Decay Rate Characterization for Positive Singular Systems with Unbounded Delays

Bohao Zhu[†] **James Lam**[†] **Yukang Cui**[*] **Jason Ying Kuen Chan**[‡] **Ka-Wai Kwok**[†]

Abstract

This paper investigates the decay rates of continuous-time singular systems with unbounded delays. By introducing an auxiliary system for the original system, the positivity and asymptotic stability conditions of the system are investigated first. Then, μ-stability criteria, which are applied to characterize the decay rates of the systems, are proposed, and the relation between the system matrices and μ-stability is studied. Those results include the stability of positive singular systems with bounded time-varying delays and time-varying delays with linear growth rate as special cases. Finally, a numerical example is given to illustrate the obtained theoretical results.

Keywords: Decay rates, positive systems, singular systems, stability, time-delay systems.

1 Introduction

In recent decades, singular systems [10] also known as implicit systems, descriptor systems, or generalized state-space systems have drawn considerable attention due to their strong practical background in electrical systems, economic systems, aerospace systems, chemical systems and robotic systems [13]. In view of the intrinsic nonnegativity of the states of some singular systems, like the current of electricity nodes, number of molecules and concentration of chemical component, those systems can be regarded as positive singular systems. Some monographs on the behavioral analysis of positive systems are available in [18, 29, 32, 33]. Many analytical approaches developed for positive systems [3, 11, 21, 34] have been applied to stability analysis [1, 8, 9, 20], input-output gain analysis [7], and state-feedback control [26] for positive singular systems.

[†]Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong (Email: zhubohao@connect.hku.hk; james.lam@hku.hk; kwokkw@hku.hk).

[*]College of Mechatronics and Control Engineering, Shenzhen University, Shenzhen, 518000, China (Email: cuiyukang@gmail.com).

[‡]Department of Otorhinolaryngology, Head and Neck Surgery, The Chinese University of Hong Kong, Prince of Wales Hospital, Shatin, NT, Hong Kong (Email: jasonchan@ent.cuhk.edu.hk).

[§]This work was supported in part by the Research Grants Council of Hong Kong under Grant 17201820, and in part by the Innovation and Technology Commission (ITC) (MRP/029/20X), and the HKSAR Government under the InnoHK initiative, Hong Kong, via Centre for Garment Production Limited.

As a consequence of the special properties for positive time-delay systems [2, 4, 22], positive singular systems with time-delay becomes a hot topic in recent years [14, 17, 24, 27, 31]. In [24], the exponential stability for the positive singular systems with fixed delays is investigated. Based on the Lyapunov function with exponential terms, a sufficient exponential stability criterion in terms of algebraic matrix inequalities is obtained. In [9], the asymptotic stability of systems with bounded time-varying delay is addressed. By analyzing the relation between the states of the initial systems and the systems with constant delay, a sufficient asymptotic stability condition is established. Under strictly positive initial conditions, the asymptotic stability condition turns into a necessary and sufficient one. In [17], the ℓ_1 stability of switched positive singular with time-varying delay is investigated. By combining the average dwell time scheme and co-positive Lyapunov function, a delay-dependent stability condition can be established. Although extensive research efforts have been focused on stability condition of positive singular systems with time-delay, the existing results are still deficient: (1) *Constraints on time delays.* For positive singular systems with time-delays, the Lyapunov-Krasovskii method and comparison methods are commonly applied. On one hand, the Lyapunov-Krasovskii functional [17, 25] requires the derivative of the unbounded delay to be less than unity. On the other hand, the comparison methods [9] require the time delay have a given upper bound. Those methods require the bound of the time delay or the bound of the derivative of the delay and fails to analyze unbounded time-delay systems with less constraints on delays; (2) *Few results on characterization of decay rate.* When analyzing the positivity and stability condition of a singular system, an auxiliary system is always introduced first. For the auxiliary system, the state of the original system is divided into two states, and the two states are characterized by differential and difference equations, respectively. How to unify the convergent speed of the original system and the auxiliary system and obtain the decay rate of the original system remains a challenging problem.

Back to positive systems, both bounded and unbounded delays of different kind of positive systems have been characterized [12, 23, 28]. Specifically, for positive systems with unbounded delays, the asymptotic stability is not affected by the time-varying delays, while the decay rates of the states depend on the value of the delays. Motivated by the above work, we endeavour to present a condition to analyze the stability and decay rate of positive singular systems with unbounded delay. In this paper, an auxiliary system for the original positive singular systems is given first. By constructing a Lyapunov function for the auxiliary system, an asymptotic stability condition for a positive singular system with unbounded delay is derived. Then, a monotonically nondecreasing function is introduced to characterize the μ-stability for the positive singular system. When the product of a nondecreasing function and the norm of the state is bounded, we can characterize the decay rate of the system via the given function. The main contributions of this paper are given as follows:

- **Asymptotic stability:** We show that the asymptotic stability condition of the given auxiliary system is equivalent to the original positive singular system. The asymptotic stability of positive singular systems with unbounded delay is only affected by the system matrices, but not the rate of change and magnitude of the delay.

- **Decay rate characterization:** By introducing a monotonically nondecreasing function $\mu(t)$, the decay rate of a positive singular system can be characterized. The results also show that the μ-stability can be applied to characterize the decay rate of positive singular systems with bounded time-varying delay and time-varying delay with linear growth rate with respect to time.

The rest of the paper is organized as follows. Problem formulation and the positivity condition of singular systems with unbounded time-delay are given in Section 2. The decay rate characterization of the system is investigated in Section 3. Based on the characterization, some positive singular systems with special time-delay, including bounded time-delay and time-varying delay with linear growth rate are given in this section. In Section 4, an example is given to illustrate the effectiveness of the obtained results. Section 5 concludes the paper.

Notation: $\mathbb{R}^n$ denotes the n-dimensional real vector space, $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices, $\mathbb{N}=\{0,1,\ldots\}$, $\mathbb{C}(-\infty,0]$ denotes the set of continuous function defined on $(-\infty,0]$, A^{T} denotes the transpose of matrix A, $v_{[i]}$ denotes the i-th element of vector v, $A_{[i,j]}$ denotes the i-th row, j-th column element of matrix A. Furthermore, some basic notations for positive system are recalled [3]. $v \succeq (\succ) 0$ or $v\in\mathbb{R}^n_{0,+}\left(\mathbb{R}^n_{+}\right)$ means a real vector v is a nonnegative (positive) vector whose entries are all nonnegative (positive). $A\succeq(\succ)0$ or $A\in\mathbb{R}^{m\times n}_{0,+}\left(\mathbb{R}^{m\times n}_{+}\right)$ means a real matrix $A\in\mathbb{R}^{m\times n}$ is a nonnegative (positive) matrix. For two nonnegative (positive) matrices A and $B\in\mathbb{R}^{m\times n}_{0,+}\left(\mathbb{R}^{m\times n}_{+}\right)$, $A\succeq(\succ)B$ means $A-B$ is a nonnegative (positive) matrix. $\mathbb{M}^{n\times n}$ denotes the set of $n\times n$ Metzler matrices whose off-diagonal entries are nonnegative.

2 Problem Formulation

A singular system with time-varying delays is given as follows:

$$\begin{aligned} E\dot{x}(t) &= Ax(t)+A_d x(t-d(t)),\\ x(s) &= \phi(s),\ s\le 0, \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^{n_x}$ is the state vector, A and $A_d\in\mathbb{R}^{n_x\times n_x}$ are real matrices. The matrix $E\in\mathbb{R}^{n_x\times n_x}$ is assumed to be singular, that is, $\mathrm{rank}(E)=r<n_x$. $\phi(\cdot)\in\mathbb{C}(-\infty,0]$ is the initial condition. According to Theorem 5.1 of [1], the initial condition $\phi(\cdot)$ should to be admissible to guarantee the uniqueness of the solution $x(t)$. In what follows, we will require $\phi(\cdot)$ to be an admissible initial condition. The time delay $d(t)$ could be unbounded and satisfies Assumption 1.

