ON GENERALIZED MITTAG-LEFFLER-TYPE FUNCTIONS OF TWO VARIABLES

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Abstract. We aim to study Mittag-Leffler type functions of two variables $D_1(x,y), ..., D_5(x,y)$ by analogy with the Appell hypergeometric functions of two variables,. Moreover, we targeted functions $E_1(x,y), ..., E_{10}(x,y)$ as limiting cases of the functions $D_1(x,y), ..., D_5(x,y)$ and studied certain properties, as well. Following Horn's method, we determine all possible cases of the convergence region of the function $D_1(x,y)$. Further, for a generalized hypergeometric function $D_1(x,y)$ (Mittag-Leffler type function) integral representations of the Euler type are proved. One-dimensional and two-dimensional Laplace transforms of the function are also defined. We have constructed a system of partial differential equations which is linked with the function $D_1(x,y)$.

1. Introduction

The Mittag-Leffler function has gained importance and popularity through its applications [24]. Namely, it appears as a solution of fractional differential equations and integral equations of fractional order. Also, the Mittag-Leffler function plays an important role in various fields of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, computer science, and signal processing[5]. In addition, the Mittag-Leffler function of many variables arises in solving some boundary value problems involving fractional Volterra type integro-differential equations [31], initial-boundary value problems for a generalized polynomial diffusion equation with fractional time [19], and also initial-boundary value problems for time-fractional diffusion equations with positive constant coefficients [17]. The definition of the classical Mittag-Leffler function is the following [24]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \ (\alpha > 0, \ z \in \mathbb{C}). \tag{1}$$

2020 *Mathematics Subject Classification.* 33E12, 33C60, 26A33, 47B38.
Key words and phrases. Mittag-Leffler type function; Euler-type integral representations; system of partial differential equations.
***.

In [40], a function with two parameters was introduced:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ (\alpha,\beta\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0). \quad (2)$$

Prabhakar [26] considered a function $E^{\gamma}_{\alpha,\beta}(z)$ with three parameters

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)\,n!},$$
$$(\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0). \quad (3)$$

The article [35] considers the function

$$E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nq} z^n}{\Gamma(\alpha n+\beta)\,n!},$$
$$(\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,\ q\in(0,1)). \quad (4)$$

In [29], [30], the properties of the following functions were studied

$$E^{\gamma,\delta}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)\,(\delta)_n},$$
$$(\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,\ \delta>0), \quad (5)$$

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nq} z^n}{\Gamma(\alpha n+\beta)\,(\delta)_{np}},\ \alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,$$
$$\mathrm{Re}(\gamma)>0,\ \delta>0,\ (p,q)>0, q\le \mathrm{Re}(\alpha)+p. \quad (6)$$

The following generalized Mittag-Leffler functions were introduced and studied in the article [32], [33]

$$E_{\gamma,K}\left[(\alpha_j,\beta_j)_{1,m};z\right]=\sum_{r=0}^{\infty}\frac{(\gamma)_{rK} z^r}{\prod_{j=1}^{m}\Gamma(\alpha_j r+\beta_j)\,r!},$$
$$\left(z,\gamma,\alpha_j,\beta_j\in\mathbb{C},\ \sum_{j=1}^{m}\mathrm{Re}(\alpha_j)>\mathrm{Re}(K)-1,\ j=1,...,\mathrm{Re}(K)>0\right), \quad (7)$$

$$E^{(\gamma_1,...,\gamma_m)}_{(\rho_1,...,\rho_m),\lambda}(z_1,...,z_m)=\sum_{k_1,...,k_m=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}\cdots(\gamma_k)_{k_m} z_1^{k_1} z_2^{k_2}\cdots z_m^{k_m}}{\Gamma\left[\lambda+\sum_{j=1}^{m}\rho_i k_i\right]k_1!k_2!\cdots k_m!},$$
$$(\lambda,\gamma_j,\rho_j,z_j\in\mathbb{C},\ \mathrm{Re}(\rho_j)>0,\ j=1,...,m). \quad (8)$$

