improved the above result [37], and get that
$$\left\|\nabla^k \mathbf{H}(t)\right\|_{L^2} \leqslant C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{2}}, \quad k = 2,\ 3.$$

Recently, Wang–Wen [39] investigated the full compressible Navier–Stokes equations with reaction diffusion, and gave the results of global well-posedness and some optimal decay estimates of the solutions in the whole space. Inspired by Wang–Wen's results, this paper expects to obtain the decay estimates of the highest-order derivatives of the strong solution of compressible MHD fluid system in the case of non-isentropic. Therefore, the goal of this paper is twofold.

- First, making use of *a priori* estimates and the continuity argument, we establish the existence result of the global-in-time solution of non-isentropic MHD system based on the local existence result.

- Then, by establishing the energy functional, we fully explore the hidden equivalent information. Meanwhile, we further use the semigroup structure of the system and spectral methods to analyze the linearized system so that we can obtain the optimal convergence rates of solutions for the Cauchy problem (1)–(2).

Before stating our main result, let us introduce some notations throughout this paper.

1.1 | Notations

(1) To begin with, we will review the notations of L^p spaces, Sobolev spaces and the corresponding norms. $L^p\left(\mathbb{R}^3\right)$ with $1 \leqslant p \leqslant \infty$ stands for the usual L^p space whose norm is expressed by $\|\cdot\|_{L^p}$; $H^s\left(\mathbb{R}^3\right)$ with $1 \leqslant s \leqslant \infty$ stands for the usual Sobolev space whose norm is expressed by $\|\cdot\|_{H^s}$; Λ^s denotes the pseudo-differential operator defined by

$$\Lambda^s f = \mathcal{F}^{-1}\left(|\xi|^s \widehat{f}\right), \text{ for } s \in \mathbb{R},$$

where $\widehat{f}$ and $\mathcal{F}^{-1}(f)$ stand for the Fourier transform and the inverse Fourier transform, respectively.

(2) We shall introduce a frequency decomposition. Choose two smooth cut-off functions $\phi_0(\xi)$ and $\phi_1(\xi)$, which satisfy $0 \leqslant \phi_0(\xi), \phi_1(\xi) \leqslant 1 \left(\xi \in \mathbb{R}^3\right)$ and

$$\phi_0(\xi) = \begin{cases} 1, & |\xi| < r_0/2\,, \\ 0, & |\xi| > r_0, \end{cases} \qquad \phi_1(\xi) = \begin{cases} 0, & |\xi| < R_0, \\ 1, & |\xi| > R_0 + 1, \end{cases}$$

where some fixed constants r_0 and R_0 satisfy $0 < r_0 \leqslant \min\left\{\frac{1}{2}\sqrt{\frac{1}{2\mu_1+\mu_2}}, \frac{1}{2}\right\}$ and

$$R_0 > \max\left\{2\sqrt{\frac{\left(\mu_1+\mu_2+1\right)}{\mu_1}}, 2\right\}. \tag{3}$$

Definition 1. Let $\phi_0\left(D_x\right)$ and $\phi_1\left(D_x\right)$ be the pseudo-differential operators with respect to $\phi_0(\xi)$ and $\phi_1(\xi)$, respectively. For any function $f(x) \in L^2\left(\mathbb{R}^3\right)$, we then define the low, medium and high frequent part of $f(x)$ by

$$f^l(x) = \phi_0\left(D_x\right) f(x), \qquad f^m(x) = \left(I - \phi_0\left(D_x\right) - \phi_1\left(D_x\right)\right) f(x)$$

and

$$f^h(x) = \phi_1\left(D_x\right) f(x),$$

respectively. Here we denote D_x as $D_x = \frac{1}{\sqrt{-1}}\left(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\right)$.

Notice that $f(x)$ can be expressed as

$$f(x) = f^l(x) + f^m(x) + f^h(x), \tag{4}$$

where we define $f^L(x) := f^l(x) + f^m(x)$ and $f^H(x) := f^m(x) + f^h(x)$.

(3) Basic notations: for any integer $l \geqslant 0$, ∇^l denotes usual l-order spatial derivatives. When $l < 0$ or l is not a positive integer, ∇^l is usually written as Λ^l. We will use $m \lesssim n$ to denote $m \leqslant cn$, where c is a positive constant. We also employ $m \approx n$ to express $m \lesssim n$ and $m \gtrsim n$. And c_i $(i = 1, 2, ..., 10)$ stand for some general positive constants, which may vary in different estimates. We also use $\langle\cdot,\cdot\rangle$ to represent the inner product in $L^2\left(\mathbb{R}^3\right)$, i.e.

$$\langle f, h\rangle = \int_{\mathbb{R}^3} f(x) \cdot h(x) dx, \quad \text{for any } f(x), h(x) \in L^2\left(\mathbb{R}^3\right).$$

For simplicity, we set $\partial_i = \partial_{x_i}$ $(i = 1, 2, 3)$ and denote $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ for multi-indices $\alpha = \left(\alpha_1, \alpha_2, \alpha_3\right)$. And let $\|(m,n)\|_Z := \|m\|_Z + \|n\|_Z$, where m and n belong to Z.

1.2 | Main results

We consider the global existence of the solutions when the initial data

$$(\rho, \mathbf{u}, \theta, \mathbf{H})(0, \mathbf{x}) = \left(\rho^0(\mathbf{x}), \mathbf{u}^0(\mathbf{x}), \theta^0(\mathbf{x}), \mathbf{H}^0(\mathbf{x})\right)$$

of the non-isentropic compressible MHD system (1) is slightly perturbed near the equilibrium state in the three-dimensional case, and obtain the optimal decay rates of the strong solutions of the system (1). Our main result is as follows:

Theorem 1. Let $\sigma^0 = \rho^0 - 1$ and $\Theta^0 = \theta^0 - 1$, assume that $\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right) \in H^2\left(\mathbb{R}^3\right)$, there exists a constant $\epsilon_0 > 0$, such that if

$$\left\|\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)\right\|_{H^2\left(\mathbb{R}^3\right)} \leqslant \epsilon_0, \tag{5}$$

then the Cauchy problem of (1)–(2) with initial data $\left(\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0\right)$ admits a unique and global-in-time solution $(\rho(t,x), \mathbf{u}(t,x), \theta(t,x), \mathbf{H}(t,x))$, which satisfies

$$\begin{aligned} &\rho - 1 \in C^0\left([0,\infty); H^2\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,\infty); H^1\left(\mathbb{R}^3\right)\right), \\ &\mathbf{u},\ \theta - 1,\ \mathbf{H} \in C^0\left([0,\infty); H^2\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,\infty); L^2\left(\mathbb{R}^3\right)\right). \end{aligned}$$

Furthermore, if the initial data satisfies

$$\left\|\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)\right\|_{L^1\left(\mathbb{R}^3\right)} < +\infty, \tag{6}$$

then there exists a constant $C > 0$, such that for any $t \geqslant 0$, we have the following decay estimates

$$\left\|\nabla^k(\rho - 1, \mathbf{u}, \theta - 1, \mathbf{H})(t)\right\|_{L^2\left(\mathbb{R}^3\right)} \leqslant C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad k = 0, 1, 2. \tag{7}$$

Remark 1. Although the standard energy mathod is essential to solve the large-time behavior of the solutions, as far as we know, the decay rates of the highest-order derivatives of the solutions for the system can not be obtained directly and effectively by using the energy estimate method alone. Therefore, this paper provides a new strategy to study the large time behavior of the strong solutions of compressible non-isentropic MHD system, and obtains the optimal decay rates of the highest-order derivatives of the system. Furthermore, we note that this method is also applicable to the decay rates of the highest-order derivatives of most fluid systems that deal with compressible situations in three-dimensional space, such as liquid crystals, viscoelastic, capillaries and other complex fluids.

Next, we will briefly describe the difficulties encountered and the corresponding methods taken in proving the main theorems.

- Specifically, the difficulty of proving Theorem 1 is how to obtain the optimal decay estimates of the highest-order derivatives of strong solutions of MHD system. However, if we do not judge the existence of the solutions in advance, but skip the global existence of the solutions to directly study the large time behavior of the strong solutions. This is obviously meaningless. Fortunately, Matsumura and Nishida gave the proof of the existence of the initial value problem for the equations of motion of viscous and heat-conductive gases as early as [33], which provided us with a guiding idea. And the techniques used can be easily applied to the case of three-dimensional MHD system. Therefore, we can use the fixed point theorem and iteration technique to prove the local existence of strong solutions.

- Here we mainly introduce the key steps to prove the global existence of strong solutions. In the first step, we establish the following two energy functionals

$$\mathcal{E}_l(t) := \frac{1}{2}\left(\|\sigma\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2 + \|\Theta\|_{H^1}^2 + \|\mathbf{H}\|_{H^1}^2\right) + \alpha_1 \int_{\mathbb{R}^3} \nabla\sigma \cdot \mathbf{u}\, dx$$

  and

$$\mathcal{E}_h(t) := \frac{1}{2}\left(\left\|\nabla^2\sigma\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^2\Theta\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{H}\right\|_{L^2}^2\right) + \alpha_2 \int_{\mathbb{R}^3} \nabla\nabla\sigma \cdot \nabla\mathbf{u}\, dx.$$

  Furthermore, the standard energy method is used to estimate the $L_t^\infty H_x^1$-norm and the norm of the 2nd-order derivative of strong solutions $(\rho(t,x), \mathbf{u}(t,x), \theta(t,x), \mathbf{H}(t,x))$ for the system (1). By using the equivalence condition

$$\mathcal{E}_l(t) + \mathcal{E}_h(t) \approx \|(\sigma, \mathbf{u}, \Theta, \mathbf{H})\|_{H^2}^2$$

  and integrating with respect to t, the upper bound of our expected *a priori* estimates can be obtained. Based on this, through the standard continuity argument, the local-in-time strong solutions of the system can be extended to the global-in-time strong solutions, and the global existence of the strong solutions can be proved.

- The above proof of the global existence of strong solutions is a conventional and standard practice. However, as we mentioned in Remark 1, the optimal decay rates of the highest-order derivatives of strong solutions for the system can not be obtained directly by the standard energy method. Therefore, it is a difficult problem to be solved urgently.
  In this regard, we will adopt the following strategies.
  Step 1: we observe the implicit equivalence relationship

$$\mathcal{E}_h(t) - \alpha_2 \int_{\mathbb{R}^3} \nabla\nabla\sigma^L \cdot \nabla\mathbf{u}\, dx \approx \left\|\nabla^2(\sigma, \mathbf{u}, \Theta, \mathbf{H})\right\|_{L^2}^2$$

by eliminating the interference term $\int_{\mathbb{R}^3} \nabla\nabla\sigma \cdot \nabla \mathbf{u} dx$ in $\mathcal{E}_h(t)$.

Step 2: by applying the above results of *a priori* estimates, we can further draw a conclusion that the L^2-norm of the highest-order derivatives of the strong solutions for the system can be controlled by the initial data $\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)$ and the L^2 estimates of the 2nd-order derivative of the low-medium-frequency parts $\left(\sigma^L, \mathbf{u}^L, \Theta^L, \mathbf{H}^L\right)$.

Step 3: we consider the linearized system, transform it into Fourier space, and then use the semigroup decomposition theory. By tedious calculation, the estimates of eigenvalues $\lambda_j\,(j=1,2,3)$ and the semigroups $e^{-t\widehat{\mathcal{A}}(|\xi|)}$ in low, medium and high frequency can be obtained.

Based on the above key steps, we can solve the optimal decay rates of the highest-order derivatives of strong solutions for the system.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries for later use. In Section 3, we reformulate the nonhomogeneous system (21), which is transformed into a perturbation form (22)–(23). In Section 4, we state the local existence of the solution for the system, establish the a priori estimates of the solution, and then prove the existence of the global-in-time solution. Moreover, the proof of Proposition 2 and Theorem 2 are given in Subsection 4.2–4.3, respectively. Finally, we establish some decay estimates for the linearized system, and thus obtain the optimal time-decay rates for the nonhomogeneous system in Section 5.

2 | PRELIMINARIES

In this section, we will introduce some important lemmas, which are frequently used in the sequel. Now let us recall some well-known Sobolev inequalities.

Lemma 1 ([40,41])**.** Let $f \in H^2\left(\mathbb{R}^3\right)$, we have
(i) $\|f\|_{L^r} \leqslant c\|f\|_{H^1}$ for $2 \leqslant r \leqslant 6$;
(ii) $\|f\|_{L^\infty} \leqslant c\,\|\nabla f\|_{L^2}^{1/2}\ \|\nabla f\|_{H^1}^{1/2} \leqslant c\|\nabla f\|_{H^1}$;
(iii) $\|f\|_{L^6} \leqslant c\|\nabla f\|_{L^2}$, where c is a positive constant.

Besides, we have the following estimate on the product of the two functions.

Lemma 2 ([43])**.** Let g and h belong to the Schwartz function class, then for $k \geqslant 0$, we have

$$\left\|\nabla^k(gh)\right\|_{L^r} \lesssim \|g\|_{L^{r_1}}\left\|\nabla^k h\right\|_{L^{r_2}} + \left\|\nabla^k g\right\|_{L^{r_3}}\|h\|_{L^{r_4}}, \tag{8}$$

where $1 < r, r_2, r_3 < \infty$ and $r_i\,(i=1,2,3,4)$ satisfy

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}. \tag{9}$$

We then recall the following Gagliardo-Nirenberg inequality.

Lemma 3 ([44])**.** If $0 \leqslant i, j \leqslant k$, then we have

$$\|\nabla^i f\|_{L^q} \lesssim \|\nabla^j f\|_{L^{q_1}}^{1-\delta}\left\|\nabla^k f\right\|_{L^{q_2}}^{\delta} \tag{10}$$

with $0 \leqslant \delta \leqslant 1$, and it satisfies

$$\frac{i}{3} - \frac{1}{q} = \left(\frac{j}{3} - \frac{1}{q_1}\right)(1-\delta) + \left(\frac{k}{3} - \frac{1}{q_2}\right)\delta. \tag{11}$$

In particular, if $q = \infty$, then $0 < \delta < 1$ is required.

By using the above lemma, we can easily prove that

Lemma 4 ([42, Lemma 4.2])**.** Let $\varphi(\sigma, \Theta)$ be a smooth function of σ, Θ with bounded derivatives of any order. If $\|(\sigma, \Theta)\|_{L^\infty(\mathbb{R}^3)} \leqslant 1$ holds, then for any integer $i \geqslant 1$, we have

$$\left\|\nabla^i(\varphi(\sigma, \Theta))\right\|_{L^p(\mathbb{R}^3)} \lesssim \left\|\nabla^i(\sigma, \Theta)\right\|_{L^p(\mathbb{R}^3)}, \tag{12}$$

where $1 \leqslant p \leqslant +\infty$.

