Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

SHORT COMMUNICATION

Minimum-Radius Criterion for a Zonotopic State Estimator based on Degrees of Freedom

Alesi A. de Paula[1] | Guilherme V. Raffo[1,2] | Bruno O. S. Teixeira[1,2]

[1]Graduate Program in Electrical Engineering, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brazil
[2]Department of Electronic Engineering, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brazil

Funding Information
This research was supported by the Brazilian agencies CNPq, under grants 465755/2014-3 (INCT project), 315695/2020-0, and 306564/2020-3; FAPESP, under grant 2014/50851-0; CAPES, under the grants 001 and 88887.136349/2017-00; and FAPEMIG.

Abstract

This paper proposes a new cost criterion to enhance the precision of a zonotopic state estimator for discrete-time descriptor linear systems. Originally, the algorithm solves a minimum-trace problem involving zonotopes, whose evolution is given by an interval observer structure containing extra design matrices, called degrees of freedom. Although the minimization of trace yields explicit solutions, it does not necessarily imply minimization of volume, and thereby, the precision of the output zonotope cannot be improved effectively. The volume measure for zonotopes is computationally expensive and, when used as cost criterion, implies nonlinear optimization problems. Motivated by such issues, we here propose a minimum-radius criterion where the smallest box enclosing the output zonotope is minimized. The resulting optimization problem is nonlinear, but its convexity is exploited to yield an equivalent linear program. The effectiveness of our approach is illustrated over two numerical examples.

KEYWORDS:
Set-based state estimators, Zonotopes, Discrete-time descriptor linear systems, Linear programming.

1 | INTRODUCTION

Zonotopes are particular convex polytopes that confer appealing properties to set-based state estimation. A brief historical motivation about zonotopic algorithms and their practical application is found in[1, Table S1]. Among those algorithms,[2] has proposed a zonotopic estimator for discrete-time descriptor linear systems, whose interval observer structure contains three time-varying design matrices, namely: a Kalman-like gain matrix and two other matrices adding degrees of freedom to obtain tight solutions. Aiming at explicit solutions, the authors take advantage from the Lagrange function to formulate a minimum-trace optimization problem, and thereby, obtain the desired solutions using matrix inversions. Although this algorithm returns more precise sets than the originally compared approaches, it does not imply volume minimization.

In the authors' best knowledge, minimum-volume approaches for zonotopes have not been actually discussed yet in the interval observers framework, since the exact volume brings up a series of challenges such as large computational burden and nonlinearity when evaluating the design matrices. Conversely, H_∞ formulations as in[3,4] make use of the so-called bounded real lemma[5] to reach stable observers with a good performance at steady state.

Motivated by the aforementioned works, we here propose a new cost criterion to the zonotopic filter from[2], whose idea is to minimize the radius of the smallest box containing the output zonotope. Though the proposed minimum-radius criterion is not equivalent to the exact volume approach, it directly influences the volume measure since the considered box is the so-called interval hull of the zonotope. The resulting optimization problem is nonlinear, but we rewrite it as a linear program (LP) to enable the use of linear solvers. This paper assumes that the possible state linear equality constraints are already embedded in the descriptor representation. Then, the linear inequality and equality constraints present in the derived LP are related to only the reformulation of the original problem.

This paper is outlined as follows. Subsection 1 presents the necessary notations and definitions. The problem here investigated is stated in Section 2. In section 3, the minimum-radius problem is expressed as an LP, and the corresponding state estimator (called DRZF) is experimented in Section 4 by means of two case studies. Section 5 concludes the paper.