Assumption 1. *The assumptions for the continuous time delay $d(t)$ are given as follows:*
(i) $d(t)\ge \underline{d}>0$ for all $t\in\mathrm{R}_{0,+}$;
(ii) $\lim_{t\to\infty}(t-d(t))=\infty$.

Some basic definitions and useful lemmas for positive singular systems in [8,9], which will be employed for deriving the main results, are given as follows.

Definition 1. ***(Positivity)*** *System (1) is said to be positive if any admissible initial condition satisfies $\phi(s)\succeq 0$, $s<0$, and unbounded delay $d(t)$ satisfies Assumption 1, one has $x(t)\succeq 0$ for all $t\ge 0$.*

Definition 2. ***(Asymptotic Stability)*** *System (1) is said to be asymptotically stable if for any $\varepsilon>0$, a scalar $\delta(\varepsilon)$ exists such that for any admissible initial condition $\phi(t)$, $t\le 0$, satisfying $\sup_{t\in(-\infty,0]}\|\phi(t)\|_\infty\le\delta(\varepsilon)$, the vector $x(t)$ satisfies $\|x(t)\|\le\varepsilon$, for all $t\ge 0$. Furthermore, when $t\to\infty$, $x(t)\to 0$.*

Definition 3. ***(μ-Stability)*** *Suppose that $\mu : \mathbb{R}_{0,+} \to \mathbb{R}_+$ is a nondecreasing function satisfying $\mu(t) \to \infty$, when $t \to \infty$. System (1) is said to be μ-stable if there exists a constant $M > 0$ such that for any admissible initial condition $\phi(\cdot) \succeq 0$, the state $x(t)$ satisfies $\|x(t)\|_\infty \leq \frac{M}{\mu(t)}$, for all $t \in \mathbb{R}_{0,+}$.*

Definition 4. ***(Regular and Impulse-free)***
(i) The pair (E,A) is said to be regular if there exists a scalar $s \in \mathbb{R}$ such that $\det(sE-A)$ *is not identically zero.*
(ii) The pair (E,A) is said to be impulse-free if there exists a scalar $s \in \mathbb{R}$ such that $\deg\{\det(sE-A)\} = \operatorname{rank}(E)$.

Definition 5. ***(Drazin Inverse)*** *For any matrix $E \in \mathrm{R}^{n\times n}$, a unique matrix E^D, called the Drazin inverse of matrix E, always exists satisfying $EE^D = E^D E$, $E^D E E^D = E^D$, $E^D E^{v+1} = E^v$, where v is the smallest nonnegative integer such that* $\operatorname{rank}(E^v) = \operatorname{rank}(E^{v+1})$*, called the index of E, and is denoted by $v = \operatorname{ind}(\mathrm{E})$.*

In the following, some properties of the Drazin inverse are recalled.

Lemma 1. [5] *Let the pair (E,A) be regular and scalar $\beta \in \mathbb{R}$ such that matrix $\beta E - A$ is nonsingular. Then the matrices $\hat{E} \triangleq (\beta E-A)^{-1}E$ and $\hat{A} \triangleq (\beta E-A)^{-1}A$ commute.*

Lemma 2. [15] *Suppose $EA = AE$. Then $EA^D = A^D E$, $E^D A = A/E^D$, $E^D A^D = A^D E^D$.*

Suppose the pair (E,A) is regular and impulse-free. Define $x_1(t) \triangleq Mx(t)$, $x_2 \triangleq (I-M)x(t)$ with $M = \hat{E}^D\hat{E}$. According to the result in [9], the auxiliary system for the singular system (1) are given as follows:

$$
\begin{aligned}
\dot{x}_1(t) &= A_1 x_1(t) + A_{d1}\left[x_1(t-d(t)) + x_2(t-d(t))\right], \\
x_2(t) &= A_{d2}\left[x_1(t-d(t)) + x_2(t-d(t))\right],
\end{aligned} \tag{2}
$$

where $A_1 \triangleq \hat{E^D}\hat{A}$, $A_{d1} \triangleq \hat{E}^D\hat{A}_d$, $A_{d2} \triangleq (M-I)\hat{A}^D\hat{A}_d$, $\hat{A} \triangleq (\beta E-A)^{-1}A$, $\hat{A}_d \triangleq (\beta E-A)^{-1}A_d$, $\hat{E} \triangleq (\beta E-A)^{-1}E$ with β satisfying matrix $\beta E - A$ is nonsingular.

Remark 1. [9] *Based on Definition 5 and Lemma 2, several properties are given below:*
(i) $M^2 = M$;
(ii) $MA_1 = A_1 M = A_1$, $MA_{d1} = A_{d1}$;
(iii) $Mx_1(t) = x_1(t)$, $Mx_2(t) = 0$.

Lemma 3. [6] *Let $F \in \mathbb{R}^{p\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$. Consider the linear system $\dot{z}(t) = Az(t) + B\omega(t)$. For $t' \in \mathbb{R}_+$, condition: $\forall z(0),\ Fz(0) \succeq 0,\ \forall\omega(t) \succeq 0 \Rightarrow Fz(t) \succeq 0,\ t \in [0,t']$ holds if and only if there exists a Metzler matrix $H \in \mathbb{R}^{p\times p}$ and a matrix $K \in \mathbb{R}^{n\times q}$, $K \succeq 0$, such that $FA = HF$, $FB = K$.*

The positivity conditions of the system (1) and the auxiliary system (2) are first investigated.

Lemma 4. ***(Positivity)*** *Suppose that the pair (E,A) is regular and impulse-free. The following statements are equivalent:*
(i) System (1) is positive.
(ii) System (2) is positive.
(iii) $A_{d1} \succeq 0$, $A_{d2} \succeq 0$ and there exists a Metzler matrix $H \in \mathbb{R}^{n_x\times n_x}$ such that $A_1 = HM$.

Proof. *(iii)→(ii)*: According to system (2), $x_1(t)$ and $x_2(t)$ are given as follows:

$$x_1(t) = e^{A_1 t}x_1(0) + \int_0^t A_{d1}e^{A_1(t-\tau)}x(\tau - d(\tau))d\tau, \tag{3}$$

$$x_2(t) = A_{d2}x(t-d(t)). \tag{4}$$

Since $x_1(t) = Mx(t)$, equation (3) can be denoted as follows:

$$x_1(t) = M\left(e^{A_1 t}x(0) + \int_0^t A_{d1}e^{A_1(t-\tau)}x(\tau - d(\tau))d\tau\right).$$