To prove the decay estimates of the solution, we further introduce the following basic inequalities

Lemma 5 ([45]). Assume $a_1, a_2, a_3 \in \mathbb{R}$ and $a_2 > 1, 0 \leqslant a_1 \leqslant a_2, a_3 > 0$, then for $t \in \mathbb{R}_+$, we have

$$\int_0^t (1+t-\tau)^{-a_1}(1+\tau)^{-a_2} d\tau \leqslant C\left(a_1, a_2\right)(1+t)^{-a_1}, \tag{13}$$

$$\int_0^t (1+\tau)^{-a_1} e^{-a_3(t-\tau)} d\tau \leqslant C\left(a_1, a_3\right)(1+t)^{-a_1}, \tag{14}$$

where $C\left(a_1, a_2\right), C\left(a_1, a_3\right)$ are positive constants that depend only on a_1, a_2, a_3.

By combining the definition of $\left(f^l(x), f^m(x), f^h(x)\right)$ and Plancherel theorem, we get the following conclusions.

Lemma 6 ([39]). Let $f \in H^m\left(\mathbb{R}^3\right)$, then for any given intergers i, j and k, we have

$$\left\|\nabla^j f^l\right\|_{L^2} \leqslant r_0^{j-i}\left\|\nabla^i f^l\right\|_{L^2}, \quad \left\|\nabla^j f^h\right\|_{L^2} \leqslant \frac{1}{R_0^{k-j}}\left\|\nabla^k f^h\right\|_{L^2}, \tag{15}$$

$$\left\|\nabla^j f^l\right\|_{L^2} \leqslant \left\|\nabla^k f\right\|_{L^2} \text{ and } \left\|\nabla^j f^h\right\|_{L^2} \leqslant \left\|\nabla^k f\right\|_{L^2}, \tag{16}$$

where $i \leqslant j \leqslant k \leqslant m$. In addition, it hold that for some constant $r_0 > 0$ and $R_0 > 0$,

$$r_0^j \|f^m\|_{L^2} \leqslant \|\nabla^j f^m\|_{L^2} \leqslant R_0^j \|f^m\|_{L^2}. \tag{17}$$

3 | REFORMULATION OF THE SYSTEM

For the convenience of proving Theorem 1, we first need to reformulate the system (1)–(2). Notice that div$\mathbf{H} = 0$, and by direct calculation, we can easily get the following identities

$$\mathrm{curl}\mathbf{H} \times \mathbf{H} = \mathbf{H} \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla^{\mathrm{T}} \mathbf{H} = \mathrm{div}\left(\mathbf{H} \otimes \mathbf{H}\right) - \frac{1}{2}\nabla\left(|\mathbf{H}|^2\right), \tag{18}$$

$$\mathrm{curlcurl}\mathbf{H} = \nabla \mathrm{div}\mathbf{H} - \Delta \mathbf{H} = -\Delta \mathbf{H} \tag{19}$$

and

$$\mathrm{curl}\left(\mathbf{u} \times \mathbf{H}\right) = \left(\mathbf{H} \cdot \nabla\right)\mathbf{u} - \left(\mathbf{u} \cdot \nabla\right)\mathbf{H} - \mathbf{H}\mathrm{div}\mathbf{u}. \tag{20}$$

Without loss of generality, we take $c_v = \kappa = \nu = 1$ and $P_\rho(1,1) = P_\theta(1,1) = 1$. Thanks to (18)–(20), the system (1) can be rewritten as follows:

$$\begin{cases} \rho_t + \mathrm{div}\left(\rho \mathbf{u}\right) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \dfrac{\mu_1}{\rho}\Delta \mathbf{u} - \dfrac{\left(\mu_1 + \mu_2\right)}{\rho}\nabla \mathrm{div}\mathbf{u} + \dfrac{P_\rho(\rho,\theta)}{\rho}\nabla\rho + \dfrac{P_\theta(\rho,\theta)}{\rho}\nabla\theta \\ \quad = \dfrac{\left(\mathbf{H} \cdot \nabla\right)\mathbf{H}}{\rho} - \dfrac{\mathbf{H} \cdot \nabla^{\mathrm{T}}\mathbf{H}}{\rho}, \\ \theta_t + \mathbf{u} \cdot \nabla\theta + \dfrac{\theta P_\theta(\rho,\theta)}{\rho}\mathrm{div}\mathbf{u} = \dfrac{1}{\rho}\Delta\theta + \dfrac{1}{\rho}\Psi(\mathbf{u}) + \dfrac{1}{\rho}(\mathrm{curl}\mathbf{H})^2, \\ \mathbf{H}_t - \left(\mathbf{H} \cdot \nabla\right)\mathbf{u} + \left(\mathbf{u} \cdot \nabla\right)\mathbf{H} + \mathbf{H}\mathrm{div}\mathbf{u} = \Delta \mathbf{H}, \\ \mathrm{div}\mathbf{H} = 0. \end{cases} \tag{21}$$

Next, let

$$\sigma = \rho - 1, \qquad \mathbf{u} = \mathbf{u}, \qquad \Theta = \theta - 1, \qquad \mathbf{H} = \mathbf{H},$$

and then the system (1)–(2) is equivalent to the following perturbation form

$$\begin{cases} \sigma_t + \mathrm{div}\mathbf{u} = \mathcal{N}_1, \\ \mathbf{u}_t - \mu_1 \Delta \mathbf{u} - \left(\mu_1 + \mu_2\right) \nabla \mathrm{div}\mathbf{u} + \nabla \sigma + \nabla \Theta = \mathcal{N}_2, \\ \Theta_t - \Delta \Theta + \mathrm{div}\mathbf{u} = \mathcal{N}_3, \\ \mathbf{H}_t - \Delta \mathbf{H} = \mathcal{N}_4, \\ \mathrm{div}\mathbf{H} = 0 \end{cases} \tag{22}$$

with the initial data

$$\begin{aligned} (\sigma, \mathbf{u}, \Theta, \mathbf{H})(0, \mathbf{x}) &= \left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)(\mathbf{x}) \\ &= \left(\rho^0 - 1, \mathbf{u}^0, \theta^0 - 1, \mathbf{H}^0\right)(\mathbf{x}), \end{aligned} \tag{23}$$

where the nonlinear terms $\mathcal{N}_i$ $(1 \leqslant i \leqslant 4)$ are defined as follows

$$\begin{cases} \mathcal{N}_1 = -\mathrm{div}\,(\sigma \mathbf{u}), \\ \mathcal{N}_2 = -\mathbf{u} \cdot \nabla \mathbf{u} - h_1(\sigma, \Theta) \nabla \sigma - h_2(\sigma, \Theta) \nabla \Theta \\ \qquad + g_1(\sigma)\left((\mathbf{H} \cdot \nabla)\mathbf{H} - \mathbf{H} \cdot \nabla^{\mathrm{T}} \mathbf{H}\right) - g_2(\sigma)\left(\mu_1 \Delta \mathbf{u} + \left(\mu_1 + \mu_2\right) \nabla \mathrm{div}\mathbf{u}\right), \\ \mathcal{N}_3 = -(\mathbf{u} \cdot \nabla)\Theta - g_2(\sigma)\Delta\Theta + g_1(\sigma)\left(\Psi(\mathbf{u}) + (\mathrm{curl}\mathbf{H})^2\right) - h_3(\sigma, \Theta)\,\mathrm{div}\mathbf{u}, \\ \mathcal{N}_4 = (\mathbf{H} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{H} - \mathbf{H}\mathrm{div}\mathbf{u} \end{cases} \tag{24}$$

and the nonlinear functions of (σ, Θ) are written as

$$\begin{cases} g_1(\sigma) = \dfrac{1}{\sigma + 1}, \\ g_2(\sigma) = \dfrac{\sigma}{\sigma + 1}, \\ h_1(\sigma, \Theta) = \dfrac{P_\sigma(\sigma + 1, \Theta + 1)}{\sigma + 1} - 1, \\ h_2(\sigma, \Theta) = \dfrac{P_\Theta(\sigma + 1, \Theta + 1)}{\sigma + 1} - 1, \\ h_3(\sigma, \Theta) = \dfrac{(\Theta + 1) P_\theta(\sigma + 1, \Theta + 1)}{\sigma + 1} - 1. \end{cases} \tag{25}$$

4 | GLOBAL WELL-POSEDNESS FOR THE NONLINEAR SYSTEM

In this section, we will prove the global well-posedness of the solution stated in Theorem 1, that is, the global existence and uniqueness of the solution for the system (1)–(2). For this problem, our strategy is to combine the local existence result and *a priori* estimates. And then, by using the standard continuity argument, we accomplish the proof of the global well-posedness. Specifically, the local existence of the solution will be given in subsection 4.1. In addition, the results of *a priori* estimates and the global existence will be proved in detail in subsection 4.2–4.3, respectively.

4.1 | The global existence of the solution

To begin with, we define the solution space for the system (22)–(23) by

$$\begin{aligned} \Omega(0, T) := \Big\{ (\sigma, \mathbf{u}, \Theta, \mathbf{H}) \Big| & \sigma \in C^0\left(0, T; H^2\left(\mathbb{R}^3\right)\right) \cap C^1\left(0, T; H^1\left(\mathbb{R}^3\right)\right), \\ & \mathbf{u}, \Theta, \mathbf{H} \in C^0\left(0, T; H^2\left(\mathbb{R}^3\right)\right) \cap C^1\left(0, T; L^2\left(\mathbb{R}^3\right)\right), \\ & \nabla \sigma \in L^2\left(0, T; H^1\left(\mathbb{R}^3\right)\right), \nabla \mathbf{u}, \nabla \Theta, \nabla \mathbf{H} \in L^2\left(0, T; H^2\left(\mathbb{R}^3\right)\right) \Big\}, \end{aligned} \tag{26}$$

for any $0 \leqslant T \leqslant +\infty$.

In the following discussion, we will present the local existence of the solution and some *a priori* estimates one by one. It is critically important to prove the global well-posedness for the system. The details are as follows:

Proposition 1 (The local existence)**.** Assume that $\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right) \in H^2\left(\mathbb{R}^3\right)$ and

$$\inf_{x\in\mathbb{R}^3} \left\{\sigma^0+1, \Theta^0+1\right\} > 0,$$

then there exists a constant $T_1 > 0$ depending only on $\left\|\left(\sigma^0, \mathbf{u^0}, \Theta^0, \mathbf{H^0}\right)\right\|_{H^2}$, such that the system (22)–(23) with initial data $\left(\sigma^0, \mathbf{u^0}, \Theta^0, \mathbf{H^0}\right)$ has a unique solution $(\sigma, \mathbf{u}, \Theta, \mathbf{H}) \in \Omega\left(0, T_1\right)$ satisfying the estimates for any $t \in \left[0, T_1\right]$

$$\inf_{x\in\mathbb{R}^3, 0\leqslant t\leqslant T_1} \{\sigma+1, \Theta+1\} > 0$$

and

$$\|(\sigma, \mathbf{u}, \Theta, \mathbf{H})(t)\|_{H^2}, \left(\int_0^t \|\nabla(\mathbf{u}, \Theta, \mathbf{H})(t)\|_{H^2}^2 d\tau\right)^{\frac{1}{2}} \leqslant \sqrt{c_1}\left\|\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)\right\|_{H^2},$$

where $c_1 > 0$ is a constant.

Proof. We can easily use iteration technique, the fixed point theorem and the maximum principle to prove the proposition, please refer to [33,2]. □

Proposition 2 (A priori estimates)**.** Let $\Omega(0, T)$ be given by (26) for some $T > 0$. Assume that the Cauchy problem of (22)–(23) with initial data $\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)$ has a solution $(\sigma, \mathbf{u}, \Theta, \mathbf{H}) \in \Omega(0, T)$, then there exist a small enough constant $\epsilon > 0$, such that if

$$\mathfrak{E} =: \sup_{0\leqslant t\leqslant T} \|(\sigma, \mathbf{u}, \Theta, \mathbf{H})(t)\|_{H^2} \leqslant \epsilon, \tag{27}$$

then for any $t \in [0, T]$, we have

$$\begin{aligned}
&\|(\sigma, \mathbf{u}, \Theta, \mathbf{H})(t)\|_{H^2}^2 + \int_0^t \left(\|\nabla\sigma(\tau)\|_{H^1}^2 + \|\nabla(\mathbf{u}, \Theta, \mathbf{H})(\tau)\|_{H^2}^2\right) d\tau \\
&\leqslant c_2\left\|\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)(t)\right\|_{H^2}^2,
\end{aligned} \tag{28}$$

where $c_2 > 0$ is a constant that does not depend on T.

Remark 2. Some remarks concerning Proposition 2 are listed as follows:

- It is worth noting that c_2 depends not only on T, but also on ϵ and ϵ_0. In addition, if the initial data $\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)$ also satisfies $\left\|\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)\right\|_{H^2} < min\left\{\epsilon/\sqrt{c_1}, \epsilon/\sqrt{c_1 c_2}\right\}$, then we can deduce the global existence of the solution, see Theorem 2.
- Under *a priori* assumption (27), by using Sobolev imbedding inequality, we can easily deduce

$$\frac{1}{2} \leqslant \rho+1,\ \theta+1 \leqslant \frac{3}{2}. \tag{29}$$

We further obtain

$$\left|g_2(\sigma)\right|, \left|h_1(\sigma, \Theta)\right|, \left|h_2(\sigma, \Theta)\right|, \left|h_3(\sigma, \Theta)\right| \leqslant c_3(|\sigma| + |\Theta|) \tag{30}$$

and for any integer $k_1 \geqslant 0$, $k_2 \geqslant 1$,

$$\left|{g_1}^{(k_1)}(\sigma)\right| \leqslant c_3, \tag{31}$$

$$\left|{g_2}^{(k_2)}(\sigma)\right|, \left|{h_1}^{(k_2)}(\sigma, \Theta)\right|, \left|{h_2}^{(k_2)}(\sigma, \Theta)\right| \leqslant c_3, \tag{32}$$

where c_3 is a positive constant.

With Propositions 1–2 in hand, we easily get the following global existence result, the proof of which will be provided in subsection 4.3.