NOTATION AND DEFINITIONS

An $(n \times 1)$-dimensional vector and an $(n \times m)$-dimensional matrix are, respectively, denoted as $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times m}$. The cost vector of an LP is denoted as ψ. The matrix and the vector of linear inequality constraint of an LP are, respectively, denoted as D^{i} and d^{i}. The matrix and the vector of linear equality constraint of an LP are, respectively, denoted as D^{e} and d^{e}. A diagonal matrix obtained from a vector is denoted as $\mathrm{diag}(\cdot)$. The rank and the transpose of a matrix are, respectively, denoted as $\mathrm{rank}(\cdot)$ and $(\cdot)^\top$. A suboptimal solution is denoted as $\hat{(\cdot)}$. The optimal solution of an optimization problem is denoted as $\check{(\cdot)}$. The absolute value operator is denoted as $|\cdot|$. An $(n \times m)$-dimensional one matrix, an $(n \times m)$-dimensional zero matrix, and an $(n \times n)$-dimensional identity matrix are, respectively, denoted as $1_{n\times m}$, $0_{n\times m}$, and I_n. The jth column of a matrix and the ith row of a matrix are, respectively, denoted as $(\cdot)_{:,j}$ and $(\cdot)_{i,:}$. An $(n \times 1)$-dimensional interval vector is defined as $[x] \triangleq \left[x^{\mathrm{L}}, x^{\mathrm{U}}\right] \subset \mathbb{R}^n$, where $x^{\mathrm{L}} \in \mathbb{R}^n$ and $x^{\mathrm{U}} \in \mathbb{R}^n$ are its known lower and upper bounds, respectively. The diameter of the box $[x] \subset \mathbb{R}^n$ is defined as $\mathrm{diam}([x]) \triangleq \left(x^{\mathrm{U}} - x^{\mathrm{L}}\right)$. The unitary box of order m is defined as $\mathcal{B}^m \triangleq [-1, 1]^m$. The generator matrix and the center of a set are, respectively, denoted as $G^{\mathrm{x}} \in \mathbb{R}^{n\times n_g}$ and $\bar{x} \in \mathbb{R}^n$. A zonotope of order $n_g \geq n$ is defined as $\mathcal{X} \triangleq \{G^{\mathrm{x}}, \bar{x}\} = \{G^{\mathrm{x}}\xi + \bar{x} : \xi \in \mathcal{B}^{n_g}\} \subset \mathbb{R}^n$[6], with $G^{\mathrm{x}} \in \mathbb{R}^{n\times n_g}$ and $\bar{x} \in \mathbb{R}^n$. The linear mapping of the zonotope $\mathcal{X} = \{G^{\mathrm{x}}, \bar{x}\} \subset \mathbb{R}^n$ is defined as $L\mathcal{X} \triangleq \{LG^{\mathrm{x}}, L\bar{x}\}$[6], where $L \in \mathbb{R}^{m\times n}$. The Minkowski sum of the zonotopes $\mathcal{X} = \{G^{\mathrm{x}}, \bar{x}\} \subset \mathbb{R}^n$ and $\mathcal{W} = \{G^{\mathrm{w}}, \bar{w}\} \subset \mathbb{R}^n$ is defined as $\mathcal{X} \oplus \mathcal{W} \triangleq \left\{\left[G^{\mathrm{x}}\ G^{\mathrm{w}}\right], \bar{x} + \bar{w}\right\}$[6]. The interval hull of the zonotope $\mathcal{X} = \{G^{\mathrm{x}}, \bar{x}\} \subset \mathbb{R}^n$ is defined as $\square\mathcal{X} \triangleq [\bar{x} - \zeta, \bar{x} + \zeta]$[6], where $\zeta = |G^{\mathrm{x}}|\, 1_{n_g\times 1}$ is the radius.

2 | PROBLEM FORMULATION

Consider the discrete-time descriptor linear time-varying dynamical system

$$E_k x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + w_{k-1}, \tag{1}$$

$$y_k = C_k x_k + v_k, \tag{2}$$

where $E_k \in \mathbb{R}^{n\times n}$, $A_{k-1} \in \mathbb{R}^{n\times n}$, $B_{k-1} \in \mathbb{R}^{n\times p}$, and $C_k \in \mathbb{R}^{m\times n}$ are the known system matrices, $u_{k-1} \in \mathbb{R}^p$ is the deterministic input vector, $y_k \in \mathbb{R}^m$ is the measured output vector, and $x_k \in \mathbb{R}^n$ is the state vector to be estimated. Since $\mathrm{rank}(E_k) \leq n$, the descriptor representation (1) already embeds the state linear equality constraints to x_k.

The noise terms $w_{k-1} \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$, as well as the initial state x_0, are here represented by the known zonotopes $\mathcal{W}_{k-1} = \left\{G^{\mathrm{w}}_{k-1}, 0_{n\times 1}\right\}$, $\mathcal{V}_k = \left\{G^{\mathrm{v}}_k, 0_{m\times 1}\right\}$, and $\hat{\mathcal{X}}_0 = \left\{\hat{G}^{\mathrm{x}}_0, \hat{x}_0\right\}$, respectively. No rank assumption is required for the generator matrices.

By assuming that $\mathrm{rank}\left(\begin{bmatrix} E_k \\ C_k \end{bmatrix}\right) = n$, there exists a full-rank matrix $\begin{bmatrix} T_k & N_k \end{bmatrix}$, where $T_k \in \mathbb{R}^{n\times n}$ and $N_k \in \mathbb{R}^{n\times m}$, such that

$$T_k E_k + N_k C_k = \mathrm{I}_n. \tag{3}$$

Post-multiplying (3) by x_k and using (1)-(2), we obtain

$$x_k = T_k A_{k-1}x_{k-1} + T_k B_{k-1}u_{k-1} + T_k w_{k-1} + N_k y_k - N_k v_k.$$

Therefore, we define the following center estimate:

$$\hat{x}_k = M_{k-1}\hat{x}_{k-1} + T_k B_{k-1}u_{k-1} + K_{k-1}y_{k-1} + N_k y_k, \tag{4}$$

where $M_{k-1} = \left(T_k A_{k-1} - K_{k-1}C_{k-1}\right)$, and $K_{k-1} \in \mathbb{R}^{n\times m}$ is the observer gain. The gain matrix K is here stated at $k-1$ because the predictor/corrector structure is based on y_{k-1}.