A new system for $z(t)$ is given as follows:

$$\dot{z}(t) = A_1 z(t) + A_{d1}x(t-d(t)), \tag{5}$$

where $z(t)$ satisfies $x_1(t) = Mz(t)$. Since function $t - d(t)$ is continuous and $-d(t) \le -\underline{d}$ for all $t \in \mathbb{R}_{0,+}$, we can always find a scalar $t^* \in (0, \underline{d})$ such that $t - d(t) < 0$ for all $t \in [0, t^*]$. When $t \in [0, t^*]$, equation $x(t-d(t)) = \phi(t-d(t))$ holds. Based on Remark 1 and condition (iii) of Lemma 4, $MA_{d1} = A_{d1} \succeq 0$ holds and there exists a Metzler matrix H such that $MA_1 = HM$ holds. According to Lemma 3, for the given $t^* \in \mathbb{R}_+$ and nonnegative initial condition $\phi(t-d(t))$, inequality $x_1(t) = Mz(t) \succeq 0$ holds for all $t \in [0, t^*]$. Based on (4), when $A_{d1} \succeq 0$ and $A_{d2} \succeq 0$, we have $x_2(t) \succeq 0$ for all $t \in [0, t^*]$. Since $t - d(t) < t^*$ when $t \in [t^*, 2t^*]$, and $x_1(t) \succeq 0$ and $x_2(t) \succeq 0$ hold for all $t \in [0, t^*]$, we have $x_1(t) \succeq 0$ and $x_2(t) \succeq 0$ for all $t \in [0, 2t^*]$. By the principle of mathematical induction, we have $x_1(t) \succeq 0$ and $x_2(t) \succeq 0$ for all $t \in \mathbb{R}_{0,+}$, and system (2) is positive.
(ii)→ (i): Since $x(t) = x_1(t) + x_2(t)$, when system (2) is positive, $x(t) \succeq 0$ for all $t \in \mathbb{R}_{0,+}$.
(i)→(iii): Theorem 1 of [9] indicates that the condition (iii) of Lemma 4 is a necessary and sufficient positivity condition of a positive singular system with bounded time-varying delay. Therefore, (i)→(iii) is proved. □

3 Decay Rate Characterization

In order to characterize the decay rate, the asymptotic stability of the system is investigated first in this section. Then we move to the analysis of decay rate of the systems. Finally, several special cases: positive singular systems with bounded time-varying delays and time-varying delays with linear growth rate are investigated and the decay rates of those systems are characterized.

3.1 μ-Stability Condition

We first propose a lemma that reveals the equivalent relation of asymptotic stability between system (1) and system (2).

Lemma 5. *System (1) is asymptotically stable if and only if system (2) is asymptotically stable.*

Therefore, in order to obtain the μ-stability condition of system (1), the stability of system (2) is analyzed. Some useful lemmas, which will be used to derive the theorem, are recalled first.

Lemma 6. [3] For a Metzler matrix $Q \in \mathbb{M}^{n\times n}$, the following statements are equivalent:
(i) Q is a Hurwitz matrix;
(ii) Q is invertible and $Q^{-1} \preceq 0$;
(iii) There exists a vector $p \in \mathbb{R}^n_+$ such that $Qp \prec 0$.

Lemma 7. *For a Metzler matrix $Q \in \mathbb{M}^{n\times n}$ and two vectors $q_1 \in \mathbb{R}_+$ and $q_2 \in \mathbb{R}_+$ satisfying*

$$(q_1)_{[i]} \le (q_2)_{[i]}, \quad i = 1,2,\ldots,n, \tag{6}$$

$$(q_1)_{[j]} = (q_2)_{[j]}, \quad j \in \{1,2,\ldots,n\}, \tag{7}$$

it follows that inequality $(Qq_1)_{[j]} \le (Qq_2)_{[j]}$.

Proof. One can find $(Qq_1)_{[j]} = \sum_{i'=1}^{n} Q_{[j,i']}(q_1)_{[i']}$. Since Q is a Metzler matrix, $Q_{[j,i']} \ge 0$ for all $i' \neq j$ and $i' \in \{1,2,\ldots,n\}$. When inequality (6) and equation (7) hold, $Q_{[j,j]}(q_1)_{[j]} = Q_{[j,j]}(q_2)_{[j]}$ holds, and we have

$$(Qq_1)_{[j]} = \sum_{i'=1}^{n} Q_{[j,i']}(q_1)_{[i']} \le \sum_{i'=1}^{n} Q_{[j,i']}(q_2)_{[i']} = (Qq_2)_{[j]}.$$

This completes the proof. □

Then Theorem 1 characterizing the asymptotic stability condition of system (1) is given below.

Theorem 1. ***(Asymptotic Stability)*** *Suppose that the pair (E,A) is regular and impulse-free. System (1) is asymptotically stable with any admissible initial condition $\phi(\cdot) \succeq 0$ if there exists a Metzler matrix H such that $A_1 = HM$ and Π is Hurwitz where*

$$\Pi = \begin{bmatrix} H + A_{d1} & A_{d1} \\ A_{d2} & A_{d2} - I_{n_x} \end{bmatrix}. \tag{8}$$

Proof. According to Lemma 5, the asymptotic stability condition of system (1) is the same as the one of system (2). In what follows, we will show that system (2) is asymptotically stable if there exists a Metzler matrix H such that $A_1 = HM$ and Π is Hurwitz. Based on (iii) of Lemma 6, when Π is Hurwitz, there exists a positive vector $\bar{v} = \begin{bmatrix} v_1^{\mathrm{T}} & v_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{2n_x}_+$ such that $\Pi\bar{v} \prec 0$. This strict inequality indicates that there exist scalars $\varepsilon \in (0,1)$ and $\gamma \in (0,1)$ such that

$$\begin{bmatrix} H + A_{d1} & A_{d1} \\ A_{d2} & A_{d2} - I_{n_x} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \prec -\varepsilon \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{9}$$

$$\begin{bmatrix} \gamma H + A_{d1} & A_{d1} \\ A_{d2} & A_{d2} - I_{n_x} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \prec -(1-\gamma) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{10}$$

A vector function $\mathscr{V}(\bar{x}(t)) = \left[\bar{V}_1\left(\bar{x}_{[1]}(t)\right), \bar{V}_2\left(\bar{x}_{[2]}(t)\right), \ldots,\ \bar{V}_{2n_x}\left(\bar{x}_{[2n_x]}(t)\right)\right]^{\mathrm{T}}$ is constructed, where $\bar{x}(t) = \begin{bmatrix} x_1^{\mathrm{T}}(t) & x_2^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, and $\bar{V}_i\left(\bar{x}_{[i]}(t)\right)$ satisfies $\bar{V}_i(\bar{x}_{[i]}(t)) \triangleq \frac{\bar{x}_{[i]}(t)}{\bar{v}_{[i]}}$, for all $i \in \{1,2,\ldots,2n_x\}$. By mathematical induction, we will prove that, for any given scalar $m \in \mathbb{N}$, one can always find a time $t_m \in \mathrm{R}_{0,+}$ such that $\bar{V}_i\left(\bar{x}_{[i]}(t)\right) \le \gamma^m \|\varphi\|_c$, where

$$\varphi(t) = \begin{bmatrix} M\phi(t) \\ (I_{n_x} - M)\,\phi(t) \end{bmatrix}. \tag{11}$$

and $\|\varphi\|_c = \sup_{t\in(-\infty,0]}\left[\max_{i\in\{1,2,\ldots,2n_x\}}\left(\varphi_{[i]}(t)/\bar{v}_{[i]}\right)\right]$, holds for all $t \geq t_m$ and $i \in \{1,2,\ldots,2n_x\}$.

***Basis Step:* Prove that $\bar{V}_i(\bar{x}_{[i]}(t)) \leq \|\varphi\|_c$ holds for all $t \geq t_0 = 0$ and $i \in \{1,2,\ldots,2n_x\}$.** The statement is proved by contradiction, and the proof can be divided into two cases.