An interesting generalization of the Mittag-Leffler function $E_\alpha(z)$ to several variables was proposed by Luchko Y. and Gorenflo R. [18], who used an operational method to solve a boundary value problem for linear fractional differential equations with constant coefficients. The solution to the boundary value problem is expressed by the functions

$$E_{(\alpha_1,...,\alpha_m),\beta}(z_1,...,z_m)=\sum_{k=0}^{\infty}\sum_{\substack{l_1+...+l_m=k\\ l_1\ge 0,...l_m\ge 0}}^{\infty}\frac{k!}{l_1!\times...\times l_m!}\frac{\prod_{i=1}^{m} z_i^{l_i}}{\Gamma\left(\beta+\sum_{j=1}^{m}\alpha_i l_i\right)}. \quad (9)$$

Srivastava H.M., Daoust Martha C. [37], [38] studied the domain of convergence and Euler-type integral representations for generalized Kampe de Feriet's functions

$$S_{C:D;D'}^{A:B;B'}\begin{pmatrix} x \\ y \end{pmatrix} = S_{C:D;D'}^{A:B;B'}\left(\begin{matrix} [(a):\theta,\phi]:[(b):\psi];[(b'):\psi'] ; \\ [(c):\delta,\varepsilon]:[(d):\eta];[(d'):\eta'] ; \end{matrix} \; x, y \right)$$
$$= \sum_{m,n=0}^{\infty} \frac{\prod\limits_{j=1}^{A}\Gamma\left[a_j+m\theta_j+n\phi_j\right]\prod\limits_{j=1}^{B}\Gamma\left[b_j+m\psi_j\right]\prod\limits_{j=1}^{B'}\Gamma\left[b'_j+n\psi'_j\right]}{\prod\limits_{j=1}^{C}\Gamma\left[c_j+m\delta_j+n\varepsilon_j\right]\prod\limits_{j=1}^{D}\Gamma\left[d_j+m\eta_j\right]\prod\limits_{j=1}^{D'}\Gamma\left[d'_j+n\eta'_j\right]}\frac{x^m}{m!}\frac{y^n}{n!},$$
$$a_j, b_j, b'_j, c_j, d_j, d'_j \in \mathbb{C}, \quad \theta_j, \phi_j, \psi_j, \psi'_j, \ \delta_j, \varepsilon_j, \eta_j, \eta'_j \in \mathbb{R}^+. \tag{10}$$

In [3] the Mittag-Leffler type function E_1 of two variables is introduced and studied, which in a particular case includes several Mittag-Leffler type functions of one variable. All possible cases are determined by the region of convergence. The system of hypergeometric equations is determined, which satisfies the function E_1, Euler type integral representations and the Mellin-Barnes contour integral, as well as the Laplace integral transformation is given

$$E_1\left(\begin{matrix} \gamma_1,\alpha_1;\gamma_2,\beta_1; \\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3; \end{matrix} \left| \begin{matrix} x \\ y \end{matrix} \right. \right) =$$
$$\sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m}(\gamma_2)_{\beta_1 n}}{\Gamma\left(\delta_1+\alpha_2 m+\beta_2 n\right)}\frac{x^m}{\Gamma\left(\delta_2+\alpha_3 m\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_3 n\right)}, \tag{11}$$
$$\gamma_1,\gamma_2,\delta_1,\delta_2,\delta_3,x,y\in\mathbb{C}, \ \ \min\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\}>0,$$

In that paper, another two-variable Mittag-Leffler type function was also introduced, but not studied:

$$E_2\left(\begin{matrix} \gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2; \\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3; \end{matrix} \left| \begin{matrix} x \\ y \end{matrix} \right. \right) =$$
$$\sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m}}{\Gamma\left(\delta_1+\alpha_3 m+\beta_2 n\right)}\frac{x^m}{\Gamma\left(\delta_2+\alpha_4 m\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_3 n\right)}, \tag{12}$$
$$\gamma_1,\gamma_2,\delta_1,\delta_2,\delta_3,x,y\in\mathbb{C}, \ \ \min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3\}>0.$$

Function (9) for $m = 2$ was studied in [21]. In this article, an attempt is made to study the generalized Mittag-Leffler function, and its various properties, including integral and operational relations with other known Mittag-Leffler functions of one variable, differential recurrence relations, Euler transform, Laplace transform, Mellin transform, Whittaker transform, Mellin-integral representation. Barnes and its connection with Wright's hypergeometric function. We also consider properties of the Mittag-Leffler function of two variables related to fractional calculus operators.