By defining the estimation error as $e_k \triangleq x_k - \hat{x}_k$, we obtain

$$e_k = M_{k-1}\left(x_{k-1} - \hat{x}_{k-1}\right) + T_k w_{k-1} - K_{k-1}v_{k-1} - N_k v_k.$$

Make explicit $x_k = \left(x^{\mathrm{z}}_k + \bar{x}_k\right) \in \mathcal{X}_k = \left\{G^{\mathrm{x}}_k, \bar{x}_k\right\}$, where $x^{\mathrm{z}}_k \in \mathcal{X}^{\mathrm{z}}_k = \left\{G^{\mathrm{x}}_k, 0_{n\times 1}\right\}$ is the error estimate around $\bar{x}_k$, and assume that the center deviation $(\bar{x}_k - \hat{x}_k)$ in e_k is shifted to x^{z}_k, yielding a new variable $\hat{x}^{\mathrm{z}}_k \in \hat{\mathcal{X}}^{\mathrm{z}}_k = \left\{\hat{G}^{\mathrm{x}}_k, 0_{n\times 1}\right\} \supseteq \mathcal{X}^{\mathrm{z}}_k$. Then, $x_k = \left(\hat{x}^{\mathrm{z}}_k + \hat{x}_k\right) \in \hat{\mathcal{X}}_k = \left\{\hat{G}^{\mathrm{x}}_k, \hat{x}_k\right\}$, and $e_k = \hat{x}^{\mathrm{z}}_k$ is such that

$$\begin{aligned}\hat{x}^{\mathrm{z}}_k &= M_{k-1}\hat{x}^{\mathrm{z}}_{k-1} + T_k w_{k-1} - K_{k-1} v_{k-1} - N_k v_k \\ &\in M_{k-1}\hat{\mathcal{X}}^{\mathrm{z}}_{k-1} \oplus T_k \mathcal{W}_{k-1} \oplus \left(-K_{k-1}\right)\mathcal{V}_{k-1} \oplus \left(-N_k\right)\mathcal{V}_k \\ &= \hat{\mathcal{X}}^{\mathrm{z}}_k,\end{aligned}$$

where

$$\hat{G}^{\mathrm{x}}_k = \left[M_{k-1}\hat{G}^{\mathrm{x}}_{k-1} \;\; T_k G^{\mathrm{w}}_{k-1} \;\; -K_{k-1}G^{\mathrm{v}}_{k-1} \;\; -N_k G^{\mathrm{v}}_k\right], \tag{5}$$

$G^{\mathrm{w}}_{k-1} \in \mathbb{R}^{n\times n^{\mathrm{w}}_g}$, $G^{\mathrm{v}}_{k-1} \in \mathbb{R}^{n\times n^{\mathrm{v}}_{g,k-1}}$, and $G^{\mathrm{v}}_k \in \mathbb{R}^{n\times n^{\mathrm{v}}_{g,k}}$.

Our goal is to determine the design matrices T_k, N_k, and K_{k-1}, for $k \in \mathbb{Z}_+$, by minimizing the radius of the interval hull of $\hat{\mathcal{X}}_k$ given by $\square\hat{\mathcal{X}}_k$.

3 | MINIMUM-RADIUS CRITERION

In this section, our proposal of minimizing the radius of the box $\square\hat{\mathcal{X}}_k$ is presented, where $\hat{\mathcal{X}}_k$ is the zonotope estimate that encloses the unknown state vector x_k. Let $F \triangleq A_{k-1}\hat{G}^{\mathrm{x}}_{k-1}$ and $H \triangleq C_{k-1}\hat{G}^{\mathrm{x}}_{k-1}$. According to (5), the ith radius of $\square\hat{\mathcal{X}}_k$, for $i = 1, \dots, n$, is given by

$$\begin{aligned}\zeta_{i,k} &\triangleq \left|\hat{G}^{\mathrm{x}}_{(i,:),k}\right| 1_{\left(n_g + n^{\mathrm{w}}_g + n^{\mathrm{v}}_{g,k-1} + n^{\mathrm{v}}_{g,k}\right)\times 1} \\ &= \sum_{j=1}^{n_g}\left|T_{(i,:),k}F_{:,j} - K_{(i,:),k-1}H_{:,j}\right| + \sum_{j=1}^{n^{\mathrm{w}}_g}\left|T_{(i,:),k}G^{\mathrm{w}}_{:,j}\right| + \sum_{j=1}^{n^{\mathrm{v}}_{g,k-1}}\left|-K_{(i,:),k-1}G^{\mathrm{v}}_{(:,j),k-1}\right| + \sum_{j=1}^{n^{\mathrm{v}}_{g,k}}\left|-N_{(i,:),k}G^{\mathrm{v}}_{(:,j),k}\right|,\end{aligned} \tag{6}$$