Case 1: Assume that there exist a scalar $j \in \{1,2,\ldots,n_x\}$ and a time $0 \leq \tau < t$ such that $\bar{V}_j(\bar{x}_{[j]}(t)) > \|\varphi\|_c$. Then one can find an index $j \in \{1,2,\ldots,n_x\}$ and a scalar $\tau \geq 0$ such that

$$\frac{\bar{x}_{[i]}(t)}{\bar{v}_{[i]}} \leq \|\varphi\|_c, \quad i = 1,2,\ldots,2n_x, \quad \forall t \in [0,\tau], \tag{12}$$

$$\frac{\bar{x}_{[j]}(\tau)}{\bar{v}_{[j]}} = \|\varphi\|_c, \tag{13}$$

$$\left.\frac{\mathrm{d}\left(\bar{x}_{[j]}(t)\right)}{\mathrm{d}t}\right|_{t=\tau} \geq 0. \tag{14}$$

According to system (2) and statement (iii) of Remark 1, the left-hand side of (14) gives

$$\left.\frac{\mathrm{d}\left(\bar{x}_{[j]}(t)\right)}{\mathrm{d}t}\right|_{t=\tau} = \{Hx_1(\tau) + A_{d1}\left[x_1(\tau - d(\tau)) + x_2(\tau - d(\tau))\right]\}_{[j]}.$$

Inequality (12) shows that $x_1(t) \preceq \|\varphi\|_c v_1$ and $x_2(t) \preceq \|\varphi\|_c v_2$ for all $t \in [0,\tau]$. According to Lemma 7, when $\tau - d(\tau) \leq \tau$ and matrix A_{d1} is nonnegative matrix, we have

$$\left.\frac{\mathrm{d}\left(\bar{x}_{[j]}(t)\right)}{\mathrm{d}t}\right|_{t=\tau} \leq \|\varphi\|_c\left[(H + A_{d1})v_1 + A_{d1}v_2\right]_{[j]}. \tag{15}$$

Based on inequality (9), inequality (15) gives $\left.\frac{\mathrm{d}\left(\bar{x}_{[j]}(t)\right)}{\mathrm{d}t}\right|_{t=\tau} < 0$, for all $j \in \{1,2,\ldots,n_x\}$. It contradicts with the assumption. Thus, $\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}} \leq \|\varphi\|_c$ holds for all $j \in \{1,2,\ldots,n_x\}$ and $t \in \mathbb{R}_{0,+}$.

Case 2: Assume that there exist a scalar $j \in \{n_x+1, n_x+2,\ldots,2n_x\}$ and a time $0 \leq \tau < t$ such that $\bar{V}_j(\bar{x}_{[j]}(t)) > \|\varphi\|_c$. Then, there exist an index $j \in \{n_x+1, n_x+2,\ldots,2n_x\}$ and scalars $\tau \geq 0$ and $\xi > 0$ such that

$$\frac{\bar{x}_{[i]}(t)}{\bar{v}_{[i]}} \leq \|\varphi\|_c, \quad i = 1,2,\ldots,2n_x, \quad \forall t \in [0,\tau], \tag{16}$$

$$\frac{\bar{x}_{[j]}(\tau)}{\bar{v}_{[j]}} = \|\varphi\|_c, \tag{17}$$

$$\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}} > \|\varphi\|_c, \quad \forall t \in (\tau, \tau+\xi). \tag{18}$$

Inequality (16) indicates that $x_1(t) \preceq \|\varphi\|_c v_1$ and $x_2(t) \preceq \|\varphi\|_c v_2$ for all $t \in [0,\tau]$. Since $\tau - d(\tau) \leq \tau$ holds and matrix A_{d2} is nonnegative, when $t = \tau$, we have

$$\begin{aligned}\bar{x}_{[j]}(\tau) &= \left[A_{d2}x_1(\tau - d(\tau)) + A_{d2}x_2(\tau - d(\tau))\right]_{[j-n_x]} \\ &\leq \|\varphi\|_c\left(A_{d2}v_1 + A_{d2}v_2\right)_{[j-n_x]}.\end{aligned} \tag{19}$$

Inequality (9) implies that $A_{d2}v_1 + A_{d2}v_2 \prec v_2$. Combining with inequality (19), we have $\bar{x}_{[j]}(\tau) < \|\varphi\|_c \bar{v}_{[j]}$, which contradicts with (17). Therefore, $\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}} \leq \|\varphi\|_c$ holds for all $j \in \{n_x+1, n_x+2,\ldots,n_x\}$ and $t \in \mathbb{R}_{0,+}$.

***Inductive Hypothesis:* Let m be an arbitrary integer satisfying $m \geq 0$. Assume that there exists a scalar t_m such that $\bar{V}_i\left(\bar{x}_{[i]}(t)\right) \leq \gamma^m \|\varphi\|_c$ holds for all $t \geq t_m$ and $i \in \{1,2,\ldots,2n_x\}$.**

<u>Inductive Step:</u> Prove that there exists a scalar $t_{m+1} \geq t_m$ such that $\bar{V}_i\left(\bar{x}_{[i]}(t)\right) \leq \gamma^{n+1}\|\varphi\|_c$ holds for all $t \geq t_{m+1}$ and $i \in \{1,2,\ldots,2n_x\}$. This proof can be divided into two cases.

Case 1: Prove that there exists a scalar t'_{m+1} such that $\bar{V}_i\left(\bar{x}_{[i]}(t)\right) \leq \gamma^{n+1}\|\varphi\|_c$ holds for all $t \geq t'_{m+1}$ and $i \in \{n_x+1, n_x+2, \ldots, 2n_x\}$. Statement (ii) of Assumption 1 implies that, for a given time t_m, we can always find a time t'_{m+1} such that $t'_{m+1} - d\left(t'_{m+1}\right) \geq t_m$. According to (10), when $t \geq t'_{m+1}$, we have

$$\begin{aligned}\bar{V}_i\left(\bar{x}_{[i]}(t)\right) &= \frac{\left[A_{d2}x_1(t-d(t)) + A_{d2}x_2(t-d(t))\right]_{[i-n_x]}}{(v_2)_{[i-n_x]}} \\ &\leq \gamma^m \|\varphi\|_c \frac{\left[A_{d2}(v_1+v_2)\right]_{[i-n_x]}}{(v_2)_{[i-n_x]}} < \gamma^{m+1}\|\varphi\|_c,\end{aligned}$$

for all $i \in \{n_x+1, \ldots, 2n_x\}$. When $t \geq t'_{m+1}$, $\bar{V}_i(\bar{x}_{[i]}(t)) \leq \gamma^{n+1}\|\varphi\|_c$ holds for all $i \in \{n_x+1, n_x+2, \ldots, 2n_x\}$.

Case 2: Prove that there exists a scalar $t_{m+1} \geq t'_{m+1}$ such that $\bar{V}_i\left(\bar{x}_{[i]}(t)\right) \leq \gamma^{n+1}\|\varphi\|_c$ holds for all $t \geq t_{m+1}$ and $i \in \{1,2,\ldots,n_x\}$. For a time $t \geq t'_{m+1}$, we assume that $\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right) = \max_{i\in\{1,2,\ldots,n_x\}} \bar{V}_i\left(\bar{x}_{[i]}(t)\right)$, where $i_{max} \in \{1,2,\ldots,n_x\}$. In what follows, we will first prove that there exists a time interval $[\tau_a, \tau_b]$, where $\tau_b > \tau_a \geq t'_{m+1}$, such that

$$\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right) \leq \gamma^{n+1}\|\varphi\|_c, \forall t \in [\tau_a, \tau_b]. \tag{20}$$

When $t = t'_{m+1}$, the value of $\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right)$ is discussed. If $\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t'_{m+1})\right) < \gamma^{n+1}\|\varphi\|_c$, the time interval $[\tau_a, \tau_b]$ satisfying condition (20) always exists due to the continuity of function $\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right)$. If $\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t'_{m+1})\right) \geq \gamma^{n+1}\|\varphi\|_c$, we have

$$\dot{\bar{V}}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right) = \frac{\left\{Hx_1(t) + A_{d1}\left[x_1(t-d(t)) + x_2(t-d(t))\right]\right\}_{[i_{max}]}}{\bar{v}_{[i_{max}]}}.$$