Theorem 2 (The global existence)**.** Let $\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right) \in H^2\left(\mathbb{R}^3\right)$. There exists a positive constant ϵ, such that if

$$\mathfrak{E}_0 < \min\left\{\epsilon/\sqrt{c_1}, \epsilon/\sqrt{c_1 c_2}\right\}, \tag{33}$$

then the Cauchy problem of (22)–(23) with initial data $\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)$ has a unique, global-in-time classical solution $(\sigma, \mathbf{u}, \Theta, \mathbf{H})$ and for any $t > 0$, we have

$$\|(\sigma, \mathbf{u}, \Theta, \mathbf{H})(t)\|_{H^2}^2 + \int_0^t \left(\|\nabla\sigma(\tau)\|_{H^1}^2 + \|\nabla(\mathbf{u}, \Theta, \mathbf{H})(\tau)\|_{H^2}^2\right) d\tau \leqslant c_2 \mathfrak{E}_0^2, \tag{34}$$

where $\mathfrak{E}_0$ is defined by

$$\mathfrak{E}_0 := \left\|\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)\right\|_{H^2} < \infty. \tag{35}$$

4.2 | Proof of Proposition 2

This subsection is devoted to the proof of Proposition 2, in which the essential step is to establish some energy estimates of the solution $(\sigma, \mathbf{u}, \boldsymbol{\Theta}, \mathbf{H})$. Thus the proof can be divided into two parts:

- First, we will pay attention to the energy estimate on the lower-order derivatives of the solution, see Lemma 7 for details.
- Second, based on the energy method, the estimate about the highest-order derivatives of the solution $(\sigma, \mathbf{u}, \Theta, \mathbf{H})$ will be established, see Lemma 8 for details.

Lemma 7. Let

$$\mathcal{E}_l(t) =: \frac{1}{2}\left(\|\sigma\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2 + \|\Theta\|_{H^1}^2 + \|\mathbf{H}\|_{H^1}^2\right) + \alpha_1 \int_{\mathbb{R}^3} \nabla\sigma \cdot \mathbf{u} dx, \tag{36}$$

then we have

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}_l(t) + \frac{\alpha_1}{4}\|\nabla\sigma\|_{L^2}^2 + \frac{\mu_1}{2}\|\nabla\mathbf{u}\|_{H^1}^2 + \frac{\left(\mu_1+\mu_2\right)}{2}\|\mathrm{div}\mathbf{u}\|_{H^1}^2 \\ &+ \frac{1}{2}\|\nabla\Theta\|_{H^1}^2 + \frac{1}{2}\|\nabla\mathbf{H}\|_{H^1}^2 \leqslant 0, \end{aligned} \tag{37}$$

where $0 < \alpha_1 \leqslant \min\left\{\frac{1}{6\left(\mu_1+\mu_2\right)}, \frac{1}{3\mu_1}, \frac{1}{3}, \frac{\left(\mu_1+\mu_2\right)}{4}\right\}$ is a given constant.

Proof. To start with, applying ∇^k to $(22)_1$–$(22)_4$, and multiplying the resulting identities by $\nabla^k\sigma$, $\nabla^k\mathbf{u}$, $\nabla^k\Theta$, $\nabla^k\mathbf{H}$ respectively, summing them up, and then integrating over $\mathbb{R}^3$ by parts, we can obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla^k\sigma\right\|_{L^2}^2 + \left\|\nabla^k\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^k\Theta\right\|_{L^2}^2 + \left\|\nabla^k\mathbf{H}\right\|_{L^2}^2\right) + \mu_1\left\|\nabla^k\nabla\mathbf{u}\right\|_{L^2}^2 \\ &+ \left(\mu_1+\mu_2\right)\left\|\nabla^k\mathrm{div}\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^k\nabla\Theta\right\|_{L^2}^2 + \left\|\nabla^k\nabla\mathbf{H}\right\|_{L^2}^2 \\ &= \left\langle\nabla^k\sigma, \nabla^k\mathcal{N}_1\right\rangle + \left\langle\nabla^k\mathbf{u}, \nabla^k\mathcal{N}_2\right\rangle + \left\langle\nabla^k\Theta, \nabla^k\mathcal{N}_3\right\rangle + \left\langle\nabla^k\mathbf{H}, \nabla^k\mathcal{N}_4\right\rangle. \end{aligned} \tag{38}$$

Next, by taking $\langle\nabla(22)_1, \mathbf{u}\rangle + \langle(22)_2, \nabla\sigma\rangle$, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3} \mathbf{u}\cdot\nabla\sigma dx + \int_{\mathbb{R}^3} |\nabla\sigma|^2 dx &= \|\mathrm{div}\mathbf{u}\|_{L^2}^2 + \mu_1\int_{\mathbb{R}^3} \Delta\mathbf{u}\cdot\nabla\sigma dx - \int_{\mathbb{R}^3} \nabla\Theta\cdot\nabla\sigma dx \\ &+ \left(\mu_1+\mu_2\right)\int_{\mathbb{R}^3} \nabla\mathrm{div}\mathbf{u}\cdot\nabla\sigma dx \\ &+ \int_{\mathbb{R}^3} \nabla\mathcal{N}_1\cdot\mathbf{u} dx + \int_{\mathbb{R}^3} \mathcal{N}_2\cdot\nabla\sigma dx \\ &=: \|\mathrm{div}\mathbf{u}\|_{L^2}^2 + \sum_{i=1}^5 \mathcal{K}_i. \end{aligned} \tag{39}$$

For a given constant $\alpha_1 > 0$, we can utilize Young's inequality to get

$$\alpha_1 \mathcal{K}_1 = \alpha_1 \mu_1 \int_{\mathbb{R}^3} \Delta \mathbf{u} \cdot \nabla \sigma dx \leqslant \frac{\alpha_1}{6} \|\nabla \sigma\|_{L^2}^2 + \frac{3\alpha_1 \mu_1^2}{2} \|\Delta \mathbf{u}\|_{L^2}^2, \tag{40}$$

$$\alpha_1 \mathcal{K}_2 = -\alpha_1 \int_{\mathbb{R}^3} \nabla \Theta \cdot \nabla \sigma dx \leqslant \frac{\alpha_1}{6} \|\nabla \sigma\|_{L^2}^2 + \frac{3\alpha_1}{2} \|\nabla \Theta\|_{L^2}^2 \tag{41}$$

and

$$\alpha_1 \mathcal{K}_3 = \alpha_1 \left(\mu_1 + \mu_2\right) \int_{\mathbb{R}^3} \nabla \mathrm{div} \mathbf{u} \cdot \nabla \sigma dx \leqslant \frac{\alpha_1}{6} \|\nabla \sigma\|_{L^2}^2 + \frac{3\alpha_1 \left(\mu_1 + \mu_2\right)^2}{2} \|\nabla \mathrm{div} \mathbf{u}\|_{L^2}^2. \tag{42}$$

And then, we add up $\alpha_1 \times (39)$ to (38) with $k = 0, 1$. This together with (40)–(42) implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|\sigma\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2 + \|\Theta\|_{H^1}^2 + \|\mathbf{H}\|_{H^1}^2 + 2\alpha_1 \int_{\mathbb{R}^3} \nabla \sigma \cdot \mathbf{u} dx\right\} \\
&+ \frac{\alpha_1}{2} \|\nabla \sigma\|_{L^2}^2 + \mu_1 \|\nabla \mathbf{u}\|_{H^1}^2 + \left(\mu_1 + \mu_2\right) \|\mathrm{div} \mathbf{u}\|_{H^1}^2 + \|\nabla \Theta\|_{H^1}^2 + \|\nabla \mathbf{H}\|_{H^1}^2 \\
&\leqslant \frac{3\alpha_1 \mu_1^2}{2} \|\Delta \mathbf{u}\|_{L^2}^2 + \frac{3\alpha_1}{2} \|\nabla \Theta\|_{L^2}^2 + \frac{3\alpha_1 \left(\mu_1 + \mu_2\right)^2}{2} \|\nabla \mathrm{div} \mathbf{u}\|_{L^2}^2 + \alpha_1 \|\mathrm{div} \mathbf{u}\|_{L^2}^2 \\
&+ \int_{\mathbb{R}^3} \nabla \sigma \cdot \nabla \mathcal{N}_1 dx + \int_{\mathbb{R}^3} \sigma \cdot \mathcal{N}_1 dx + \int_{\mathbb{R}^3} \nabla \mathbf{u} \cdot \nabla \mathcal{N}_2 dx \\
&+ \int_{\mathbb{R}^3} \mathbf{u} \cdot \mathcal{N}_2 dx + \int_{\mathbb{R}^3} \nabla \Theta \cdot \nabla \mathcal{N}_3 dx + \int_{\mathbb{R}^3} \Theta \cdot \mathcal{N}_3 dx \\
&+ \int_{\mathbb{R}^3} \nabla \mathbf{H} \cdot \nabla \mathcal{N}_4 dx + \int_{\mathbb{R}^3} \mathbf{H} \cdot \mathcal{N}_4 dx + \alpha_1 \int_{\mathbb{R}^3} \mathbf{u} \cdot \nabla \mathcal{N}_1 dx + \alpha_1 \int_{\mathbb{R}^3} \nabla \sigma \cdot \mathcal{N}_2 dx \\
&=: \frac{3\alpha_1 \mu_1^2}{2} \|\Delta \mathbf{u}\|_{L^2}^2 + \frac{3\alpha_1}{2} \|\nabla \Theta\|_{L^2}^2 + \frac{3\alpha_1 \left(\mu_1 + \mu_2\right)^2}{2} \|\nabla \mathrm{div} \mathbf{u}\|_{L^2}^2 + \alpha_1 \|\mathrm{div} \mathbf{u}\|_{L^2}^2 + \sum_{i=1}^{10} \mathcal{S}_i.
\end{aligned} \tag{43}$$

Now, we turn to estimate the nonlinear terms $\mathcal{S}_i \, (1 \leqslant i \leqslant 10)$ on the right hand side of (43). For the term $\mathcal{S}_1$, by applying integration by parts, Hölder's inequality, Young's inequality, Lemma 1–2 and (27), we deduce that

$$\begin{aligned}
\mathcal{S}_1 =& - \int_{\mathbb{R}^3} \nabla \sigma \cdot \nabla \left(\mathrm{div}\left(\sigma \mathbf{u}\right)\right) dx \\
\leqslant& c \left\|\nabla^2 \sigma\right\|_{L^2} \|\nabla \left(\sigma \mathbf{u}\right)\|_{L^2} \\
\leqslant& c \left\|\nabla^2 \sigma\right\|_{L^2} \left(\|\nabla \sigma\|_{L^2} \|\mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{u}\|_{L^2} \|\sigma\|_{L^\infty}\right) \\
\leqslant& c\epsilon \left\|\nabla^2 \sigma\right\|_{L^2} \|\nabla \left(\sigma, \mathbf{u}\right)\|_{L^2} \\
\leqslant& c\epsilon \left(\left\|\nabla^2 \sigma\right\|_{L^2}^2 + \|\nabla \left(\sigma, \mathbf{u}\right)\|_{L^2}^2\right).
\end{aligned} \tag{44}$$

Similarly, $\mathcal{S}_2$ can be estimated as follows

$$\begin{aligned}
\mathcal{S}_2 =& - \int_{\mathbb{R}^3} \sigma \mathrm{div} \left(\sigma \mathbf{u}\right) dx \\
\leqslant& c \|\sigma\|_{L^6} \|\nabla \left(\sigma \mathbf{u}\right)\|_{L^{\frac{6}{5}}} \\
\leqslant& c \|\sigma\|_{L^6} \left(\|\nabla \sigma\|_{L^2} \|\mathbf{u}\|_{L^3} + \|\sigma\|_{L^3} \|\nabla \mathbf{u}\|_{L^2}\right) \\
\leqslant& c\epsilon \|\nabla \left(\sigma, \mathbf{u}\right)\|_{L^2}^2.
\end{aligned} \tag{45}$$

Next, for the term $\mathcal{S}_3$, by using the definition of $\mathcal{N}_2$, Hölder's inequality, Young's inequality, Lemma 1–2, (27) and (30)–(31), we get

$$\begin{aligned}
\mathcal{S}_3 =& \int_{\mathbb{R}^3} \nabla\mathbf{u}\cdot\nabla\left(-\mathbf{u}\cdot\nabla\mathbf{u} - h_1(\sigma,\Theta)\nabla\sigma - h_2(\sigma,\Theta)\nabla\Theta\right)dx \\
&+\int_{\mathbb{R}^3} \nabla\mathbf{u}\cdot\nabla\left(g_1(\sigma)\left(\mathbf{H}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla^{\mathrm{T}}\mathbf{H}\right)\right)dx \\
&+\int_{\mathbb{R}^3} \nabla\mathbf{u}\cdot\nabla\left(-g_2(\sigma)\left(\mu_1\Delta\mathbf{u} + \left(\mu_1+\mu_2\right)\nabla\mathrm{div}\mathbf{u}\right)\right)dx \\
\leqslant& c\left\|\nabla^2\mathbf{u}\right\|_{L^2}\left(\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^\infty} + \left\|h_1(\sigma,\Theta)\right\|_{L^\infty}\|\nabla\sigma\|_{L^2} + \left\|h_2(\sigma,\Theta)\right\|_{L^\infty}\|\nabla\Theta\|_{L^2}\right) \\
&+c\left\|\nabla^2\mathbf{u}\right\|_{L^2}\left(\left\|g_1(\sigma)\right\|_{L^\infty}\|\mathbf{H}\|_{L^\infty}\|\nabla\mathbf{H}\|_{L^2} + \left\|g_2(\sigma)\right\|_{L^\infty}\left\|\nabla^2\mathbf{u}\right\|_{L^2}\right) \\
\leqslant& c\epsilon\left(\|\nabla(\sigma,\mathbf{u},\Theta,\mathbf{H})\|_{L^2}^2 + \left\|\nabla^2\mathbf{u}\right\|_{L^2}^2\right).
\end{aligned} \tag{46}$$

Similarly to (46), $\mathcal{S}_4$ can be estimated as follows

$$\begin{aligned}
\mathcal{S}_4 =& \int_{\mathbb{R}^3} \mathbf{u}\cdot\left(-\mathbf{u}\cdot\nabla\mathbf{u} - h_1(\sigma,\Theta)\nabla\sigma - h_2(\sigma,\Theta)\nabla\Theta\right)dx \\
&+\int_{\mathbb{R}^3} \mathbf{u}\cdot\left(g_1(\sigma)\left(\mathbf{H}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla^{\mathrm{T}}\mathbf{H}\right)\right)dx \\
&+\int_{\mathbb{R}^3} \mathbf{u}\cdot\left(-g_2(\sigma)\left(\mu_1\Delta\mathbf{u} + \left(\mu_1+\mu_2\right)\nabla\mathrm{div}\mathbf{u}\right)\right)dx \\
\leqslant& c\|\mathbf{u}\|_{L^6}\left(\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^3} + \left\|h_1(\sigma,\Theta)\right\|_{L^3}\|\nabla\sigma\|_{L^2} + \left\|h_2(\sigma,\Theta)\right\|_{L^3}\|\nabla\Theta\|_{L^2}\right) \\
&+c\|\mathbf{u}\|_{L^6}\left(\left\|g_1(\sigma)\right\|_{L^3}\|\mathbf{H}\|_{L^\infty}\|\nabla\mathbf{H}\|_{L^2} + \left\|g_2(\sigma)\right\|_{L^3}\left\|\nabla^2\mathbf{u}\right\|_{L^2}\right) \\
\leqslant& c\epsilon\left(\|\nabla(\sigma,\mathbf{u},\Theta,\mathbf{H})\|_{L^2}^2 + \left\|\nabla^2\mathbf{u}\right\|_{L^2}^2\right).
\end{aligned} \tag{47}$$