with $T_{(i,:),k}E_k + N_{(i,:),k}C_k = \left(\mathrm{I}_n\right)_{i,:}$ being the corresponding linear equality constraint (3). Thus, the minimization of ζ_k consists of n constrained optimization problems, from which the matrices T_k, N_k, and K_{k-1} are optimally specified. In principle, due to the fact that $\zeta_{i,k}$ is nonlinear, the prior problem should be solved using nonlinear solvers. However, by employing some artifices presented in[7, Section 1.3], we transform the original optimization problem in an LP. In doing so, efficient solvers such as CPLEX and Gurobi can be used. The derived LP is presented next in Proposition 1.

Proposition 1. The constrained optimization problem given by the minimization of (6) subject to $T_{(i,:),k}E_k + N_{(i,:),k}C_k = \left(\mathrm{I}_n\right)_{i,:}$ is equivalent to the LP

$$\min_z \; \psi^\top z \quad \text{s.t.} \quad D^{\mathrm{i}} z \leq d^{\mathrm{i}}, \quad D^{\mathrm{e}} z = d^{\mathrm{e}}, \tag{7}$$

where $z \in \mathbb{R}^{n^{\mathrm{z}}}$ is the variable vector such that

$$z^\top_{1:n} \triangleq T_{(i,:),k} \tag{8}$$

$$z^\top_{n+1:n+m} \triangleq N_{(i,:),k}, \tag{9}$$

$$z^\top_{n+m+1:n+2m} \triangleq K_{(i,:),k-1}, \tag{10}$$

$n^{\mathrm{z}} = \left(n + 2m + n_g + n^{\mathrm{w}}_g + n^{\mathrm{v}}_{g,k-1} + n^{\mathrm{v}}_{g,k}\right)$, $\psi \in \mathbb{R}^{n^{\mathrm{z}}}$ is the cost vector, $D^{\mathrm{i}} \in \mathbb{R}^{n^{\mathrm{i}}\times n^{\mathrm{z}}}$ and $d^{\mathrm{i}} \in \mathbb{R}^{n^{\mathrm{i}}}$ compose the inequality constraints, $n^{\mathrm{i}} = 2\left(n_g + n^{\mathrm{w}}_g + n^{\mathrm{v}}_{g,k-1} + n^{\mathrm{v}}_{g,k}\right)$, and $D^{\mathrm{e}} \in \mathbb{R}^{n\times n^{\mathrm{z}}}$ and $d^{\mathrm{e}} \in \mathbb{R}^{n}$ compose the equality constraints. These parameters are defined as

$$\psi \triangleq \left[0_{1\times(n+2m)} \;\; 1_{1\times\left(n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}+n_{g,k}^{\mathrm{v}}\right)}\right]^\top, \tag{11}$$

$$D^{\mathrm{i}}_{1:n_g,:} \triangleq \left[-F^\top \;\; 0_{n_g\times m} \;\; H^\top \;\; -\mathrm{I}_{n_g} \;\; 0_{n_g\times\left(n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}+n_{g,k}^{\mathrm{v}}\right)}\right], \tag{12}$$

$$D^{\mathrm{i}}_{n_g+1:2n_g,:} \triangleq \left[F^\top \;\; 0_{n_g\times m} \;\; -H^\top \;\; -\mathrm{I}_{n_g} \;\; 0_{n_g\times\left(n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}+n_{g,k}^{\mathrm{v}}\right)}\right], \tag{13}$$

$$D^{\mathrm{i}}_{2n_g+1:2n_g+n_g^{\mathrm{w}},:} \triangleq \left[-\left(G_{k-1}^{\mathrm{w}}\right)^\top \;\; 0_{n_g^{\mathrm{w}}\times(2m+n_g)} \;\; -\mathrm{I}_{n_g^{\mathrm{w}}} \;\; 0_{n_g^{\mathrm{w}}\times\left(n_{g,k-1}^{\mathrm{v}}+n_{g,k}^{\mathrm{v}}\right)}\right], \tag{14}$$

$$D^{\mathrm{i}}_{2n_g+n_g^{\mathrm{w}}+1:2\left(n_g+n_g^{\mathrm{w}}\right),:} \triangleq \left[\left(G_{k-1}^{\mathrm{w}}\right)^\top \;\; 0_{n_g^{\mathrm{w}}\times(2m+n_g)} \;\; -\mathrm{I}_{n_g^{\mathrm{w}}} \;\; 0_{n_g^{\mathrm{w}}\times\left(n_{g,k-1}^{\mathrm{v}}+n_{g,k}^{\mathrm{v}}\right)}\right], \tag{15}$$