According to Lemma 7, when $t \geq t'_{m+1}$, we have

$$\begin{aligned}\dot{\bar{V}}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right) &\leq \frac{\left[Hx_1(t) + \gamma^m\|\varphi\|_c A_{d1}(v_1+v_2)\right]_{[i_{max}]}}{\bar{v}_{[i_{max}]}} \\ &\leq \frac{\left[\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right)Hv_1 + \gamma^m\|\varphi\|_c A_{d1}(v_1+v_2)\right]_{[i_{max}]}}{\bar{v}_{[i_{max}]}}.\end{aligned} \tag{21}$$

Since $Hv_1 \prec 0$, $\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right)Hv_1 \preceq \gamma^{n+1}\|\varphi\|_c Hv_1$ holds, and the left-hand side of (21) satisfies

$$\dot{\bar{V}}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right) \leq \gamma^m\|\varphi\|_c \frac{\left[\gamma Hv_1 + A_{d1}(v_1+v_2)\right]_{[i_{max}]}}{\bar{v}_{[i_{max}]}} < -(1-\gamma)\gamma^m\|\varphi\|_c. \tag{22}$$

Therefore, we can find a scalar τ_a such that $\bar{V}_{i_{max}}(\bar{x}_{[i_{max}]}(\tau_a)) = \gamma^{n+1}\|\varphi\|_c$, and $\dot{\bar{V}}_{i_{max}}\left(\bar{x}_{[i_{max}]}(\tau_a)\right) < 0$, where τ_a satisfies

$$t'_{m+1} \leq \tau_a \leq \frac{\gamma^m\|\varphi\|_c(1-\gamma)}{\gamma^m\|\varphi\|_c(1-\gamma)} + t'_{m+1} = t'_{m+1} + 1. \tag{23}$$

Due to the continuity of function $\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right)$, we can find a time interval $[\tau_a, \tau_b]$ such that inequality (20) holds.

Then, we will prove that if there exists a time interval $[\tau_a, \tau_b]$ such that inequality (20) holds, inequality

$$\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right) \leq \gamma^{m+1} \|\varphi\|_c$$

holds for all $t \in [\tau_a, \infty)$. This statement is proved by contradiction. Assume that there exist an index $j \in \{1, 2, \ldots, n_x\}$ and a time $\tau_b \geq \tau_a$ such that

$$\frac{\bar{x}_{[i]}(\tau_b)}{\bar{v}_{[i]}} \leq \gamma^{m+1} \|\varphi\|_c, \ i = 1, 2, \ldots, n_x, \ \forall t \in [\tau_a, \tau_b], \tag{24}$$

$$\frac{\bar{x}_{[j]}(\tau_b)}{\bar{v}_{[j]}} = \gamma^{m+1} \|\varphi\|_c, \tag{25}$$

$$\left.\frac{\mathrm{d}\left(\bar{x}_{[j]}(t)\right)}{\mathrm{d}t}\right|_{t=\tau_b} \geq 0. \tag{26}$$

Then the derivative of $\bar{x}_{[j]}(t)$ satisfies

$$\dot{\bar{x}}_{[j]}(\tau_b) \leq \gamma^m \|\varphi\|_c \left(\gamma H v_1 + A_{d1} v_1 + A_{d1} v_2\right)_{[j]} < 0. \tag{27}$$

Inequality (27) contradicts with inequality (26). Therefore, inequality $\bar{V}_{i_{max}}\left(\bar{x}_{[i_{max}]}(t)\right) \leq \gamma^{m+1} \|\varphi\|_c$ holds for all $t \in [\tau_a, \infty)$. Let $t_{m+1} = \tau_a$, Case 2 is proved.

Therefore, for a scalar $m \in \mathbb{N}$, we can always find a scalar $t_m \in \mathbb{R}_{0,+}$ such that $\bar{V}_i\left(\bar{x}_{[i]}(t)\right) \leq \gamma^m \|\varphi\|_c$ holds for all $t \geq t_m$ and $i \in \{1, 2, \ldots, 2n_x\}$. When $m \to \infty$, $\bar{V}_i\left(\bar{x}_{[i]}(t)\right) \to 0$, and Theorem 1 is proved. □

Remark 2. *If strictly positive initial conditions $x_1(t) \succ 0$ and $x_2(t) \succ 0$ are given, the asymptotic stability condition becomes a necessary and sufficient one. The proof can be found in Theorem 3 of [9].*

Theorem 2. ***(μ-stability)*** *Suppose that the pair (E, A) is regular and impulse-free, and there exist a Metzler matrix H such that $A_1 = HM$ and a positive vector $\bar{v} = \begin{bmatrix} v_1^{\mathrm{T}} & v_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}_+^{2n_x}$ such that*

$$\begin{bmatrix} H + A_{d1} & A_{d1} \\ A_{d2} & A_{d2} - I_{n_x} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \prec 0. \tag{28}$$

If there exists a function $\mu : \mathbb{R}_{0,+} \to \mathbb{R}_+$ such that the following conditions hold:
(i) $\mu(t) > 0$ for all $t \geq 0$;
(ii) $\mu(t)$ is a nondecreasing function;
(iii) $\mu(t) \to \infty$, when $t \to \infty$;
(iv) for all $i \in \{1, 2, \ldots, n_x\}$,

$$\lim_{t\to\infty}\left(\frac{\dot{\mu}(t)}{\mu(t)}\right) v_1 + H v_1 + \lim_{t\to\infty} \upsilon(t) A_{d1}(v_1 + v_2) \prec 0, \tag{29}$$

$$\lim_{t\to\infty} \upsilon(t) A_{d2}(v_1 + v_2) \prec v_2, \tag{30}$$

where $\upsilon(t) = \frac{\mu(t)}{\mu(t - d(t))}$. Then system (1) is μ-stable with any admissible initial condition $\phi(\cdot) \succeq 0$.

Proof. Before analyzing the μ-stability of system (1), we first investigate system (2). A Lyapunov function for $\bar{x}(t)=\begin{bmatrix}x_1^{\mathrm{T}}(t) & x_2^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}}$ is given as follows:

$$\bar{V}(\bar{x}(t)) \triangleq \max_{i\in\{1,2,\ldots,2n_x\}}\left(\frac{\bar{x}_{[i]}(t)}{\bar{v}_{[i]}}\right), \tag{31}$$

where $\bar{v}=\begin{bmatrix}v_1^{\mathrm{T}} & v_2^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$. According to Assumption 1, we can always find a scalar $T>0$ such that $T-d(T)\geq 0$. Conditions (29) and (30) indicate that we can always find a scalar $T'>0$ such that

$$\frac{\dot{\mu}(t)}{\mu(t)}+\frac{(Hv_1)_{[i]}}{(v_1)_{[i]}}+\frac{\mu(t)}{\mu(t-d(t))}\left(\frac{[A_{d1}(v_1+v_2)]_{[i]}}{(v_1)_{[i]}}\right)<0,$$

$$\left(\frac{\mu(t)}{\mu(t-d(t))}\right)\frac{[A_{d2}(v_1+v_2)]_{[i]}}{(v_2)_{[i]}}<1,$$

hold for all $i\in\{1,2,\ldots,n_x\}$ and $t>T'$. Since $\mu(t)$ is monotonically non-decreasing, and $\bar{V}(\bar{x}(t))\leq\|\varphi\|_c$, where $\|\varphi\|_c=\sup_{t\in(-\infty,0]}\left[\max_{i\in\{1,2,\ldots,2n_x\}}\left(\varphi_{[i]}(t)/\bar{v}_{[i]}\right)\right]$, and $\varphi(t)$ satisfies (11) for all $t\in\mathrm{R}_{0,+}$, we have

$$\mu(t)\bar{V}(\bar{x}(t))\leq M',$$

for all $t\in[0,T_{max}]$, where $T_{max}=\max\{T,T'\}$ and $M'=\mu(T_{max})\|\varphi\|_c$.