Thanks to Hölder's inequality, Young's inequality, Lemma 1–2, (27) and (30)–(31), the estimates of $\mathcal{S}_5$ and $\mathcal{S}_6$ can be given as follows:

$$\begin{aligned}
\mathcal{S}_5 =& \int_{\mathbb{R}^3} \nabla\Theta\cdot\nabla\left(-(\mathbf{u}\cdot\nabla)\Theta - g_2(\sigma)\Delta\Theta + g_1(\sigma)\left(\Psi(\mathbf{u}) + (\mathrm{curl}\mathbf{H})^2\right) - h_3(\sigma,\Theta)\mathrm{div}\mathbf{u}\right)dx \\
\leqslant& c\left\|\nabla^2\Theta\right\|_{L^2}\left(\|\mathbf{u}\|_{L^\infty}\|\nabla\Theta\|_{L^2} + \left\|g_2(\sigma)\right\|_{L^\infty}\left\|\nabla^2\Theta\right\|_{L^2} + \left\|h_3(\sigma,\Theta)\right\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\right) \\
&+c\left\|\nabla^2\Theta\right\|_{L^2}\left(\left\|g_1(\sigma)\right\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}\|_{L^6} + \left\|g_1(\sigma)\right\|_{L^\infty}\|\nabla\mathbf{H}\|_{L^6}\|\nabla\mathbf{H}\|_{L^3}\right) \\
\leqslant& c\epsilon\left(\left\|\nabla^2(\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2 + \|\nabla(\mathbf{u},\Theta)\|_{L^2}^2\right)
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
\mathcal{S}_6 =& \int_{\mathbb{R}^3} \Theta\cdot\left(-(\mathbf{u}\cdot\nabla)\Theta - g_2(\sigma)\Delta\Theta + g_1(\sigma)\left(\Psi(\mathbf{u}) + (\mathrm{curl}\mathbf{H})^2\right) - h_3(\sigma,\Theta)\mathrm{div}\mathbf{u}\right)dx \\
\leqslant& c\|\Theta\|_{L^6}\left(\|\mathbf{u}\|_{L^3}\|\nabla\Theta\|_{L^2} + \left\|g_2(\sigma)\right\|_{L^3}\left\|\nabla^2\Theta\right\|_{L^2} + \left\|h_3(\sigma,\Theta)\right\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\right) \\
&+c\|\Theta\|_{L^6}\left(\left\|g_1(\sigma)\right\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3} + \left\|g_1(\sigma)\right\|_{L^\infty}\|\nabla\mathbf{H}\|_{L^2}\|\nabla\mathbf{H}\|_{L^3}\right) \\
\leqslant& c\epsilon\left(\|\nabla(\mathbf{u},\Theta,\mathbf{H})\|_{L^2}^2 + \left\|\nabla^2\Theta\right\|_{L^2}^2\right).
\end{aligned} \tag{49}$$

For terms $\mathcal{S}_7$ and $\mathcal{S}_8$, we similarly derive that

$$\begin{aligned}
\mathcal{S}_7 =& \int_{\mathbb{R}^3} \nabla \mathbf{H} \cdot \nabla \left(\mathbf{H} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{H} - \mathbf{H} \mathrm{div} \mathbf{u} \right) dx \\
\leqslant & c \left\| \nabla^2 \mathbf{H} \right\|_{L^2} \left(\|\mathbf{H}\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{L^\infty} \|\nabla \mathbf{H}\|_{L^2} \right) \\
\leqslant & c\epsilon \left(\|\nabla (\mathbf{u}, \mathbf{H})\|_{L^2}^2 + \left\| \nabla^2 \mathbf{H} \right\|_{L^2}^2 \right)
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\mathcal{S}_8 =& \int_{\mathbb{R}^3} \mathbf{H} \cdot \left(\mathbf{H} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{H} - \mathbf{H} \mathrm{div} \mathbf{u} \right) dx \\
\leqslant & c \|\mathbf{H}\|_{L^6} \left(\|\mathbf{H}\|_{L^3} \|\nabla \mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{L^3} \|\nabla \mathbf{H}\|_{L^2} \right) \\
\leqslant & c\epsilon \|\nabla (\mathbf{u}, \mathbf{H})\|_{L^2}^2 .
\end{aligned} \tag{51}$$

Next, by Hölder's inequality, Young's inequality and Lemma 1–2, we can conclude

$$\begin{aligned}
\mathcal{S}_9 =& -\alpha_1 \int_{\mathbb{R}^3} (\mathrm{div} \mathbf{u}) \cdot \mathcal{N}_1 dx \\
\leqslant & c\alpha_1 \|\mathrm{div} \mathbf{u}\|_{L^2} \left\| \mathcal{N}_1 \right\|_{L^2} \\
\leqslant & c\alpha_1 \|\nabla \mathbf{u}\|_{L^2} \left(\|\nabla \mathbf{u}\|_{L^2} \|\sigma\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} \|\nabla \sigma\|_{L^2} \right) \\
\leqslant & c\alpha_1 \epsilon \|\nabla (\sigma, \mathbf{u})\|_{L^2}^2
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
\mathcal{S}_{10} \leqslant & c\alpha_1 \|\nabla \sigma\|_{L^2} \left\| \mathcal{N}_2 \right\|_{L^2} \\
\leqslant & c\alpha_1 \|\nabla \sigma\|_{L^2} \left(\|\mathbf{u}\|_{L^\infty} \|\nabla \mathbf{u}\|_{L^2} + \left\| h_1 (\sigma, \Theta) \right\|_{L^\infty} \|\nabla \sigma\|_{L^2} + \left\| h_2 (\sigma, \Theta) \right\|_{L^\infty} \|\nabla \Theta\|_{L^2} \right) \\
& + c\alpha_1 \|\nabla \sigma\|_{L^2} \left(\left\| g_1 (\sigma) \right\|_{L^\infty} \|\mathbf{H}\|_{L^\infty} \|\nabla \mathbf{H}\|_{L^2} + \left\| g_2 (\sigma) \right\|_{L^\infty} \left\| \nabla^2 \mathbf{u} \right\|_{L^2} \right) \\
\leqslant & c\alpha_1 \epsilon \left(\|\nabla (\sigma, \mathbf{u}, \Theta, \mathbf{H})\|_{L^2}^2 + \left\| \nabla^2 \mathbf{u} \right\|_{L^2}^2 \right),
\end{aligned} \tag{53}$$

where the definitions of $\mathcal{N}_1$ and $\mathcal{N}_2$ are used.

Finally, putting the estimates (44)–(53) into (43) yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left\{ \|\sigma\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2 + \|\Theta\|_{H^1}^2 + \|\mathbf{H}\|_{H^1}^2 + 2\alpha_1 \int_{\mathbb{R}^3} \nabla \sigma \cdot \mathbf{u} dx \right\} \\
& + \frac{\alpha_1}{2} \|\nabla \sigma\|_{L^2}^2 + \mu_1 \|\nabla \mathbf{u}\|_{H^1}^2 + \left(\mu_1 + \mu_2 \right) \|\mathrm{div} \mathbf{u}\|_{H^1}^2 + \|\nabla \Theta\|_{H^1}^2 + \|\nabla \mathbf{H}\|_{H^1}^2 \\
\leqslant & \frac{3\alpha_1 \mu_1^2}{2} \|\Delta \mathbf{u}\|_{L^2}^2 + \frac{3\alpha_1}{2} \|\nabla \Theta\|_{L^2}^2 + \frac{3\alpha_1 \left(\mu_1 + \mu_2 \right)^2}{2} \|\nabla \mathrm{div} \mathbf{u}\|_{L^2}^2 + \alpha_1 \|\mathrm{div} \mathbf{u}\|_{L^2}^2 \\
& + c \left(1 + \alpha_1 \right) \epsilon \left(\|\nabla (\sigma, \mathbf{u}, \Theta, \mathbf{H})\|_{L^2}^2 + \left\| \nabla^2 (\sigma, \mathbf{u}, \Theta, \mathbf{H}) \right\|_{L^2}^2 \right),
\end{aligned} \tag{54}$$

where α_1 is a fixed constant satisfying

$$0 < \alpha_1 \leqslant \min \left\{ \frac{1}{6 \left(\mu_1 + \mu_2 \right)}, \frac{1}{3\mu_1}, \frac{1}{3}, \frac{\left(\mu_1 + \mu_2 \right)}{4} \right\} . \tag{55}$$

Obviously, this implies that (36) holds. □

Now we turn to the energy estimate on the highest-order derivatives of the solution.

Lemma 8. Let

$$\mathcal{E}_h (t) =: \frac{1}{2} \left(\left\| \nabla^2 \sigma \right\|_{L^2}^2 + \left\| \nabla^2 \mathbf{u} \right\|_{L^2}^2 + \left\| \nabla^2 \Theta \right\|_{L^2}^2 + \left\| \nabla^2 \mathbf{H} \right\|_{L^2}^2 \right) + \alpha_2 \int_{\mathbb{R}^3} \nabla \nabla \sigma \cdot \nabla \mathbf{u} dx, \tag{56}$$

it holds that

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_h(t)+\frac{\alpha_2}{4}\|\nabla\nabla\sigma\|_{L^2}^2+\frac{(\mu_1+\mu_2)}{2}\left\|\nabla^2\mathrm{div}\mathbf{u}\right\|_{L^2}^2\\
&+\frac{\mu_1}{2}\left\|\nabla^2\nabla\mathbf{u}\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla^2\nabla\Theta\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla^2\nabla\mathbf{H}\right\|_{L^2}^2\\
&\leqslant\frac{1}{4}\|\nabla\nabla\Theta\|_{L^2}^2+\frac{(\mu_1+\mu_2)}{4}\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+c\epsilon\left\|\nabla^2(\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2,
\end{aligned}
\tag{57}
$$

where $0<\alpha_2\leqslant\min\left\{\frac{1}{6\mu_1},\frac{1}{6(\mu_1+\mu_2)},\frac{\mu_1+\mu_2}{4},\frac{1}{6}\right\}$ is a given constant.

Proof. Firstly, applying ∇^2 to $(22)_1$–$(22)_4$, and multiplying the resulting identities by $\nabla^2\sigma$, $\nabla^2\mathbf{u}$, $\nabla^2\Theta$, $\nabla^2\mathbf{H}$ respectively, summing them up, and then integrating over $\mathbb{R}^3$ by parts, we can obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\left\|\nabla^2\sigma\right\|_{L^2}^2+\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^2\Theta\right\|_{L^2}^2+\left\|\nabla^2\mathbf{H}\right\|_{L^2}^2\right\}+\mu_1\left\|\nabla^2\nabla\mathbf{u}\right\|_{L^2}^2\\
&+(\mu_1+\mu_2)\left\|\nabla^2\mathrm{div}\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^2\nabla\Theta\right\|_{L^2}^2+\left\|\nabla^2\nabla\mathbf{H}\right\|_{L^2}^2\\
&=\left\langle\nabla^2\sigma,\nabla^2\mathcal{N}_1\right\rangle+\left\langle\nabla^2\mathbf{u},\nabla^2\mathcal{N}_2\right\rangle+\left\langle\nabla^2\Theta,\nabla^2\mathcal{N}_3\right\rangle+\left\langle\nabla^2\mathbf{H},\nabla^2\mathcal{N}_4\right\rangle.
\end{aligned}
\tag{58}
$$

Secondly, we multiply $\nabla(22)_2$ by $\nabla\nabla\sigma$. By using $(22)_1$ and Young's inequality, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla\nabla\sigma\cdot\nabla\mathbf{u}dx+\int_{\mathbb{R}^3}|\nabla\nabla\sigma|^2dx\\
&=\mu_1\int_{\mathbb{R}^3}\nabla\nabla\sigma\cdot\nabla\Delta\mathbf{u}dx+(\mu_1+\mu_2)\int_{\mathbb{R}^3}\nabla\nabla\sigma\cdot\nabla\nabla\mathrm{div}\mathbf{u}dx\\
&-\int_{\mathbb{R}^3}\nabla\nabla\sigma\cdot\nabla\nabla\Theta dx+\int_{\mathbb{R}^3}|\nabla\mathrm{div}\mathbf{u}|^2dx+\int_{\mathbb{R}^3}\left(\nabla\nabla\sigma\cdot\nabla\mathcal{N}_2+\nabla\mathbf{u}\cdot\nabla\nabla\mathcal{N}_1\right)dx\\
&\leqslant\frac{1}{2}\|\nabla\nabla\sigma\|_{L^2}^2+\frac{3\mu_1^2}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2+\frac{3(\mu_1+\mu_2)^2}{2}\|\nabla\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2\\
&+\frac{3}{2}\|\nabla\nabla\Theta\|_{L^2}^2+\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\int_{\mathbb{R}^3}\left(\nabla\nabla\sigma\cdot\nabla\mathcal{N}_2+\nabla\mathbf{u}\cdot\nabla\nabla\mathcal{N}_1\right)dx.
\end{aligned}
\tag{59}
$$

By taking a fixed constant $\alpha_2>0$, we then sum up $\alpha_2\times(59)$ and (58) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\left\|\nabla^2\sigma\right\|_{L^2}^2+\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^2\Theta\right\|_{L^2}^2+\left\|\nabla^2\mathbf{H}\right\|_{L^2}^2+2\alpha_2\int_{\mathbb{R}^3}\nabla\nabla\sigma\cdot\nabla\mathbf{u}dx\right\}\\
&+\mu_1\left\|\nabla^2\nabla\mathbf{u}\right\|_{L^2}^2+(\mu_1+\mu_2)\left\|\nabla^2\mathrm{div}\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^2\nabla\Theta\right\|_{L^2}^2+\left\|\nabla^2\nabla\mathbf{H}\right\|_{L^2}^2+\frac{\alpha_2}{2}\|\nabla\nabla\sigma\|_{L^2}^2\\
&\leqslant\frac{3\alpha_2\mu_1^2}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2+\frac{3\alpha_2(\mu_1+\mu_2)^2}{2}\|\nabla\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\frac{3\alpha_2}{2}\|\nabla\nabla\Theta\|_{L^2}^2\\
&+\alpha_2\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\left\langle\nabla^2\sigma,\nabla^2\mathcal{N}_1\right\rangle+\left\langle\nabla^2\mathbf{u},\nabla^2\mathcal{N}_2\right\rangle+\left\langle\nabla^2\Theta,\nabla^2\mathcal{N}_3\right\rangle\\
&+\left\langle\nabla^2\mathbf{H},\nabla^2\mathcal{N}_4\right\rangle+\alpha_2\int_{\mathbb{R}^3}\nabla\mathbf{u}\cdot\nabla\nabla\mathcal{N}_1dx+\alpha_2\int_{\mathbb{R}^3}\nabla\nabla\sigma\cdot\nabla\mathcal{N}_2dx\\
&=:\frac{3\alpha_2\mu_1^2}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2+\frac{3\alpha_2(\mu_1+\mu_2)^2}{2}\|\nabla\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2\\
&+\frac{3\alpha_2}{2}\|\nabla\nabla\Theta\|_{L^2}^2+\alpha_2\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\sum_{i=1}^{6}\mathcal{Q}_i.
\end{aligned}
\tag{60}
$$