$$D^{\mathrm{i}}_{2\left(n_g+n_g^{\mathrm{w}}\right)+1:2\left(n_g+n_g^{\mathrm{w}}\right)+n_{g,k-1}^{\mathrm{v}},:} \triangleq \left[0_{n_{g,k-1}^{\mathrm{v}}\times(n+m)} \;\; \left(G_{k-1}^{\mathrm{v}}\right)^\top \;\; 0_{n_{g,k-1}^{\mathrm{v}}\times\left(n_g+n_g^{\mathrm{w}}\right)} \;\; -\mathrm{I}_{n_{g,k-1}^{\mathrm{v}}} \;\; 0_{n_{g,k-1}^{\mathrm{v}}\times n_{g,k}^{\mathrm{v}}}\right], \tag{16}$$

$$D^{\mathrm{i}}_{2\left(n_g+n_g^{\mathrm{w}}\right)+n_{g,k-1}^{\mathrm{v}}+1:2\left(n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}\right),:} \triangleq \left[0_{n_{g,k-1}^{\mathrm{v}}\times(n+m)} \;\; -\left(G_{k-1}^{\mathrm{v}}\right)^\top \;\; 0_{n_{g,k-1}^{\mathrm{v}}\times\left(n_g+n_g^{\mathrm{w}}\right)} \;\; -\mathrm{I}_{n_{g,k-1}^{\mathrm{v}}} \;\; 0_{n_{g,k-1}^{\mathrm{v}}\times n_{g,k}^{\mathrm{v}}}\right], \tag{17}$$

$$D^{\mathrm{i}}_{2\left(n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}\right)+1:2\left(n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}\right)+n_{g,k}^{\mathrm{v}},:} \triangleq \left[0_{n_{g,k}^{\mathrm{v}}\times n} \;\; \left(G_{k}^{\mathrm{v}}\right)^\top \;\; 0_{n_{g,k}^{\mathrm{v}}\times\left(m+n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}\right)} \;\; -\mathrm{I}_{n_{g,k}^{\mathrm{v}}}\right], \tag{18}$$

$$D^{\mathrm{i}}_{2\left(n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}\right)+n_{g,k}^{\mathrm{v}}+1:2\left(n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}+n_{g,k}^{\mathrm{v}}\right),:} \triangleq \left[0_{n_{g,k}^{\mathrm{v}}\times n} \;\; -\left(G_{k}^{\mathrm{v}}\right)^\top \;\; 0_{n_{g,k}^{\mathrm{v}}\times\left(m+n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}\right)} \;\; -\mathrm{I}_{n_{g,k}^{\mathrm{v}}}\right], \tag{19}$$

$$d^{\mathrm{i}} \triangleq 0_{n^{\mathrm{i}}\times 1}, \tag{20}$$

$$D^{\mathrm{e}} \triangleq \left[E_k^\top \;\; C_k^\top \;\; 0_{n\times\left(m+n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}+n_{g,k}^{\mathrm{v}}\right)}\right], \tag{21}$$

$$d^{\mathrm{e}} \triangleq \left(\mathrm{I}_n\right)_{:,i}. \tag{22}$$

Proof. Since the objective function (6) is composed of piecewise linear convex functions, and the constraints (3) are linear, we here employ the methodology discussed in[7, Example 1.5] to replace each absolute value by a new variable and two additional linear inequality constraints. To achieve that, we first define the variable vector $z \in \mathbb{R}^{n^z}$ whose first $(n+2m)$ elements correspond the ith row of matrices T_k, N_k, and K_{k-1}, respectively, whereas its $\left(n_g + n_g^{\mathrm{w}} + n_{g,k-1}^{\mathrm{v}} + n_{g,k}^{\mathrm{v}}\right)$ remaining elements correspond to the absolute values. Thereby, only the additional variables contribute to $\zeta_{i,k}$, yielding the cost vector ψ in (11).

From (6), we define the following inequality constraints:

$$\left.\begin{aligned} -z_{1:n}^\top F_{:,j} + z_{n+m+1:n+2m}^\top H_{:,j} - z_{n+2m+j} \leq 0 \\ z_{1:n}^\top F_{:,j} - z_{n+m+1:n+2m}^\top H_{:,j} - z_{n+2m+j} \leq 0 \end{aligned}\right\},\; j = 1,\dots,n_g,$$

$$\left.\begin{aligned} -z_{1:n}^\top G^{\mathrm{w}}_{(:,j),k-1} - z_{n+2m+n_g+j} \leq 0 \\ z_{1:n}^\top G^{\mathrm{w}}_{(:,j),k-1} - z_{n+2m+n_g+j} \leq 0 \end{aligned}\right\},\; j = 1,\dots,n_g^{\mathrm{w}},$$