Then our goal is to prove that $\mu(t)\bar{V}(\bar{x}(t))\leq M'$ for all $t\in[T_{max},\infty)$, when conditions (i)–(iv) of Theorem a hold. By contradiction, we assume that there exist an index $j\in\{1,2,\ldots,2n_x\}$ and scalars $t_1\geq T_{max}$ and $\tau>0$ such that that

$$\mu(t)\frac{\bar{x}_{[i]}(t)}{\bar{v}_{[i]}}\leq M',\quad i=1,2,\ldots,2n_x,\quad t\in[0,t_1], \tag{32}$$

$$\mu(t_1)\frac{\bar{x}_{[j]}(t_1)}{\bar{v}_{[j]}}=M', \tag{33}$$

$$\mu(t)\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}}>M',\quad t\in(t_1,t_1+\tau). \tag{34}$$

Based on the value of index j, the proof can be divided into two cases.

Case 1: $j\in\{1,2,\ldots,n_x\}$. Then, we have $\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}}=\frac{(x_1(t))_{[j]}}{(v_1)_{[j]}}$, and inequality (34) implies $\left.\frac{\mathrm{d}\left(\mu(t)\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}}\right)}{\mathrm{d}t}\right|_{t=t_1}\geq 0$.

According to system (2), the derivative of $\mu(t)\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}}$ can be written as

$$\left.\frac{\mathrm{d}\left(\mu(t)\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}}\right)}{\mathrm{d}t}\right|_{t=t_1}=\dot{\mu}(t_1)\frac{\bar{x}_{[j]}(t_1)}{\bar{v}_{[j]}}+\mu(t_1)\frac{\{Hx_1(t_1)+A_{d1}[x_1(t_1-d(t_1))+x_2(t_1-d(t_1))]\}_{[j]}}{\bar{v}_{[j]}}. \tag{35}$$

Due to the nonnegativity of $\mu(t)$, $\dot{\mu}(t)$ and matrix A_{d1} and Lemma 7, we have $\left.\frac{\mathrm{d}\left(\mu(t)\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}}\right)}{\mathrm{d}t}\right|_{t=t_1}<0$, which contradicts with the assumption.

Case 2: $j\in\{n_x+1,n_x+2,\ldots,2n_x\}$. Then, we have $\frac{\bar{x}_{[j]}(t)}{\bar{v}_{[j]}}=\frac{(x_2(t))_{[j-n_x]}}{(v_2)_{[j-n_x]}}$. According to system (2), the left-hand side of equation (33) can be written as

$$\mu(t_1)\frac{\bar{x}_{[j]}(t_1)}{\bar{v}_{[j]}}=\mu(t_1)\frac{[A_{d2}(x_1(t_1-d(t_1)+x_2(t_1-d(t_1)))]_{[j-n_x]}}{\bar{v}_{[j]}}.$$

When A_{d2} is nonnegative, one can find

$$\mu(t_1)\frac{\bar{x}_{[j]}(t_1)}{\bar{v}_{[j]}} \leq \frac{M'\mu(t_1)}{\mu(t_t - d(t_1))}\frac{[A_{d2}(v_1+v_2)]_{[j-n_x]}}{\bar{v}_{[j]}} < M'.$$

The above inequality contradicts with assumption (33).

Therefore, for all time $t \in \mathbb{R}_{0,+}$, inequality $\mu(t)\bar{V}(\bar{x}(t)) \leq M'$ holds. It implies that inequality

$$\|\bar{x}(t)\|_\infty \leq \frac{M'}{\max_{i\in\{1,2,\ldots,2n_x\}} \bar{v}_{[i]}\mu(t)}$$

holds, for all $t \in \mathbb{R}_{0,+}$, and system (2) is μ-stable.

Then the μ-stability of system (1) is investigated. The Lyapunov function of $x(t)$ is given as

$$V(x(t)) \triangleq \max_{i\in\{1,2,\ldots,n_x\}} \left(\frac{x_{[i]}(t)}{v_{[i]}}\right),$$

where $v_{[i]} = (v_1)_{[i]} + (v_2)_{[i]}$ for all $i \in \{1,2,\ldots,n_x\}$. Since $\mu(t)\bar{V}(\bar{x}(t)) \leq M'$, we have

$$\begin{aligned}\mu(t)V(x(t)) &= \max_{i\in\{1,2,\ldots,n_x\}} \left(\frac{\mu(t)x_{[i]}(t)}{v_{[i]}}\right)\\ &= \max_{i\in\{1,2,\ldots,n_x\}} \left(\frac{\mu(t)\,(x_1(t))_{[i]} + \mu(t)\,(x_2(t))_{[i]}}{(v_1)_{[i]} + (v_2)_{[i]}}\right)\\ &\leq M'. \end{aligned} \tag{36}$$

Let $v_{max} = \max_{i\in\{1,2,\ldots,n_x\}} v_{[i]}$. The ∞-norm of $x(t)$ satisfies

$$\|x(t)\|_\infty \leq \frac{M'v_{max}}{\mu(t)}$$

for all $t \in \mathbb{R}_{0,+}$. When conditions (i)–(iv) hold, there always exists a constant $M = M'v_{max}$ such that for any compatible condition $\phi(t)$, vector $x(t)$ satisfies $\|x(t)\|_\infty \leq \frac{M}{\mu(t)}$ for all $t \in \mathbb{R}_{0,+}$. This completes the proof. $\square$

3.2 Special Cases

In this section, we will apply the obtained results on μ-stability to some special kinds of positive singular systems with time-delay. Two special cases are taken into consideration: one is the system with bounded time-varying delays, and the other is the system with time-varying delays subject to linear growth rate.

Case I: Bounded Time-varying Delays. When the time delay $d(t)$ satisfies $d(t) \in \left(0,\overline{d}\right]$, we choose $\mu(t) = e^{\lambda t}$ and analyze the λ-exponential stability of the system. According to Theorem 2, we have Corollary 1 to characterize the λ-exponential stability of system (1).

Corollary 1. *Suppose that the pair (E,A) is regular and impulse-free. For a given positive scalar $\lambda > 0$, if there exist a Metzler matrix H satisfying $A_1 = HM$ and a vector $v = \begin{bmatrix} v_1^{\mathrm{T}} & v_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ such that*

$$\begin{bmatrix} H + \lambda I_{n_x} + e^{\lambda\overline{d}}A_{d1} & e^{\lambda\overline{d}}A_{d1} \\ e^{\lambda\overline{d}}A_{d2} & e^{\lambda\overline{d}}A_{d2} - I_{n_x} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \prec 0, \tag{37}$$

then system (1) with bounded time-delay $d(t)$ such that $0 < d \leq \overline{d}$ is λ-exponentially stable with any admissible initial condition $\phi(\cdot) \succeq 0$.