Next, we will continue to estimate the nonlinear terms $\mathcal{Q}_i\,(i=1,2,...,6)$ on the right hand side of (60). By integration by parts, Hölder's inequality, Lemma 1–3 and Young's inequality, $\mathcal{Q}_1$ can be bounded as

$$
\begin{aligned}
\mathcal{Q}_1 \leqslant & c\left(\int_{\mathbb{R}^3} \nabla^2\sigma\cdot\nabla^2(\sigma\mathrm{div}\mathbf{u})dx + \int_{\mathbb{R}^3} \nabla^2\sigma\cdot\nabla^2(\mathbf{u}\cdot\nabla\sigma)dx\right) \\
\leqslant & c\left\|\nabla^2\sigma\right\|_{L^2}\left(\left\|\nabla^2\sigma\right\|_{L^2}\|\mathrm{div}\mathbf{u}\|_{L^\infty} + \left\|\nabla^2\mathrm{div}\mathbf{u}\right\|_{L^2}\|\sigma\|_{L^\infty}\right) \\
& + c\left\|\nabla^2\sigma\right\|_{L^2}^2\|\mathrm{div}\mathbf{u}\|_{L^\infty} + c\left\|\nabla^2\sigma\right\|_{L^2}\left\|\nabla^2(\mathbf{u}\cdot\nabla\sigma) - \nabla^2\nabla\sigma\cdot\mathbf{u}\right\|_{L^2} \\
\leqslant & c\left\|\nabla^2\sigma\right\|_{L^2}^2\|\mathrm{div}\mathbf{u}\|_{L^\infty} + c\left\|\nabla^2\sigma\right\|_{L^2}\|\sigma\|_{H^2}\left\|\nabla^2\mathrm{div}\mathbf{u}\right\|_{L^2} \\
& + c\left\|\nabla^2\sigma\right\|_{L^2}\left(\left\|\nabla^2\mathbf{u}\right\|_{L^6}\|\nabla\sigma\|_{L^3} + \|\nabla\mathbf{u}\|_{L^\infty}\left\|\nabla^2\sigma\right\|_{L^2}\right) \\
\leqslant & c\epsilon\left(\left\|\nabla^3\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^2(\sigma,\mathbf{u})\right\|_{L^2}^2\right).
\end{aligned} \tag{61}
$$

Making use of (30)–(32), (8) in Lemma 3, integration by parts, Hölder's inequality and Young's inequality, $\mathcal{Q}_2$ can be estimated as follows:

$$
\begin{aligned}
\mathcal{Q}_2 \leqslant & c\left(\left|\left\langle\nabla^3\mathbf{u},\nabla(\mathbf{u}\cdot\nabla\mathbf{u})\right\rangle\right| + \left|\left\langle\nabla^3\mathbf{u},\nabla\left(h_1(\sigma,\Theta)\nabla\sigma\right)\right\rangle\right| + \left|\left\langle\nabla^3\mathbf{u},\nabla\left(h_2(\sigma,\Theta)\nabla\Theta\right)\right\rangle\right|\right) \\
& + c\left(\left|\left\langle\nabla^3\mathbf{u},\nabla\left(\mu_1 g_2(\sigma)\Delta\mathbf{u}\right)\right\rangle\right| + \left|\left\langle\nabla^3\mathbf{u},\nabla\left((\mu_1+\mu_2)g_2(\sigma)\nabla\mathrm{div}\mathbf{u}\right)\right\rangle\right|\right) \\
& + c\left(\left|\left\langle\nabla^3\mathbf{u},\nabla\left(g_1(\sigma)\mathbf{H}\cdot\nabla\mathbf{H}\right)\right\rangle\right| + \left|\left\langle\nabla^3\mathbf{u},\nabla\left(g_1(\sigma)\mathbf{H}\cdot\nabla^{\mathrm{T}}\mathbf{H}\right)\right\rangle\right|\right) \\
\leqslant & c\left\|\nabla^3\mathbf{u}\right\|_{L^2}\left(\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + \|\mathbf{u}\|_{L^\infty}\left\|\nabla^2\mathbf{u}\right\|_{L^2}\right) \\
& + c\left\|\nabla^3\mathbf{u}\right\|_{L^2}\left(\left\|h_1(\sigma,\Theta)\right\|_{L^\infty}\left\|\nabla^2\sigma\right\|_{L^2} + \left\|\nabla h_1(\sigma,\Theta)\right\|_{L^6}\|\nabla\sigma\|_{L^3}\right) \\
& + c\left\|\nabla^3\mathbf{u}\right\|_{L^2}\left(\left\|h_2(\sigma,\Theta)\right\|_{L^\infty}\left\|\nabla^2\Theta\right\|_{L^2} + \left\|\nabla h_2(\sigma,\Theta)\right\|_{L^6}\|\nabla\Theta\|_{L^3}\right) \\
& + c\left\|\nabla^3\mathbf{u}\right\|_{L^2}\left(\left\|g_2(\sigma)\right\|_{L^\infty}\|\nabla\Delta\mathbf{u}\|_{L^2} + \left\|\nabla g_2(\sigma)\right\|_{L^6}\|\Delta\mathbf{u}\|_{L^3}\right) \\
& + c\left\|\nabla^3\mathbf{u}\right\|_{L^2}\left(\left\|g_2(\sigma)\right\|_{L^\infty}\|\nabla\nabla\mathrm{div}\mathbf{u}\|_{L^2} + \left\|\nabla g_2(\sigma)\right\|_{L^6}\|\nabla\mathrm{div}\mathbf{u}\|_{L^3}\right) \\
& + c\left\|\nabla^3\mathbf{u}\right\|_{L^2}\left(\left\|g_1(\sigma)\right\|_{L^\infty}\|\nabla(\mathbf{H}\cdot\nabla\mathbf{H})\|_{L^2} + \left\|\nabla g_1(\sigma)\right\|_{L^6}\|\mathbf{H}\cdot\nabla\mathbf{H}\|_{L^3}\right) \\
\leqslant & c\epsilon\left(\left\|\nabla^2(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2 + \left\|\nabla^3\mathbf{u}\right\|_{L^2}^2\right).
\end{aligned} \tag{62}
$$

By a argument similar to (62), we can deduce that

$$
\begin{aligned}
\mathcal{Q}_3 \leqslant & c\left(\left|\left\langle\nabla^3\Theta,\nabla(\mathbf{u}\cdot\nabla\Theta)\right\rangle\right| + \left|\left\langle\nabla^3\Theta,\nabla\left(g_2(\sigma)\Delta\Theta\right)\right\rangle\right| + \left|\left\langle\nabla^3\Theta,\nabla\left(g_1(\sigma)\Psi(\mathbf{u})\right)\right\rangle\right|\right) \\
& + c\left(\left|\left\langle\nabla^3\Theta,\nabla\left(g_1(\sigma)(\mathrm{curl}\mathbf{H})^2\right)\right\rangle\right| + \left|\left\langle\nabla^3\Theta,\nabla\left(h_3(\sigma,\Theta)\,\mathrm{div}\mathbf{u}\right)\right\rangle\right|\right) \\
\leqslant & c\left\|\nabla^3\Theta\right\|_{L^2}\left(\|\nabla\mathbf{u}\|_{L^6}\|\nabla\Theta\|_{L^3} + \|\mathbf{u}\|_{L^\infty}\left\|\nabla^2\Theta\right\|_{L^2}\right) \\
& + c\left\|\nabla^3\Theta\right\|_{L^2}\left(\left\|\nabla g_2(\sigma)\right\|_{L^6}\|\Delta\Theta\|_{L^3} + \left\|g_2(\sigma)\right\|_{L^\infty}\|\nabla\Delta\Theta\|_{L^2}\right) \\
& + c\left\|\nabla^3\Theta\right\|_{L^2}\left(\left\|\nabla g_1(\sigma)\right\|_{L^6}\left\|(\nabla\mathbf{u})^2\right\|_{L^3} + \left\|g_1(\sigma)\right\|_{L^\infty}\left\|\nabla(\nabla\mathbf{u})^2\right\|_{L^2}\right) \\
& + c\left\|\nabla^3\Theta\right\|_{L^2}\left(\left\|\nabla g_1(\sigma)\right\|_{L^6}\left\|(\mathrm{curl}\mathbf{H})^2\right\|_{L^3} + \left\|g_1(\sigma)\right\|_{L^\infty}\|\nabla\mathrm{curl}\mathbf{H}\|_{L^6}\|\nabla\mathbf{H}\|_{L^3}\right) \\
& + c\left\|\nabla^3\Theta\right\|_{L^2}\left(\left\|\nabla h_3(\sigma,\Theta)\right\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + \left\|h_3(\sigma,\Theta)\right\|_{L^\infty}\left\|\nabla^2\mathbf{u}\right\|_{L^2}\right) \\
\leqslant & c\epsilon\left(\left\|\nabla^2(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2 + \left\|\nabla^3(\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2\right)
\end{aligned} \tag{63}
$$

and

$$
\begin{aligned}
\mathcal{Q}_4 \leqslant& c\left|\left\langle\nabla^3\mathbf{H},\nabla\mathcal{N}_4\right\rangle\right| \\
\leqslant& c\left(\left|\left\langle\nabla^3\mathbf{H},\nabla\left(\mathbf{H}\cdot\nabla\mathbf{u}\right)\right\rangle\right|+\left|\left\langle\nabla^3\mathbf{H},\nabla\left(\mathbf{u}\cdot\nabla\mathbf{H}\right)\right\rangle\right|+\left|\left\langle\nabla^3\mathbf{H},\nabla\left(\mathbf{H}\mathrm{div}\mathbf{u}\right)\right\rangle\right|\right) \\
\leqslant& c\left\|\nabla^3\mathbf{H}\right\|_{L^2}\left(\|\nabla\mathbf{H}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}+\|\mathbf{H}\|_{L^\infty}\left\|\nabla^2\mathbf{u}\right\|_{L^2}\right) \\
&+c\left\|\nabla^3\mathbf{H}\right\|_{L^2}\left(\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{H}\|_{L^3}+\|\mathbf{u}\|_{L^\infty}\left\|\nabla^2\mathbf{H}\right\|_{L^2}\right) \\
\leqslant& c\epsilon\left(\left\|\nabla^3\mathbf{H}\right\|_{L^2}^2+\left\|\nabla^2\left(\mathbf{u},\mathbf{H}\right)\right\|_{L^2}^2\right). \qquad (64)
\end{aligned}
$$

For the term $\mathcal{Q}_5$, we also have

$$
\begin{aligned}
\mathcal{Q}_5 =& -\alpha_2\int_{\mathbb{R}^3}\nabla\mathrm{div}\mathbf{u}\cdot\nabla\mathcal{N}_1 dx \\
\leqslant& c\alpha_2\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}\left\|\nabla\mathcal{N}_1\right\|_{L^2} \\
\leqslant& c\alpha_2\left\|\nabla^2\mathbf{u}\right\|_{L^2}\left(\left\|\nabla^2\mathbf{u}\right\|_{L^2}\|\sigma\|_{L^\infty}+\left\|\nabla^2\sigma\right\|_{L^2}\|\mathbf{u}\|_{L^\infty}\right) \\
\leqslant& c\alpha_2\epsilon\left\|\nabla^2\left(\sigma,\mathbf{u}\right)\right\|_{L^2}^2. \qquad (65)
\end{aligned}
$$

For the term $\mathcal{Q}_6$, we use (8) in Lemma 3, integration by parts, Hölder's inequality and Young's inequality. This together with (30)–(32) deduces that

$$
\begin{aligned}
\mathcal{Q}_6 \leqslant& c\alpha_2\|\nabla\nabla\sigma\|_{L^2}\left\|\nabla\mathcal{N}_2\right\|_{L^2} \\
\leqslant& c\alpha_2\left\|\nabla^2\sigma\right\|_{L^2}\left(\Big\|\nabla\left(\mathbf{u}\cdot\nabla\mathbf{u}+h_1\left(\sigma,\Theta\right)\nabla\sigma+h_2\left(\sigma,\Theta\right)\nabla\Theta\right.\right. \\
&\left.\left. g_1\left(\sigma\right)\left(\mathbf{H}\cdot\nabla\mathbf{H}+\mathbf{H}\cdot\nabla^{\mathrm{T}}\mathbf{H}\right)+g_2\left(\sigma\right)\left(\Delta\mathbf{u}+\nabla\mathrm{div}\mathbf{u}\right)\right)\Big\|_{L^2}\right) \\
\leqslant& c\alpha_2\left\|\nabla^2\sigma\right\|_{L^2}\left(\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}+\|\mathbf{u}\|_{L^\infty}\left\|\nabla^2\mathbf{u}\right\|_{L^2}\right) \\
&+c\alpha_2\left\|\nabla^2\sigma\right\|_{L^2}\left(\left\|h_1\left(\sigma,\Theta\right)\right\|_{L^\infty}\left\|\nabla^2\sigma\right\|_{L^2}+\left\|\nabla h_1\left(\sigma,\Theta\right)\right\|_{L^6}\|\nabla\sigma\|_{L^3}\right) \\
&+c\alpha_2\left\|\nabla^2\sigma\right\|_{L^2}\left(\left\|h_2\left(\sigma,\Theta\right)\right\|_{L^\infty}\left\|\nabla^2\Theta\right\|_{L^2}+\left\|\nabla h_2\left(\sigma,\Theta\right)\right\|_{L^6}\|\nabla\Theta\|_{L^3}\right) \\
&+c\alpha_2\left\|\nabla^2\sigma\right\|_{L^2}\left(\left\|g_1\left(\sigma\right)\right\|_{L^\infty}\|\nabla\mathbf{H}\|_{L^6}\|\nabla\mathbf{H}\|_{L^3}+\left\|g_1\left(\sigma\right)\right\|_{L^\infty}\|\mathbf{H}\|_{L^\infty}\left\|\nabla^2\mathbf{H}\right\|_{L^2}\right) \\
&+c\alpha_2\left\|\nabla^2\sigma\right\|_{L^2}\left(\left\|\nabla g_1\left(\sigma\right)\right\|_{L^6}\|\mathbf{H}\|_{L^\infty}\|\nabla\mathbf{H}\|_{L^3}\right) \\
&+c\alpha_2\left\|\nabla^2\sigma\right\|_{L^2}\left(\left\|g_2\left(\sigma\right)\right\|_{L^\infty}\left\|\nabla^3\mathbf{u}\right\|_{L^2}+\left\|\nabla g_2\left(\sigma\right)\right\|_{L^6}\left\|\nabla^2\mathbf{u}\right\|_{L^3}\right) \\
\leqslant& c\alpha_2\epsilon\left(\left\|\nabla^2\left(\sigma,\mathbf{u},\Theta,\mathbf{H}\right)\right\|_{L^2}^2+\left\|\nabla^3\mathbf{u}\right\|_{L^2}^2\right). \qquad (66)
\end{aligned}
$$