$$\left.\begin{aligned} z_{n+m+1:n+2m}^\top G^{\mathrm{v}}_{(:,j),k-1} - z_{n+2m+n_g+n_g^{\mathrm{w}}+j} \leq 0 \\ -z_{n+m+1:n+2m}^\top G^{\mathrm{v}}_{(:,j),k-1} - z_{n+2m+n_g+n_g^{\mathrm{w}}+j} \leq 0 \end{aligned}\right\},\; j = 1,\dots,n_{g,k-1}^{\mathrm{v}},$$

$$\left.\begin{aligned} z_{n+1:n+m}^\top G^{\mathrm{v}}_{(:,j),k} - z_{n+2m+n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}+j} \leq 0 \\ -z_{n+1:n+m}^\top G^{\mathrm{v}}_{(:,j),k} - z_{n+2m+n_g+n_g^{\mathrm{w}}+n_{g,k-1}^{\mathrm{v}}+j} \leq 0 \end{aligned}\right\},\; j = 1,\dots,n_{g,k}^{\mathrm{v}}.$$

After transposing the prior inequalities, we extract the matrix D^{i} in (12)-(19) and the vector d^{i} in (20). Finally, by transposing the equality constraint $z_{1:n}^\top E_k + z_{n+1:n+m}^\top C_k = \left(\mathrm{I}_n\right)_{i,:}$, we obtain the matrix D^{e} in (21) and the vector d^{e} in (22). ■

3.1 | State-Estimation Algorithm

Next, we present in Algorithm 1 the steps to execute a loop of state estimation. As usual in zonotopic filtering, we add an order-reduction step to fix the number of generators n_g of $\hat{G}^{\mathrm{x}}_{k-1}$ in φ_g. The resulting algorithm is here called *minimum-radius zonotopic filter for descriptor systems* (DRZF).

Algorithm 1 $\hat{\mathcal{X}}_k = \text{DRZF}\left(\hat{\mathcal{X}}_{k-1}, E_k, A_{k-1}, B_{k-1}, C_{k-1}, C_k, \mathcal{W}_{k-1}, \mathcal{V}_{k-1}, \mathcal{V}_k, u_{k-1}, y_{k-1}, y_k, \varphi_g\right)$

Initialization: $\check{T}_k \in \mathbb{R}^{n\times n}$, $\check{N}_k \in \mathbb{R}^{n\times m}$, $\check{K}_{k-1} \in \mathbb{R}^{n\times m}$

1: Apply [1, Algorithm S8] to $\hat{G}^{\mathrm{x}}_{k-1}$ to fix its order in φ_g

2: Define $F = A_{k-1}\hat{G}^{\mathrm{x}}_{k-1}$, $H = C_{k-1}\hat{G}^{\mathrm{x}}_{k-1}$, ψ in (11), D^{i} in (12)-(19), d^{i} in (20), and D^{e} in (21)

3: **For** $i = 1, \dots, n$, **do**

4: Define d^{e} in (22) and solve the LP (7) to obtain the optimal vector $\check{z}$

5: Do $\check{T}_{(i,:),k} = \check{z}^{\top}_{1:n}$, $\check{N}_{(i,:),k} = \check{z}^{\top}_{n+1:n+m}$, and $\check{K}_{(i,:),k-1} = \check{z}^{\top}_{n+m+1:n+2m}$

6: **end**

7: Calculate the generator matrix $\hat{G}^{\mathrm{x}}_k$ in (5) and the center $\hat{x}_k$ in (4) to obtain the estimated zonotope $\hat{\mathcal{X}}_k = \left\{\hat{G}^{\mathrm{x}}_k, \hat{x}_k\right\}$

4 | NUMERICAL RESULTS

In this section, we experiment the DRZF algorithm in two three-state numerical examples. In order to fairly compare it with literature results, we implement two algorithms that use a similar observer structure to DRZF. The first one is the minimum H_∞ norm zonotopic filter for LTI state-space systems proposed in[3], which is here modified with the matrix $\Theta = \begin{bmatrix} E \\ C \end{bmatrix}$ to enable the treatment for descriptor systems; the resulting algorithm is called DH_∞ZF. The second one is the minimum-trace zonotopic filter for LTV descriptor systems proposed in[2], here called DTZF. These algorithms are executed in different examples. For comparison purposes, we compute two performance indexes, namely: (i) the *average area ratio of box* ($A^{\square}$), given by $A^{\square} \triangleq \dfrac{1}{m_s}\dfrac{1}{k_f}\sum_{j=1}^{m_s}\sum_{k=1}^{k_f}\prod_{i=1}^{n} \operatorname{diam}\left([x]_{i,k,j}\right)$, where $k_f = 80$ is the time horizon and $m_s = 1000$ is the number of Monte Carlo simulations; and (ii) the *average largest radius ratio of box* ($r^{\square}$), given by $r^{\square} \triangleq \dfrac{1}{m_s}\dfrac{1}{k_f}\sum_{j=1}^{m_s}\sum_{k=1}^{k_f}\max_i \operatorname{rad}\left(\square\hat{\mathcal{X}}_{k,j}\right)$. The reduction order $\varphi_g = 4$ is set to become the state estimation more challenging.