Proof. According to inequality (37), the following two inequalities hold:

$$Hv_1 + \lambda v_1 + e^{\lambda \bar{d}} A_{d1}(v_1 + v_2) \prec 0, \tag{38}$$

$$e^{\lambda \bar{d}} A_{d2}(v_1 + v_2) \prec v_2. \tag{39}$$

Let $\mu(t) = e^{\lambda t}$, $\frac{\dot{\mu}(t)}{\mu(t)} = \lambda$ and $1 < e^{\lambda \underline{d}} \leq \frac{\mu(t)}{\mu(t-d(t))} = e^{\lambda d(t)} \leq e^{\lambda \bar{d}}$ hold. When (38) and (39) hold, we have

$$Hv_1 + \frac{\dot{\mu}(t)}{\mu(t)} v_1 + \frac{\mu(t)}{\mu(t-d(t))} A_{d1}(v_1 + v_2) \preceq Hv_1 + \frac{\dot{\mu}(t)}{\mu(t)} v_1 + e^{\lambda \bar{d}} A_{d1}(v_1 + v_2) \prec 0 \tag{40}$$

$$\frac{\mu(t)}{\mu(t-d(t))} A_{d2}(v_1 + v_2) \preceq e^{\lambda \bar{d}} A_{d2}(v_1 + v_2) \prec v_2 \tag{41}$$

Based on Theorem 2, when inequalities (40) and (41) hold, we can always find a scalar $M > 0$ such that $\|x(t)\|_\infty \leq Me^{-\lambda t}$ for all $t \in \mathbb{R}_{0,+}$, and the system (1) with bounded time-varying delays is λ-exponentially stable. □

Case II: Time-varying Delays with Linear Growth Rate. When the time delay $d(t) = \alpha t + d$, where $\alpha \in [0, 1)$ and $d \in (0, \infty)$, we choose $\mu(t) = (1 + at)^b$, where $a, b > 0$, and analyze the decay rates of the system.

Corollary 2. *Suppose that the pair (E, A) is regular and impulse-free. For a given positive scalar $\lambda > 0$, if there exist a Metzler matrix H satisfying $A_1 = HM$ and a vector $v = \begin{bmatrix} v_1^{\mathrm{T}} & v_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ such that*

$$\begin{bmatrix} H + (1-\alpha)^{-b} A_{d1} & (1-\alpha)^{-b} A_{d1} \\ (1-\alpha)^{-b} A_{d2} & (1-\alpha)^{-b} A_{d2} - I_{n_x} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \prec 0, \tag{42}$$

then system (1) is μ-stable, where $\mu(t) = (1 + at)^b$ and $a \in \left(0, \frac{1}{d}\right)$, with any admissible initial condition $\phi(\cdot) \succeq 0$.

Proof. When $a \in \left(0, \frac{1}{d}\right)$, we have $1 + a(t - d(t)) = 1 + a((1-\alpha)t - d) \geq 1 - ad > 0$. In other words, $\mu(t - d(t)) > 0$ for all $t \in \mathbb{R}_{0,+}$. According to inequality (42), the following two inequalities hold:

$$Hv_1 + (1-\alpha)^{-b} A_{d1}(v_1 + v_2) \prec 0, \tag{43}$$

$$(1-\alpha)^{-b} A_{d2}(v_1 + v_2) \prec v_2. \tag{44}$$

When $\mu(t) = (1 + at)^b$,

$$\lim_{t \to \infty} \frac{\dot{\mu}(t)}{\mu(t)} = \lim_{t \to \infty} \frac{b}{1 + at} = 0$$

and

$$\lim_{t \to \infty} \frac{\mu(t)}{\mu(t - d(t))} = \lim_{t \to \infty} \frac{(1 + at)^b}{(1 + a(t - \alpha t - d))^b} = (1-\alpha)^{-b}$$

hold. Therefore, when (43) and (44) hold, we have

$$\begin{aligned} & Hv_1 + \lim_{t \to \infty} \frac{\dot{\mu}(t)}{\mu(t)} v_1 + \lim_{t \to \infty} \frac{\mu(t)}{\mu(t - d(t))} A_{d1}(v_1 + v_2) \\ &= Hv_1 + (1-\alpha)^{-b} A_{d1}(v_1 + v_2) \prec 0, \end{aligned} \tag{45}$$

$$\begin{aligned} & \lim_{t \to \infty} \frac{\mu(t)}{\mu(t - d(t))} A_{d2}(v_1 + v_2) \\ &= (1-\alpha)^{-b} A_{d2}(v_1 + v_2) \prec v_2. \end{aligned} \tag{46}$$

Based on Theorem 2, system (1) is μ-stable with $\mu(t) = (1+at)^b$. □

4 Illustrative Example

Delay equations of neutral type often appear in the the practical model involving the noninstant connection [16], such as food-limited population model [19], partial element equivalent circuit (PEEC) [30]. A neutral delay differential equation (NDDE) could be represented as follows:

$$\dot{y}(t) = Py(t) + Qy(t-d(t)) + L\dot{y}(t-d(t)),$$

where $y \in \mathbb{R}^{n_y}$. By letting $x(t) = [y^{\mathrm{T}}(t),\ \dot{y}^{\mathrm{T}}(t) - y^{\mathrm{T}}(t)L^{\mathrm{T}}]^{\mathrm{T}}$, the NDDE can be seen as the following singular system:

$$E\dot{x}(t) = Ax(t) + A_d x(t-d(t)), \tag{47}$$

where

$$E = \begin{bmatrix} I_{n_y} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} P & I_{n_y} \\ 0 & -I_{n_y} \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ Q+LP & L \end{bmatrix}.$$

Let $P = -0.7$, $Q = 0.5$ and $L = 0.2$. One can find that, when $\beta = 4$, we have $\beta E - A$ is nonsingular, and matrices $\hat{E}$ and $\hat{A}$ commute. Then matrices A_1, A_{d1} and A_{d2} are given as follows:

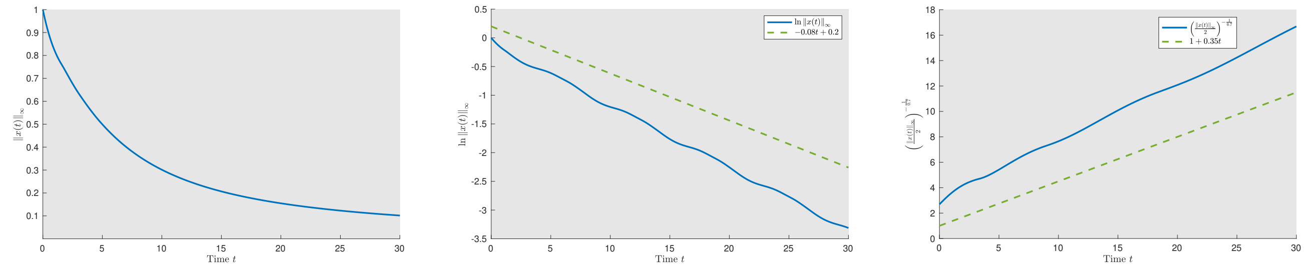
$$A_1 = \begin{bmatrix} -0.7 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.36 & 0.2 \\ 0 & 0 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0 & 0 \\ 0.36 & 0.2 \end{bmatrix}.$$

Since matrices A_{d1} and A_{d2} are nonnegative matrices, and there exists a Metzler matrix

$$H = \begin{bmatrix} -0.7 & 0.1427 \\ 0 & -0.1427 \end{bmatrix}$$

such that $A_1 = HM$. Therefore, system (47) is positive with admissible initial condition $\phi(t) = [1,\ 0.45]^{\mathrm{T}}$ for all $t \leq 0$. We can calculate the eigenvalues of Π is -0.9234, -0.1427, -1 and -0.2166, and the matrix Π is Hurwitz. Therefore, by Theorem 1, system (47) is asymptotically stable. When the delay is chosen to be $d_1(t) = 0.3t - \ln(t+1) + 2$, which satisfies Assumption 1, the variation of $\|x(t)\|_\infty$ are given in Fig.1-(a). It shows that, when $t \to \infty$, the values of $\|x(t)\|_\infty$ converge to zero.