Hence, by putting (61)–(66) into (60), it yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\left\|\nabla^2\sigma\right\|_{L^2}^2+\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^2\Theta\right\|_{L^2}^2+\left\|\nabla^2\mathbf{H}\right\|_{L^2}^2+2\alpha_2\int_{\mathbb{R}^3}\nabla\nabla\sigma\cdot\nabla\mathbf{u}dx\right\} \\
&+\frac{\alpha_2}{4}\|\nabla\nabla\sigma\|_{L^2}^2+\frac{\mu_1}{2}\left\|\nabla^2\nabla\mathbf{u}\right\|_{L^2}^2+\frac{\left(\mu_1+\mu_2\right)}{2}\left\|\nabla^2\mathrm{div}\mathbf{u}\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla^2\nabla\Theta\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla^2\nabla\mathbf{H}\right\|_{L^2}^2 \\
\leqslant& \frac{1}{4}\|\nabla\nabla\Theta\|_{L^2}^2+\frac{\left(\mu_1+\mu_2\right)}{4}\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+c\epsilon\left\|\nabla^2\left(\mathbf{u},\Theta,\mathbf{H}\right)\right\|_{L^2}^2, \qquad (67)
\end{aligned}
$$

where $\alpha_2>0$ is a fixed constant satisfying

$$
0<\alpha_2\leqslant\min\left\{\frac{1}{6\mu_1},\frac{1}{6\left(\mu_1+\mu_2\right)},\frac{\mu_1+\mu_2}{4},\frac{1}{6}\right\}. \qquad (68)
$$

This completes the proof of Lemma 8. □

With the help of Lemma 7–8, Proposition 2 can be proved as follows

Exploiting the definitions of $\mathcal{E}_l$ and $\mathcal{E}_h$, and Young's inequality, we can easily deduce that there exists a constant $c_4 > 0$, such that

$$\frac{1}{c_4}\left\|(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{H^2}^2 \leqslant \mathcal{E}_l(t)+\mathcal{E}_h(t) \leqslant c_4\left\|(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{H^2}^2, \tag{69}$$

which implies

$$\mathcal{E}_l(t)+\mathcal{E}_h(t) \approx \left\|(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{H^2}^2. \tag{70}$$

Summing up (37) and (57), and then integrating the resulting inequality over $[0,t]$, we obtain (28) for the sufficiently small ϵ. This completes the proof of Proposition 2.

4.3 | Proof of Theorem 2

Under a smallness assumption (5) and *a priori* estimates given in Propositon 2, we can extend the local existence of the solution in Proposition 1 to the global-in-time solution. The idea of proof is similar to[37]. Next we give the proof for reader's convenience.

Step 1: First, we assume that the initial data satisfy $\mathfrak{E}_0 < \epsilon/\sqrt{c_1}$. With Proposition 1 in hand, there exists a unique solution $(\sigma,\mathbf{u},\Theta,\mathbf{H})$, which satisfies

$$\mathfrak{E}_1 := \sup_{0\leqslant t\leqslant T^*}\left\|(\sigma,\mathbf{u},\Theta,\mathbf{H})(t)\right\|_{H^2} \leqslant \sqrt{c_1}\mathfrak{E}_0 < \epsilon. \tag{71}$$

More importantly, if $\mathfrak{E}_0$ also satisfies $\mathfrak{E}_0 < \epsilon/\sqrt{c_1c_2}$, then by Proposition 2, we have

$$\mathfrak{E}_1 \leqslant \sqrt{c_2}\,\mathfrak{E}_0 < \epsilon/\sqrt{c_1}.$$

Step 2: Note that T^* depends only on $\mathfrak{E}_0$, we take the initial time as T^*, then the system (22) with the initial data $(\sigma,\mathbf{u},\Theta,\mathbf{H})(T^*)$ still has a unique solution on $t\in[T^*,2T^*]$. Also, c_2 does not depend on t. Using Proposition 1 again, we get

$$\mathfrak{E}_2 := \sup_{T^*\leqslant t\leqslant 2T^*}\left\|(\sigma,\mathbf{u},\Theta,\mathbf{H})(t)\right\|_{H^2} \leqslant \sqrt{c_1}\mathfrak{E}_1 < \epsilon. \tag{72}$$

Making use of (71) and Proposition 2, we further get

$$\sup_{T^*\leqslant t\leqslant 2T^*}\left\|(\sigma,\mathbf{u},\Theta,\mathbf{H})(t)\right\|_{H^2} \leqslant \sqrt{c_2}\,\mathfrak{E}_0 < \epsilon/\sqrt{c_1}.$$

Obviously for $0\leqslant t\leqslant nT^*(3\leqslant n)$, we repeat the process above.

Consequently, under the condition (33), we can extend the local-in-time solution to the global-in-time solution. This completes the proof of Theorem 2. Since the proof of uniqueness is standard, we ignore it here. Interested readers can also refer to[33].

5 | PROOF OF DECAY-IN-TIME ESTIMATES

Under the premise that the global existence and uniqueness of the solution are guaranteed, it is meaningful to study the decay-in-time of the system (1)–(2). Next, we turn to discuss the long time behavior of the solution for the system.

5.1 | Some decay estimates for the linear system

Based on the observation of cancelling the low-medium frequent part of the solution, the following lemma can be obtained

Lemma 9. It holds that

$$\begin{aligned}\left\|\nabla^2(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2 \leqslant & ce^{-c_3t}\left\|\nabla^2\left(\sigma^0,\mathbf{u}^0,\Theta^0,\mathbf{H}^0\right)\right\|_{L^2}^2\\ &+c\int_0^t e^{-c_3(t-\tau)}\left\|\nabla^2\left(\sigma^L,\mathbf{u}^L,\Theta^L,\mathbf{H}^L\right)(\tau)\right\|_{L^2}^2 d\tau,\end{aligned} \tag{73}$$

where $c>0$ is a constant.

Proof. To begin with, we multiply $\nabla(22)_2$ by $\nabla\nabla\sigma^L$, use $(22)_1$ and integration over $\mathbb{R}^3$ by parts to obtain

$$\begin{aligned}\frac{d}{dt}\int\limits_{\mathbb{R}^3}\nabla\nabla\sigma^L\cdot\nabla\mathbf{u}dx =&\mu_1\int\limits_{\mathbb{R}^3}\nabla\Delta\mathbf{u}\cdot\nabla\nabla\sigma^L dx+\left(\mu_1+\mu_2\right)\int\limits_{\mathbb{R}^3}\nabla\nabla\mathrm{div}\mathbf{u}\cdot\nabla\nabla\sigma^L dx\\&-\int\limits_{\mathbb{R}^3}\nabla\nabla\Theta\cdot\nabla\nabla\sigma^L dx+\int\limits_{\mathbb{R}^3}\left(\nabla\mathrm{div}\mathbf{u}\cdot\nabla\mathrm{div}\mathbf{u}^L-\nabla\nabla\sigma\cdot\nabla\nabla\sigma^L\right)dx\\&-\int\limits_{\mathbb{R}^3}\left(\nabla\mathcal{N}_1^L\cdot\nabla\mathrm{div}\mathbf{u}-\nabla\mathcal{N}_2\cdot\nabla\nabla\sigma^L\right)dx.\end{aligned}\tag{74}$$

Similarly to (40)–(42), by using Young's inequality, it yields

$$\begin{aligned}-\frac{d}{dt}\int\limits_{\mathbb{R}^3}\nabla\nabla\sigma^L\cdot\nabla\mathbf{u}dx\leqslant&\frac{\mu_1}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2+\frac{\left(\mu_1+\mu_2\right)}{2}\|\nabla\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\frac{1}{2}\|\nabla\nabla\Theta\|_{L^2}^2\\&+\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\frac{1}{2}\left\|\nabla\mathrm{div}\mathbf{u}^L\right\|_{L^2}^2+\left(\frac{2\mu_1+\mu_2}{2}+3\right)\left\|\nabla\nabla\sigma^L\right\|_{L^2}^2\\&+\frac{1}{8}\|\nabla\nabla\sigma\|_{L^2}^2+\frac{1}{2}\left\|\nabla\mathcal{N}_1^L\right\|_{L^2}^2+\frac{1}{2}\|\nabla\mathcal{N}_2\|_{L^2}^2.\end{aligned}\tag{75}$$

From Plancherel theorem, (66) and Lemma 2, we can get

$$\left\|\nabla\mathcal{N}_1^L\right\|_{L^2}^2+\|\nabla\mathcal{N}_2\|_{L^2}^2\leqslant c\epsilon\left(\left\|\nabla^2(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2+\left\|\nabla^3\mathbf{u}\right\|_{L^2}^2\right).\tag{76}$$

By taking a fixed constant α_2, we add up $\alpha_2\times(75)$ to (57). This together with (76) and Lemma 6 yields that

$$\begin{aligned}&\frac{d}{dt}\left(\mathcal{E}_h(t)-\alpha_2\int\limits_{\mathbb{R}^3}\nabla\nabla\sigma^L\cdot\nabla\mathbf{u}dx\right)+\frac{\alpha_2}{8}\left\|\nabla^2\sigma\right\|_{L^2}^2+\frac{\mu_1}{4}R_0^2\left\|\nabla^2\mathbf{u}^h\right\|_{L^2}^2+\frac{\mu_1}{4}\left\|\nabla^3\mathbf{u}\right\|_{L^2}^2\\&+\frac{\left(\mu_1+\mu_2\right)}{2}\left\|\nabla^2\mathrm{div}\mathbf{u}\right\|_{L^2}^2+\frac{1}{4}R_0^2\left\|\nabla^2\Theta^h\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla^3\Theta\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla^2\nabla\mathbf{H}\right\|_{L^2}^2\\&\leqslant\left(\frac{\alpha_2}{2}+\frac{1}{4}\right)\|\nabla\nabla\Theta\|_{L^2}^2+\left(\frac{\left(\mu_1+\mu_2\right)}{4}+\alpha_2\right)\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\frac{\alpha_2\mu_1}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2\\&+\frac{\alpha_2\left(\mu_1+\mu_2\right)}{2}\|\nabla\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+c\alpha_2\left(\left\|\nabla\nabla\sigma^L\right\|_{L^2}^2+\left\|\nabla\mathrm{div}\mathbf{u}^L\right\|_{L^2}^2\right)\\&+c\epsilon\left(1+\alpha_2\right)\left\|\nabla^2(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2.\end{aligned}\tag{77}$$

With the frequency decomposition (4) in hand, we further put $\frac{\mu_1}{4}R_0^2\left\|\nabla^2\mathbf{u}^L\right\|_{L^2}^2+\frac{1}{4}R_0^2\left\|\nabla^2\Theta^L\right\|_{L^2}^2$ on the both sides of (77) to get

$$\begin{aligned}&\frac{d}{dt}\left(\mathcal{E}_h(t)-\alpha_2\int\limits_{\mathbb{R}^3}\nabla\nabla\sigma^L\cdot\nabla\mathbf{u}dx\right)+\frac{\alpha_2}{8}\left\|\nabla^2\sigma\right\|_{L^2}^2+\frac{\mu_1}{8}R_0^2\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2+\frac{\mu_1}{4}\left\|\nabla^3\mathbf{u}\right\|_{L^2}^2\\&+\frac{\left(\mu_1+\mu_2\right)}{2}\left\|\nabla^2\mathrm{div}\mathbf{u}\right\|_{L^2}^2+\frac{1}{8}R_0^2\left\|\nabla^2\Theta\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla^3\Theta\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla^2\nabla\mathbf{H}\right\|_{L^2}^2\\&\leqslant\left(\frac{\alpha_2}{2}+\frac{1}{4}\right)\|\nabla\nabla\Theta\|_{L^2}^2+\left(\frac{\left(\mu_1+\mu_2\right)}{4}+\alpha_2\right)\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+\frac{\alpha_2\mu_1}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2\\&+\frac{\alpha_2\left(\mu_1+\mu_2\right)}{2}\|\nabla\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2+c\alpha_2\left\|\nabla\nabla\sigma^L\right\|_{L^2}^2+\left(\frac{\mu_1}{4}R_0^2+c\alpha_2\right)\left\|\nabla^2\mathbf{u}^L\right\|_{L^2}^2\\&+\frac{1}{4}R_0^2\left\|\nabla^2\Theta^L\right\|_{L^2}^2+c\epsilon\left(1+\alpha_2\right)\left\|\nabla^2(\sigma,\mathbf{u},\Theta,\mathbf{H})\right\|_{L^2}^2.\end{aligned}\tag{78}$$

Furthermore, notice that ϵ is small and R_0^2 satisfies $R_0^2 > \max\left\{\frac{4(\mu_1+\mu_2+1)}{\mu_1}, 4\right\}$, hence we have

$$\frac{d}{dt}\left(\mathcal{E}_h(t) - \alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma^L \cdot \nabla \mathbf{u} dx\right) + \frac{\alpha_2}{16}\left\|\nabla^2\sigma\right\|_{L^2}^2 + \frac{\mu_1}{16}R_0^2\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2 + \frac{\mu_1}{8}\left\|\nabla^3\mathbf{u}\right\|_{L^2}^2$$
$$+ \frac{(\mu_1+\mu_2)}{4}\left\|\nabla^2 \mathrm{div}\mathbf{u}\right\|_{L^2}^2 + \frac{1}{16}R_0^2\left\|\nabla^2\Theta\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla^3\Theta\right\|_{L^2}^2 + \frac{1}{4}\left\|\nabla^2\nabla\mathbf{H}\right\|_{L^2}^2$$
$$\leqslant c\left\|\nabla^2\left(\sigma^L, \mathbf{u}^L, \Theta^L, \mathbf{H}^L\right)\right\|_{L^2}^2. \tag{79}$$

On the other hand, by using the frequency decomposition (4) again, we obtain

$$\mathcal{E}_h(t) - \alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma^L \cdot \nabla \mathbf{u} dx$$
$$= \frac{1}{2}\left(\left\|\nabla^2\sigma\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^2\Theta\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{H}\right\|_{L^2}^2\right) + \alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma^h \cdot \nabla\mathbf{u} dx, \tag{80}$$

where the definition of $\mathcal{E}_h(t)$ have been used.