The following computer configuration was used: 8 GB RAM 1333 MHz, Windows 10 Pro, and AMD FX-6300 CPU 3.50 GHz. All implementations were executed in MATLAB 9.11 with Gurobi 9.1 and MPT3[8].

4.1 | LTI System

Consider an LTI descriptor system[9] of the form (1)-(2), where

$$E = \operatorname{diag}\left(\begin{bmatrix}1 & 1 & 0\end{bmatrix}^{\top}\right),\ A = \begin{bmatrix}0.5 & 0 & 0 \\ 0.8 & 0.95 & 0 \\ -1 & 0.5 & 1\end{bmatrix},\ B = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix},\ C = \begin{bmatrix}1 & 0 & 1 \\ 1 & -1 & 0\end{bmatrix},$$

$G^{\mathrm{w}} = \operatorname{diag}\left(\begin{bmatrix}0.1 & 1.5 & 0.6\end{bmatrix}^{\top}\right)$, and $G^{\mathrm{v}} = \operatorname{diag}\left(\begin{bmatrix}0.5 & 1.5\end{bmatrix}^{\top}\right)$. The elements w_{k-1} and v_k are taken from uniform distributions, while $u_k = \cos(0.15k)\, 1_{2\times 1}$. The simulations are executed with $x_0 = \begin{bmatrix}0.5 & 0.5 & 0.25\end{bmatrix}^{\top} \in \hat{\mathcal{X}}_0 = \left\{\mathrm{I}_3, 0_{3\times 1}\right\}$.

In Figure 1, we illustrate the guaranteed state estimation using the DH_∞ZF and DRZF algorithms in one separate simulation; interval hulls are sketched by computational simplicity. As expected, DRZF returns the smallest sets, mainly during the transient, since the design matrices T_k, N_k, K_{k-1} are computed online and since the minimum-radius criterion aims at reducing $\square\hat{\mathcal{X}}_k$. Then, as shown in Table 1, DRZF returns better precision (smaller values of $A^{\square}$ and $r^{\square}$).

4.2 | LTV System

Consider an LTV descriptor system[2] of the form (1)-(2), where

$$E_k = \begin{bmatrix}1 + 0.2\sin(0.1k) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix},\ A_k = \begin{bmatrix}0.6 - 0.1\sin(0.1k) & 0 & 0.3 \\ -0.2\sin(0.2k) & 0.4 & 0 \\ 0 & 1 & 1\end{bmatrix},\ B_k = \begin{bmatrix}1 \\ \sin(0.1k) \\ 0\end{bmatrix},$$

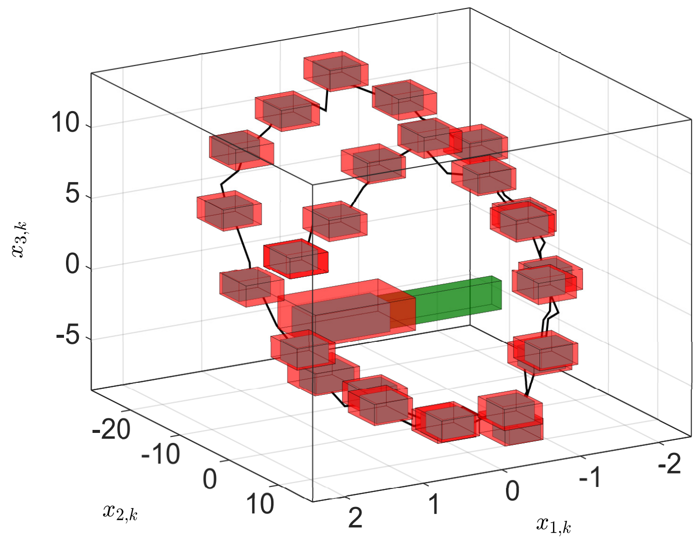

FIGURE 1 Time evolution of states (black —) and interval hulls computed by DH$_\infty$ZF (red box) and DRZF (cyan box). Green box is the common initial box.