Then the decay rates of system (47) with different type of time delays are discussed. Delay $d_2(t) = 2 - \sin t$ is a bounded time-varying delay with an upper-bound $\overline{d}$ equals 3. According to (37) in Corollary 1, we can get the largest feasible value of $\lambda = 0.081$ by iteration. Fig.1-(b) characterized the variation of $\ln\|x(t)\|_\infty$. It shows that the curve of $\ln\|x(t)\|_\infty$ is always beneath the curve $-0.081t + 0.2$, which indicates that system (1) with bounded time-varying delays is λ-exponentially stable with $\lambda = 0.081$. Then, a time-varying delay with linear growth rate $d_3(t) = 0.4t + 2$ is given. According to (42) in Corollary 2, we can choose $\alpha = 0.35$ and $b = 0.7$. The curve of $\ln\|x(t)\|_\infty$ is drawn in Fig.1-(c). One can find the value of $\|x(t)\|_\infty$ is less than $\frac{2}{(1+0.35t)^{0.7}}$ for all $t \geq 0$, which verifies Corollary 2.



(a) Trajectory of $\|x(t)\|_\infty$ in system (47) with delay $d_1(t)$

(b) Evolution of function $\ln\|x(t)\|_\infty$ with delay $d_2(t)$

(c) Evolution of function $\left(\frac{\|x(t)\|_\infty}{2}\right)^{-\frac{1}{0.7}}$ with delay $d_3(t)$

Figure 1: Trajectory of $\|x(t)\|_\infty$ with different time delays

5 Conclusion

In this paper, the asymptotic stability and decay rate characterization of a singular system with unbounded delays have been studied. By introducing an auxiliary system, the original singular system is converted to a differential-difference system. It has been shown that the positivity and stability conditions for these two systems are equivalent. Furthermore, the decay rate of the system is characterized by a non-decreasing function $\mu(t)$. By choosing different $\mu(t)$, the decay rates of positive singular systems with bounded time-varying delays and time-varying delays subject to linear growth have been characterized. This work has generalized previous work on positive singular systems with bounded time-delays.

References

[1] M. Ait Rami and D. Napp. Positivity of discrete singular systems and their stability: An LP-based approach. *Automatica*, 50(1):84–91, 2014.

[2] M. Ait Rami, M. Schönlein, and J. Jordan. Estimation of linear positive systems with unknown time-varying delays. *European Journal of Control*, 19(3):179–187, 2013.

[3] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Philadephia, PA: SIAM, 1994.

[4] V. S. Bokharaie and O. Mason. On delay-independent stability of a class of nonlinear positive time-delay systems. *IEEE Transactions on Automatic Control*, 59(7):1974–1977, 2014.

[5] S. L. Campbell. *Singular Systems of Differential Equations*. San Francisco: Pitman, 1980.

[6] E. B. Castelan and J. Hennet. On invariant polyhedra of continuous-time linear systems. *IEEE Transactions on Automatic control*, 38(11):1680–1685, 1993.

[7] Y. Cui, Z. Feng, J. Shen, and Y. Chen. L_∞-gain analysis for positive singular time-delay systems. *Journal of the Franklin Institute*, 354(13):5162–5175, 2017.

[8] Y. Cui, J. Shen, and Y. Chen. Stability analysis for positive singular systems with distributed delays. *Automatica*, 94:170–177, 2018.

[9] Y. Cui, J. Shen, Z. Feng, and Y. Chen. Stability analysis for positive singular systems with time-varying delays. *IEEE Transactions on Automatic Control*, 63(5):1487–1494, 2018.

[10] L. Dai. *Singular Control Systems*. Springer, 1989.

[11] L. Farina and S. Rinaldi. *Positive Linear Systems: Theory and Applications*. New York: Wiley-Interscience, 2000.

[12] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson. Asymptotic stability and decay rates of homogeneous positive systems with bounded and unbounded delays. *SIAM Journal on Control and Optimization*, 52(4):2623–2650, 2014.

[13] J. K. Hale and S. M. V. Lunel. *Introduction to Functional Differential Equations*. Springer Science & Business Media, 2013.

[14] D. C. Huong and N. H. Sau. Stability and ℓ_∞-gain analysis for discrete-time positive singular systems with unbounded time-varying delays. *IET Control Theory & Applications*, 14(17):2507–2513, 2020.

[15] P. Kunkel and V. Mehrmann. *Differential-algebraic Equations: Analysis and Numerical Solution*. European Mathematical Society, 2006.

[16] Y. Kyrychko and S. Hogan. On the use of delay equations in engineering applications. *Journal of Vibration and Control*, 16(7-8):943–960, 2010.

[17] S. Li and H. Lin. On ℓ_1 stability of switched positive singular systems with time-varying delay. *International Journal of Robust and Nonlinear Control*, 27(16):2798–2812, 2017.

[18] S. Li, Z. Xiang, and J. Zhang. Robust stability and stabilization conditions for uncertain switched positive systems under mode-dependent dwell-time constraints. *International Journal of Robust and Nonlinear Control*, 31(17):8569–8604, 2021.

[19] M. Liu, I. Dassios, and F. Milano. On the stability analysis of systems of neutral delay differential equations. *Circuits, Systems, and Signal Processing*, 38(4):1639–1653, 2019.

[20] T. Liu, B. Wu, L. Liu, and Y.-E. Wang. Finite-time stability of discrete switched singular positive systems. *Circuits, Systems, and Signal Processing*, 36(6):2243–2255, 2017.

[21] X. Liu, W. Yu, and L. Wang. Stability analysis of positive systems with bounded time-varying delays. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 56(7):600–604, 2009.

[22] O. Mason. Diagonal Riccati stability and positive time-delay systems. *Systems & Control Letters*, 61(1):6–10, 2012.

[23] P. N. Pathirana, P. T. Nam, and H. M. Trinh. Stability of positive coupled differential-difference equations with unbounded time-varying delays. *Automatica*, 92:259–263, 2018.

[24] V. N. Phat and N. H. Sau. On exponential stability of linear singular positive delayed systems. *Applied Mathematics Letters*, 38:67–72, 2014.

[25] V. N. Phat and N. H. Sau. Exponential stabilisation of positive singular linear discrete-time delay systems with bounded control. *IET Control Theory & Applications*, 13(7):905–911, 2018.

[26] W. Qi and X. Gao. State feedback controller design for singular positive markovian jump systems with partly known transition rates. *Applied mathematics letters*, 46:111–116, 2015.

[27] N. H. Sau, D. C. Huong, and M. V. Thuan. New results on reachable sets bounding for delayed positive singular systems with bounded disturbances. *Journal of the Franklin Institute*, 358(1):1044–1069, 2021.

[28] H. The Tuan, H. Trinh, and J. Lam. Positivity and stability of mixed fractional-order systems with unbounded delays: Necessary and sufficient conditions. *International Journal of Robust and Nonlinear Control*, 31(1):37–50, 2021.

[29] H. M. Trinh, P. T. Nam, and P. N. Pathirana. Linear functional state bounding for positive systems with disturbances varying within a bounded set. *Automatica*, 111:108644, 2020.

[30] D. Yue and Q.-L. Han. A delay-dependent stability criterion of neutral systems and its application to a partial element equivalent circuit model. In *Proceedings Of the American Control Conference*, volume 6, 2004.

[31] D. Zhang, Q. Zhang, and B. Du. Positivity and stability of positive singular Markovian jump time-delay systems with partially unknown transition rates. *Journal of the Franklin Institute*, 354(2):627–649, 2017.

[32] X. Zhao, X. Liu, S. Yin, and H. Li. Improved results on stability of continuous-time switched positive linear systems. *Automatica*, 50(2):614–621, 2014.

[33] X. Zhao, Y. Yin, L. Liu, and X. Sun. Stability analysis and delay control for switched positive linear systems. *IEEE Transactions on Automatic Control*, 63(7):2184–2190, 2017.

[34] B. Zhu, J. Lam, and Y. Ebihara. Input–output gain analysis of positive periodic systems. *International Journal of Robust and Nonlinear Control*, 31(8):2928–2945, 2021.