For the second term on the right hand side of (80), by utilizing integration by parts, Young's inequality and Lemma 6, we get

$$\alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma^h \cdot \nabla\mathbf{u} dx = -\alpha_2 \int\limits_{\mathbb{R}^3} \nabla\sigma^h \cdot \nabla \mathrm{div}\mathbf{u} dx$$
$$\leqslant \frac{\alpha_2}{2}\left\|\nabla\sigma^h\right\|_{L^2}^2 + \frac{\alpha_2}{2}\left\|\nabla \mathrm{div}\mathbf{u}\right\|_{L^2}^2$$
$$\leqslant \frac{\alpha_2}{2}\left\|\nabla^2\sigma\right\|_{L^2}^2 + \frac{\alpha_2}{2}\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2, \tag{81}$$

where the fact $0 < \alpha_2 < \frac{1}{6}$ have been used.

More importantly, by combining (79) with (80), it is easy to deduce that

$$\mathcal{E}_h(t) - \alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma^L \cdot \nabla\mathbf{u} dx \approx \left\|\nabla^2(\sigma, \mathbf{u}, \Theta, \mathbf{H})\right\|_{L^2}^2. \tag{82}$$

Thanks to (79) and (82), there exists a suitable constant $c_5 > 0$, such that

$$\frac{d}{dt}\left(\mathcal{E}_h(t) - \alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma^L \cdot \nabla\mathbf{u} dx\right) + c_5\left(\mathcal{E}_h(t) - \alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma^L \cdot \nabla\mathbf{u} dx\right)$$
$$\leqslant c\left\|\nabla^2\left(\sigma^L, \mathbf{u}^L, \Theta^L, \mathbf{H}^L\right)\right\|_{L^2}^2. \tag{83}$$

Thus, by using Gronwall's inequality, we immediately obtain

$$\mathcal{E}_h(t) - \alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma^L \cdot \nabla\mathbf{u} dx$$
$$\leqslant e^{-c_3 t}\left(\mathcal{E}_h(0) - \alpha_2 \int\limits_{\mathbb{R}^3} \nabla\nabla\sigma_0^L \cdot \nabla\mathbf{u}_0 dx\right)$$
$$+ c\int_0^t e^{-c_3(t-\tau)}\left\|\nabla^2\left(\sigma^L, \mathbf{u}^L, \Theta^L, \mathbf{H}^L\right)(\tau)\right\|_{L^2}^2 d\tau, \tag{84}$$

which implies (73) . □

In this subsection, what left is to calculate the estimate of the low-medium frequent part of the solution. To this end, we need to analyse the asymptotic expansions about $\lambda_j(|\xi|)$ and $e^{t\widehat{\mathcal{A}}(|\xi|)}$ in the low, medium and high frequency, respectively.

First of all, in view of the clear understanding of the system (22)–(23), we decompose the velocity $\mathbf{u}$, and let $m = \Lambda^{-1}\mathrm{div}\mathbf{u}$, $M = \Lambda^{-1}\mathrm{curl}\mathbf{u}$ with $(\mathrm{curl}\mathbf{u})_{ij} = \partial_j \mathbf{u}^i - \partial_i \mathbf{u}^j$. Then the following system can be derived from (22)–(23)

$$\begin{cases} \sigma_t + \Lambda m = \mathcal{N}_1, \\ m_t - \left(2\mu_1 + \mu_2\right)\Delta m - \Lambda\sigma - \Lambda\Theta = P_2, \\ \Theta_t - \Delta\Theta + \Lambda m = \mathcal{N}_3, \\ \mathbf{H}_t - \Delta\mathbf{H} = \mathcal{N}_4, \\ (\sigma, m, \Theta, \mathbf{H})(0, \mathbf{x}) = \left(\sigma^0, m^0, \Theta^0, \mathbf{H}^0\right)(\mathbf{x}). \end{cases} \tag{85}$$

Besides, $M = \Lambda^{-1}\mathrm{curl}\mathbf{u}$ satisfies

$$\begin{cases} M_t - \mu_1 \Delta M = \Lambda^{-1}\mathrm{curl}\mathcal{N}_2, \\ M(0, \mathbf{x}) = M^0(\mathbf{x}), \end{cases} \tag{86}$$

where $P_2 := \Lambda^{-1}\mathrm{div}\mathcal{N}_2$, $m^0 := \Lambda^{-1}\mathrm{div}\mathbf{u}^0$ and $M^0 := \Lambda^{-1}\mathrm{curl}\mathbf{u}^0$. Due to the identity $\Delta = \nabla\mathrm{div} - \mathrm{curlcurl}$, we get the relation

$$\mathbf{u} = \Delta^{-1}\left(\nabla\mathrm{div}\mathbf{u} - \mathrm{curlcurl}\mathbf{u}\right) = -\Lambda^{-1}\nabla m + \Lambda^{-1}\mathrm{curl}M. \tag{87}$$

Moreover, we observe that M satisfies the form of standard heat equation. For the heat equation, by direct calculations, we can obtain the following lemma, see[39,46].

Lemma 10. For the solution M of the linearized equation of (86), then there exists a constant $c > 0$, such that

$$\left|\widehat{M}(t, \xi)\right|^2 \leqslant c e^{-\mu_1 |\xi|^2 t}\left|\widehat{M}(0, \xi)\right|^2, \tag{88}$$

for all $|\xi|^2 \geqslant 0$, where $\widehat{M}$ stands for the Fourier transform of M.

Now let's go back to the linear system of (85). First, by performing Fourier transform at x on both sides of the linear system, we have

$$\begin{cases} \widehat{\sigma}_t + |\xi|\,\widehat{m} = 0, \\ \widehat{m}_t + \left(2\mu_1 + \mu_2\right)|\xi|^2\widehat{m} - |\xi|\,\widehat{\sigma} - |\xi|\,\widehat{\Theta} = 0, \\ \widehat{\Theta}_t + |\xi|^2\widehat{\Theta} + |\xi|\,\widehat{m} = 0, \\ \widehat{\mathbf{H}}_t + |\xi|^2\widehat{\mathbf{H}} = 0, \end{cases} \tag{89}$$

that is,

$$\widehat{\mathbf{V}}_t + \widehat{\mathcal{A}}(|\xi|)\,\widehat{\mathbf{V}} = \mathbf{0} \tag{90}$$

with the initial data $\widehat{\mathbf{V}}(0) = \left(\widehat{\sigma}^0, \widehat{m}^0, \widehat{\Theta}^0, \widehat{\mathbf{H}}^0\right)^{\mathrm{T}}$, where $\widehat{\mathbf{V}} = \left(\widehat{\sigma}, \widehat{m}, \widehat{\Theta}, \widehat{\mathbf{H}}\right)^{\mathrm{T}}$ and

$$\widehat{\mathcal{A}}(|\xi|) = \begin{pmatrix} 0 & |\xi| & 0 & 0 \\ -|\xi| & \left(2\mu_1 + \mu_2\right)|\xi|^2 & -|\xi| & 0 \\ 0 & |\xi| & |\xi|^2 & 0 \\ 0 & 0 & 0 & |\xi|^2 \end{pmatrix}. \tag{91}$$

By solving the ordinary differential equations, the solution of the system (90) can be expressed by

$$\widehat{\mathbf{V}} = e^{-t\widehat{\mathcal{A}}(|\xi|)}\widehat{\mathbf{V}}(0). \tag{92}$$

And then, by performing inverse Fourier transform, we immediately have

$$\mathbf{V}(t) = \mathbf{A}(t)\,\mathbf{V}(0), \tag{93}$$

which satisfies $\mathbf{A}(t)\,\mathbf{V} =: \mathcal{F}^{-1}\left(e^{-t\widehat{\mathcal{A}}(|\xi|)}\widehat{\mathbf{V}}(\xi)\right)$. Thus, we get the solution of the linear system of (85).

On the other hand, let the eigenvalues of the matrix (91) be $\lambda_j\,(j = 1, 2, 3)$ and $-|\xi|^2$, according to the semigroup decomposition theory proposed in[48], we can rewrite the semigroup $e^{-t\widehat{\mathcal{A}}(|\xi|)}$ as follows

$$e^{-t\widehat{\mathcal{A}}(|\xi|)} = \sum_{j=1}^{3} e^{\lambda_j t}\mathcal{P}_j(\xi) + e^{-|\xi|^2 t}\mathcal{P}_4(\xi), \tag{94}$$

where the projectors $\mathcal{P}_j$ $(j = 1, 2, 3)$ and $\mathcal{P}_4$ satisfy

$$\mathcal{P}_1 = \begin{pmatrix} \frac{\lambda_2\lambda_3-|\xi|^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{\gamma|\xi|^3+(\lambda_2+\lambda_3)|\xi|}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{-|\xi|^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & 0 \\ \frac{-\gamma|\xi|^3-(\lambda_2+\lambda_3)|\xi|}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{\gamma^2|\xi|^4+[\gamma(\lambda_2+\lambda_3)-2]|\xi|^2+\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{-(\lambda_2+\lambda_3)|\xi|-(\gamma+1)|\xi|^3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & 0 \\ \frac{-|\xi|^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{(\lambda_2+\lambda_3)|\xi|+(\gamma+1)|\xi|^3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{\lambda_2\lambda_3+(\lambda_2+\lambda_3-1)|\xi|^2+|\xi|^4}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{95}$$

$$\mathcal{P}_2 = \begin{pmatrix} \frac{\lambda_1\lambda_3-|\xi|^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{\gamma|\xi|^3+(\lambda_1+\lambda_3)|\xi|}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{-|\xi|^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & 0 \\ \frac{-\gamma|\xi|^3-(\lambda_1+\lambda_3)|\xi|}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{\gamma^2|\xi|^4+[\gamma(\lambda_1+\lambda_3)-2]|\xi|^2+\lambda_1\lambda_3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{-(\lambda_1+\lambda_3)|\xi|-(\gamma+1)|\xi|^3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & 0 \\ \frac{-|\xi|^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{(\lambda_1+\lambda_3)|\xi|+(\gamma+1)|\xi|^3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{\lambda_1\lambda_3+(\lambda_1+\lambda_3-1)|\xi|^2+|\xi|^4}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{96}$$

$$\mathcal{P}_3 = \begin{pmatrix} \frac{\lambda_1\lambda_2-|\xi|^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{\gamma|\xi|^3+(\lambda_1+\lambda_2)|\xi|}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{-|\xi|^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & 0 \\ \frac{-\gamma|\xi|^3-(\lambda_1+\lambda_2)|\xi|}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{\gamma^2|\xi|^4+[\gamma(\lambda_1+\lambda_2)-2]|\xi|^2+\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{-(\lambda_1+\lambda_2)|\xi|-(\gamma+1)|\xi|^3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & 0 \\ \frac{-|\xi|^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{(\lambda_1+\lambda_2)|\xi|+(\gamma+1)|\xi|^3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{\lambda_1\lambda_2+(\lambda_1+\lambda_2-1)|\xi|^2+|\xi|^4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{97}$$

and

$$\mathcal{P}_4 = \begin{pmatrix} 0_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 1 \end{pmatrix}. \tag{98}$$

with a positive constant $\gamma := 2\mu_1 + \mu_2$.

With the help of (94)–(97), by tedious calculations, we also obtain the asymptotic expansions of λ_j $(j = 1, 2, 3)$, see[47] for details.

Lemma 11. (1) If $|\boldsymbol{\xi}|$ satisfies $|\xi| < r_0$, then the eigenvalues $\lambda_j (j = 1, 2, 3)$ of $\widehat{\mathcal{A}}(|\boldsymbol{\xi}|)$ have the following expansion

$$\begin{cases} \lambda_1 = -b_1|\xi|^2 + \mathrm{O}\left(|\xi|^4\right), \\ \lambda_2 = -b_2|\xi|^2 + i\left(b_3|\xi| + \mathrm{O}\left(|\xi|^3\right)\right), \\ \lambda_3 = -b_2|\xi|^2 - i\left(b_3|\xi| + \mathrm{O}\left(|\xi|^3\right)\right). \end{cases} \tag{99}$$

(2) If $|\boldsymbol{\xi}|$ satisfies $r_0 \leqslant |\boldsymbol{\xi}| \leqslant R_0$, then for some constant $c > 0$, the eigenvalues $\lambda_j (j = 1, 2, 3)$ of $\widehat{\mathcal{A}}(|\boldsymbol{\xi}|)$ have the following spectrum gap property

$$Re\left(\lambda_j\right) \leqslant -c. \tag{100}$$

(3) If $|\boldsymbol{\xi}|$ satisfies $|\xi| > R_0$, then the eigenvalues $\lambda_j (j = 1, 2, 3)$ of $\widehat{\mathcal{A}}(|\boldsymbol{\xi}|)$ have the following expansion

$$\begin{cases} \lambda_1 = -\dfrac{1}{2\mu_1 + \mu_2} + \mathrm{O}\left(|\xi|^{-2}\right), \\ \lambda_2 = -|\xi|^2 + \mathrm{O}(1), \\ \lambda_3 = -\left(2\mu_1 + \mu_2\right)|\xi|^2 + \mathrm{O}(1), \end{cases} \tag{101}$$

where r_0, R_0 are fixed constants defined in (3), and all b_i $(i = 1, 2, 3)$ are real constants.

Next, with Lemma 11 in hand, we have

Lemma 12. Let r_0, R_0 be given in (3), then there exists some constants $c_7, c_8 > 0$, such that

$$\left|e^{-t\mathcal{A}(|\xi|)}\right| \leqslant \begin{cases} ce^{-c_7|\xi|^2 t}, & if\ |\xi| \leqslant r_0, \\ ce^{-c_8 t}, & if\ r_0 \leqslant |\xi| \leqslant R_0, \\ ce^{-c_8 t}, & if\ |\xi| \geqslant R_0, \end{cases} \tag{102}$$

where c_7, c_8 depend only on r_0, R_0, μ_1 and μ_2.

Proof. Please refer to[47] for the proof. □

Based on the above estimates on λ_j $(j = 1, 2, 3)$ and $e^{-t\mathcal{A}(|\xi|)}$, we can further estimate for the low-medium frequent part of the solution.

Lemma 13. Assume $1 \leqslant p \leqslant 2$, then we have

$$\left\|\nabla^k \left(\mathbf{A}(t)\mathbf{V}^L(0)\right)\right\|_{L^2} \leqslant c(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{k}{2}}\|\mathbf{V}(0)\|_{L^p}, \tag{103}$$

for any integer $k \geqslant 0$.