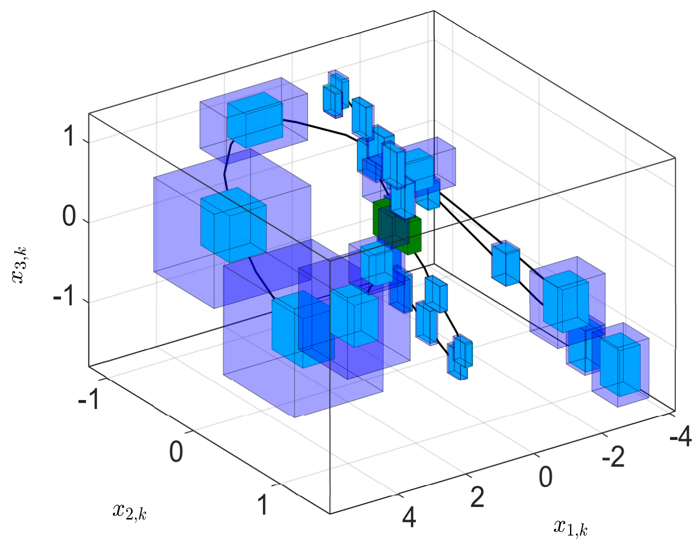

FIGURE 2 Time evolution of states (black —) and interval hulls computed by DTZF (blue box) and DRZF (cyan box). Green box is the common initial box.

$C_k = \begin{bmatrix}0 & 1+0.2\sin(0.2k) & 0.5\end{bmatrix}$, $G^{\mathrm{w}} = 0.04\mathrm{I}_3$ and $G^{\mathrm{v}} = 0.01$. The elements w_{k-1} and v_k are taken from uniform distributions, while $u_k = 0.75\sin(0.15k)$. The simulations are executed with $x_0 = \begin{bmatrix}0.1 & 0 & 0\end{bmatrix}^{\mathsf{T}} \in \hat{\mathcal{X}}_0 = \{0.2\mathrm{I}_3, 0_{3\times 1}\}$.

In Figure 2, we illustrate both the application of DRZF in an LTV case and the increase of precision with respect to DTZF. Both algorithms compute online design matrices, but the trace minimization (DTZF) is not as efficient as the radius minimization (DRZF) to reduce the uncertainty of zonotopes, implying more conservative results as shown in Table 2.

5 | CONCLUSIONS

This paper proposed a new cost criterion to the zonotopic filter from[2], where the minimization of trace is replaced by the minimization of radius. Thereby, we gave a minimum-volume interpretation to the resulting algorithm, called DRZF. The proposed criterion leads to a nonlinear optimization problem, whose piecewise linear convex nature is here exploited to yield an equivalent LP. Over two numerical examples, we illustrate the effectiveness of DRZF in reducing the size of the output sets with respect to the algorithms proposed in [2,3]. Differently from these state estimators, DRZF needs linear solvers, resulting in a slight increase of computational cost.

References

1. de Paula AA, Raffo GV, Teixeira BOS. Zonotopic Filtering for Uncertain Nonlinear Systems: Fundamentals, Implementation Aspects, and Extensions [Applications of Control]. *IEEE Control Systems* 2022; 42(1): 19–51.
2. Tang W, Wang Z, Zhang Q, Shen Y. Set-membership estimation for linear time-varying descriptor systems. *Automatica* 2020; 115: 108867.
3. Tang W, Wang Z, Shen Y. Interval estimation for discrete-time linear systems: A two-step method. *Systems & Control Letters* 2019; 123: 69–74.
4. Tang, Wentao and Wang, Zhenhua and Wang, Ye and Raïssi, Tarek and Shen, Yi. Interval estimation methods for discrete-time linear time-invariant systems. *IEEE Transactions on Automatic Control* 2019; 64(11): 4717–4724.
5. Boyd, Stephen and El Ghaoui, Laurent and Feron, Eric and Balakrishnan, Venkataramanan. *Linear matrix inequalities in system and control theory*. 15. Siam. 1994.
6. Kühn, Wolfgang. Rigorously computed orbits of dynamical systems without the wrapping effect. *Computing* 1998; 61(6): 47–67.
7. Bertsimas, Dimitris and Tsitsiklis, J N. *Introduction to linear optimization*. Athena Scientific, Belmont, Massachusetts. 1997.
8. Herceg M, Kvasnica M, Jones C, Morari M. Multi-parametric toolbox 3.0, in: *Proc. of the European Control Conference*, Zürich, Switzerland, 2013, 502–510, http://control.ee.ethz.ch/~mpt.
9. Rego, Brenner S and Raimondo, Davide M and Raffo, Guilherme V. Set-based state estimation and fault diagnosis of linear discrete-time descriptor systems using constrained zonotopes. *IFAC-PapersOnLine* 2020; 53(2): 4291–4296.

TABLE 1 Results of $A^{\square}$ and $r^{\square}$ for the system of Subsection 4.1.

Algorithms	$A^{\square}$	$r^{\square}$
$DH_{\infty}ZF$	4.27	2.67
DRZF	2.12 (↓**50.4**%)	1.70 (↓**36.3**%)

TABLE 2 Results of $A^{\square}$ and $r^{\square}$ for the system of Subsection 4.2.

Algorithms	$A^{\square}$	$r^{\square}$
DTZF	0.304	0.431
DRZF	0.0453 (↓**85.1**%)	0.257 (↓**40.4**%)