Proof. From the Plancherel theorem, (92) and Lemma 12, we can deduce that

$$\begin{aligned}
\left\|\nabla^k\left(\sigma^L, m^L, \Theta^L, \mathbf{H}^L\right)(t)\right\|_{L^2} =& \left\|(i\xi)^k\left(\widehat{\sigma^L}, \widehat{m^L}, \widehat{\Theta^L}, \widehat{\mathbf{H}^L}\right)\right\|_{L^2_\xi} \\
=& \left(\int_{\mathbb{R}^3} \left|(i\xi)^k\left(\widehat{\sigma^L}, \widehat{m^L}, \widehat{\Theta^L}, \widehat{\mathbf{H}^L}\right)(t,\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\
\leqslant& c\left(\int_{|\xi|\leqslant R_0} |\xi|^{2k}\left|\left(\widehat{\sigma}, \widehat{m}, \widehat{\Theta}, \widehat{\mathbf{H}}\right)(t,\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\
\leqslant& c\left(\int_{r_0\leqslant|\xi|\leqslant R_0} |\xi|^{2k}e^{-c_8 t}\left|\left(\widehat{\sigma}, \widehat{m}, \widehat{\Theta}, \widehat{\mathbf{H}}\right)(0,\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\
&+ c\left(\int_{|\xi|\leqslant r_0} |\xi|^{2k}e^{-c_7|\xi|^2 t}\left|\left(\widehat{\sigma}, \widehat{m}, \widehat{\Theta}, \widehat{\mathbf{H}}\right)(0,\xi)\right|^2 d\xi\right)^{\frac{1}{2}}.
\end{aligned} \tag{104}$$

By applying Hölder's inequality and Hausdorff–Young's inequality to (104), it is easy to get

$$\begin{aligned}
\left\|\nabla^k\left(\sigma^L, m^L, \Theta^L, \mathbf{H}^L\right)(t)\right\|_{L^2} \leqslant& c(1+t)^{-\frac{3}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{k}{2}}\left\|\left(\widehat{\sigma}, \widehat{m}, \widehat{\Theta}, \widehat{\mathbf{H}}\right)(0,\xi)\right\|_{L^q_\xi} \\
\leqslant& c(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{k}{2}}\left\|\left(\sigma^0, \mathbf{u}^0, \Theta^0, \mathbf{H}^0\right)\right\|_{L^p}.
\end{aligned} \tag{105}$$

By a similar argument, by using (88) in Lemma 10, we have

$$\left\|\nabla^k M^L(t)\right\|_{L^2} \leqslant c(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{k}{2}}\left\|u^0\right\|_{L^p}. \tag{106}$$

Obviously, combining (105) with (106), we get (103). □

5.2 | Optimal time-decay rates for the nonlinear system

In this subsection, we will establish the optimal time-decay rates of the solution for the nonlinear system (22)–(23). To this end, we denote $\mathbf{V}(t) = (\sigma(t), \mathbf{u}(t), \Theta(t), \mathbf{H}(t))^{\mathrm{T}}$, then the nonlinear system (22)–(23) is equivalent to the following form

$$\mathbf{V}_t + \mathbf{A}\mathbf{V} = \mathcal{N}(\mathbf{V}) \tag{107}$$

with the initial data

$$\mathbf{V}\big|_{t=0} = \mathbf{V}(0),$$

where $\mathcal{N}(\mathbf{V}) =: \left(\mathcal{N}_1, P_2, \mathcal{N}_3, \mathcal{N}_4\right)^{\mathrm{T}}$.

By using Duhamel's principle, the solution of the nonlinear system can be rewritten as below

$$\mathbf{V}(t) = \mathbf{A}(t)\mathbf{V}(0) + \int_0^t \mathbf{A}(t-\tau)\mathcal{N}(\mathbf{V})(\tau)d\tau. \tag{108}$$

Moreover, with the help of Lemma 13, we immediately have the following conclusion.

Lemma 14. Let the assumption of Lemma 13 be satisfied, then there exists a constant $c_6 > 0$ such that

$$\left\|\nabla^k \mathbf{V}^L(t)\right\|_{L^2} \leqslant c_6(1+t)^{-\left(\frac{3}{4}+\frac{k}{2}\right)}\|\mathbf{V}(0)\|_{L^1} + c_6\int_0^{\frac{t}{2}}(1+t-\tau)^{-\left(\frac{3}{4}+\frac{k}{2}\right)}\|\mathcal{N}(\mathbf{V})(\tau)\|_{L^1}d\tau$$
$$+c_6\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{k}{2}}\|\mathcal{N}(\mathbf{V})(\tau)\|_{L^2}d\tau, \tag{109}$$

for any integer $k \geqslant 0$.

In what follows, we will employ Lemma 9 and Lemma 14 to get the optimal time-decay rates of the solution. The details are listed as follows

Lemma 15 (Optimal time-decay rates)**.** Let the assumption of Theorem 1 be satisfied. Then the global solution $(\sigma, \mathbf{u}, \Theta, \mathbf{H})(x,t)$ of the system (22)–(23) obtained from Theorem 2 enjoys

$$\left\|\nabla^k(\sigma, \mathbf{u}, \Theta, \mathbf{H})(t)\right\|_{L^2} \leqslant c(1+t)^{-\left(\frac{3}{4}+\frac{k}{2}\right)}, \quad k = 0, 1, 2 \tag{110}$$

for any $t \in [0, \infty)$.

Proof. To begin with, we denote

$$\mathcal{H}(t) := \sup_{0\leqslant\tau\leqslant t}\sum_{k=0}^{2}(1+\tau)^{\frac{3}{4}+\frac{k}{2}}\left\|\nabla^k(\sigma, \mathbf{u}, \Theta, \mathbf{H})(\tau)\right\|_{L^2}. \tag{111}$$

Clearly, $\mathcal{H}(t)$ is non-decreasing, and for $0 \leqslant \tau \leqslant t$, $0 \leqslant k \leqslant 2$, we have

$$\left\|\nabla^k(\sigma, \mathbf{u}, \Theta, \mathbf{H})(\tau)\right\|_{L^2} \leqslant c_9(1+\tau)^{-\left(\frac{3}{4}+\frac{k}{2}\right)}\mathcal{H}(t), \tag{112}$$

where $c_9 > 0$ is a constant that doesn't depend on ϵ.

Step 1. Now we estimate the terms on the right side of (109). From Hölder's inequality, (112), we can get

$$\begin{aligned}\|\mathcal{N}(\mathbf{V})(\tau)\|_{L^1} \lesssim&\|(\sigma, \mathbf{u}, \Theta, \mathbf{H})\|_{L^2}\|\nabla(\sigma, \mathbf{u}, \Theta, \mathbf{H})\|_{L^2} + \|\sigma\|_{L^2}\left\|\nabla^2(\mathbf{u}, \Theta)\right\|_{L^2}\\ &+\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{H}\|_{L^2}^2\\ \lesssim&\epsilon\mathcal{H}(t)(1+\tau)^{-\frac{5}{4}}.\end{aligned} \tag{113}$$

Similarly to the above estimate, it is easy to calculate that

$$\begin{aligned}\|\mathcal{N}(\mathbf{V})(\tau)\|_{L^2} \lesssim&\|(\sigma, \mathbf{u}, \Theta, \mathbf{H})\|_{L^3}\|\nabla(\sigma, \mathbf{u}, \Theta, \mathbf{H})\|_{L^6} + \|\sigma\|_{L^\infty}\left\|\nabla^2(\mathbf{u}, \Theta)\right\|_{L^2}\\ &+\|\nabla(\mathbf{u}, \mathbf{H})\|_{L^3}\|\nabla(\mathbf{u}, \mathbf{H})\|_{L^6}\\ \lesssim&\|(\sigma, \mathbf{u}, \Theta, \mathbf{H})\|_{H^1}\left\|\nabla^2(\sigma, \mathbf{u}, \Theta, \mathbf{H})\right\|_{L^2} + \|\sigma\|_{H^2}\left\|\nabla^2(\mathbf{u}, \Theta)\right\|_{L^2}\\ &+\|\nabla(\mathbf{u}, \mathbf{H})\|_{H^1}\left\|\nabla^2(\mathbf{u}, \mathbf{H})\right\|_{L^2}\\ \lesssim&\epsilon^{1-\varpi_1}\mathcal{H}^{1+\varpi_1}(t)(1+\tau)^{-\left(\frac{7}{4}+\frac{3}{4}\varpi_1\right)},\end{aligned} \tag{114}$$

where $\varpi_1 \in \left(0, \frac{1}{2}\right)$ is a fixed constant.

This together with (109) and Lemma 5 deduces that

$$\begin{aligned}\left\|\nabla^k \mathbf{V}^L(t)\right\|_{L^2} \leqslant & c(1+t)^{-\left(\frac{3}{4}+\frac{k}{2}\right)}\|\mathbf{V}(0)\|_{L^1}+c\int_0^{\frac{t}{2}}(1+t-\tau)^{-\left(\frac{3}{4}+\frac{k}{2}\right)}\epsilon\mathcal{H}(t)(1+\tau)^{-\frac{5}{4}}d\tau \\ & +c\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{k}{2}}\epsilon^{1-\varpi_1}\mathcal{H}^{1+\varpi_1}(t)(1+\tau)^{-\left(\frac{7}{4}+\frac{3}{4}\varpi_1\right)}d\tau \\ \leqslant & c(1+t)^{-\left(\frac{3}{4}+\frac{k}{2}\right)}\left(\|\mathbf{V}(0)\|_{L^1}+\epsilon\mathcal{H}(t)+\epsilon^{1-\varpi_1}\mathcal{H}^{1+\varpi_1}(t)\right),\end{aligned} \tag{115}$$

for $0 \leqslant k \leqslant 2$. Putting (115) into (73) in Lemma 9 yields

$$\begin{aligned}\left\|\nabla^2\mathbf{V}(t)\right\|_{L^2}^2 \leqslant & ce^{-c_3 t}\left\|\nabla^2\mathbf{V}(0)\right\|_{L^2}^2+c\left(\|\mathbf{V}(0)\|_{L^1}^2+\epsilon^2\mathcal{H}^2(t)\right)\int_0^t e^{-c_3(t-\tau)}(1+t)^{-\frac{7}{2}}d\tau \\ & +c\epsilon^{2-2\varpi_1}\mathcal{H}^{2+2\varpi_1}(t)\int_0^t e^{-c_3(t-\tau)}(1+t)^{-\frac{7}{2}}d\tau.\end{aligned} \tag{116}$$

From (116), we use Lemma 5 again to obtain

$$\left\|\nabla^2\mathbf{V}(t)\right\|_{L^2}^2 \leqslant c(1+t)^{-\frac{7}{2}}\left(\|\mathbf{V}(0)\|_{H^2\cap L^1}^2+\epsilon^2\mathcal{H}^2(t)+\epsilon^{2-2\varpi_1}\mathcal{H}^{2+2\varpi_1}(t)\right). \tag{117}$$

By using (4) and (16) in Lemma 6, one has

$$\begin{aligned}\left\|\nabla^k\mathbf{V}(t)\right\|_{L^2}^2 \leqslant & c\left\|\nabla^k\mathbf{V}^L(t)\right\|_{L^2}^2+c\left\|\nabla^k\mathbf{V}^h(t)\right\|_{L^2}^2 \\ \leqslant & c\left\|\nabla^k\mathbf{V}^L\right\|_{L^2}^2+c\left\|\nabla^2\mathbf{V}\right\|_{L^2}^2.\end{aligned} \tag{118}$$

Hence, by putting (115) and (117) into (118), we have

$$\left\|\nabla^k\mathbf{V}(t)\right\|_{L^2}^2 \leqslant c(1+t)^{-\left(\frac{3}{2}+k\right)}\left(\|\mathbf{V}(0)\|_{H^2\cap L^1}^2+\epsilon^2\mathcal{H}^2(t)+\epsilon^{2-2\varpi_1}\mathcal{H}^{2+2\varpi_1}(t)\right) \tag{119}$$

for $0 \leqslant k \leqslant 2$. Since ϵ is small, by the definition (111) of $\mathcal{H}(t)$, there exists a constant c_{10}, such that

$$\mathcal{H}^2(t) \leqslant \frac{c_{10}}{2}\left(\|(\sigma,\mathbf{u},\Theta,\mathbf{H})(0)\|_{H^2\cap L^1}^2+\epsilon^2\mathcal{H}^2(t)+\epsilon^{2-2\varpi_1}\mathcal{H}^{2+2\varpi_1}(t)\right), \tag{120}$$

where c_{10} is independent of ϵ.

For the last term on the right side of (120), by Young's inequality, we get

$$\begin{aligned}c_{10}\epsilon^{2-2\varpi_1}\mathcal{H}^{2+2\varpi_1}(t) \leqslant & \frac{1-\varpi_1}{2}c_{10}^{\frac{2}{1-\varpi_1}}+\frac{1+\varpi_1}{2}\epsilon^{\frac{4(1-\varpi_1)}{1+\varpi_1}}\mathcal{H}^4(t) \\ =: & \frac{1-\varpi_1}{2}c_{10}^{\frac{2}{1-\varpi_1}}+C_\epsilon\mathcal{H}^4(t).\end{aligned} \tag{121}$$

For convenience, we define $\mathcal{W}_0$ as follows

$$\mathcal{W}_0 := c_{10}\|(\sigma,\mathbf{u},\Theta,\mathbf{H})(0)\|_{H^2\bigcap L^1}^2+\frac{1-\varpi_1}{2}c_{10}^{\frac{2}{1-\varpi_1}}. \tag{122}$$

Thus, by combining with (120)–(121), we can obtain

$$\mathcal{H}^2(t) \leqslant \mathcal{W}_0+C_\epsilon\mathcal{H}^4(t). \tag{123}$$

Step 2. Next, we only need to prove $\mathcal{H}(t) \leqslant c$. Assume $\mathcal{H}^2(t) > 2\mathcal{W}_0$ for any $t \geqslant t_1$ with a constant $t_1 > 0$. For one thing, we note that $\mathcal{H}(t) \in C^0[0,+\infty)$ and $\mathcal{H}^2(0)$ is small, then there exists $t_0 \in \left(0, t_1\right)$, such that

$$\mathcal{H}^2\left(t_0\right) = 2\mathcal{W}_0. \tag{124}$$

In addition, by (123), we can get

$$\mathcal{H}^2\left(t_0\right) \leqslant \mathcal{W}_0+C_\epsilon\mathcal{H}^4\left(t_0\right),$$

that is,

$$\mathcal{H}^2\left(t_0\right) \leqslant \frac{\mathcal{W}_0}{1-C_\epsilon \mathcal{H}^2\left(t_0\right)}. \tag{125}$$

We take a small constant ϵ, such that $C_\epsilon < \frac{1}{4\mathcal{W}_0}$. In other words, $C_\epsilon \mathcal{H}^2\left(t_0\right) < \frac{1}{2}$ holds. This fact together with (125) implies that

$$\mathcal{H}^2\left(t_0\right) < 2\mathcal{W}_0. \tag{126}$$

Obviously, (124) contradicts with (126). Thus, for any $t \geqslant t_1$, We always have $\mathcal{H}^2(t) \leqslant 2\mathcal{W}_0$. And because $\mathcal{H}(t)$ is non-decreasing, for any $t \in [0,+\infty)$, we have $\mathcal{H}(t) \leqslant c$. Finally, by combining with (112), We complete the proof of Lemma 15. □

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