The paper [9] considered the equations

$$f(x) = \begin{cases} {}_C D_{0t}^{\alpha} u(x,t) - u_{xx}(x,t), & t > 0, \\ {}_C D_{t0}^{\beta} u(x,t) - u_{xx}(x,t), & t < 0, \end{cases} \tag{13}$$

in a domain $\Omega = \{(x,t) : 0 < x < 1, -p < t < q\}$ where $\alpha, \beta, p, q \in \mathbb{R}^+$, $0 < \alpha < 1$, $1 < \beta < 2$. For equation (13), the boundary value problem is considered and the solution to this problem is expressed by the functions E_1.

A boundary value problem in a domain $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t < T\}$ for the diffusion equation with a fractional time derivative is considered [8]

$$ {}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta} u(t,x) - u_{xx}(t,x) = f(t,x),\ \alpha,\beta,\gamma,\delta \in \mathbb{C},\ \ \mathrm{Re}\alpha > 0, \tag{14}$$

where

$$ {}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta} y(t) = {}^{P}I_{0t}^{\alpha,m-\beta,-\gamma,\delta} \frac{d^m}{dt^m} y(t),\ m - 1 < \mathrm{Re}\beta < m,\ m \in \mathbb{N},$$
$$ {}^{P}I_{0t}^{\alpha,\beta,\gamma,\delta} y(t) = \int_0^t (t-\xi)^{\beta-1} E_{\alpha,\beta}^{\gamma} \left[\delta (t-\xi)^{\alpha}\right] y(\xi)\, d\xi. \tag{15}$$

The solution to the problem is expressed by functions E_2.

We note that special functions are closely related to fractional calculi (see [7], [10], [14], [16], [25], [31]), as well as to generalized fractional calculi (see, for example, [13], [14]). Special functions can be represented as fractional order integration or differentiation operators of some basic elementary special functions. Relations of this kind also provide some alternative definitions for special functions. An example of such unified approaches to special functions can be seen in Kiryakova ([14], ch. 4) and [13]. Many recent works on special functions and their application in solving problems from control theory, mechanics, physics, engineering, economics, etc. can be found in the specialized journal "Fractional Calculus and Applied Analysis" (ISSN 1311-0454), available at the website http://www.math.bas.bg/ fcaa. See also the following works [4], [7], [15], [22], [23], [27], [36], [39].

2. Definitions of some functions of the Mittag-Leffler-type

Having carefully studied the definitions of generalized hypergeometric functions and the Mittag-Leffler type functions, we understand the similarity of these functions with the Horn functions [2]. Given this situation, we define the following functions. Note that the parameters introduced in the functions satisfy the conditions $\gamma_i, \delta_i, x, y \in \mathbb{C}$, and

$\alpha_i, \beta_i \in \mathbb{R}, \quad \min\{\alpha_i, \beta_i\} > 0$:

$$D_1\left(\begin{array}{c|c} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2; \gamma_3, \beta_2; & x \\ \delta_1, \alpha_3, \beta_3; \delta_2, \alpha_4; \delta_3, \beta_4; & y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m}(\gamma_3)_{\beta_2 n}}{\Gamma(\delta_1+\alpha_3 m+\beta_3 n)} \frac{x^m}{\Gamma(\delta_2+\alpha_4 m)} \frac{y^n}{\Gamma(\delta_3+\beta_4 n)}, \tag{16}$$

$$D_2\left(\begin{array}{c|c} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2; \gamma_3, \beta_2; & x \\ \delta_1, \alpha_3; \delta_2, \beta_3; \delta_3, \alpha_4; \delta_4, \beta_4; & y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m}(\gamma_3)_{\beta_2 n}}{\Gamma(\delta_1+\alpha_3 m)\Gamma(\delta_2+\beta_3 n)} \frac{x^m}{\Gamma(\delta_3+\alpha_4 m)} \frac{y^n}{\Gamma(\delta_4+\beta_4 n)}, \tag{17}$$

$$D_3\left(\begin{array}{c|c} \gamma_1, \alpha_1; \gamma_2, \beta_1; \gamma_3, \alpha_2; \gamma_4, \beta_2; & x \\ \delta_1, \alpha_3, \beta_3; \delta_2, \alpha_4; \delta_3, \beta_4; & y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m}(\gamma_2)_{\beta_1 n}(\gamma_3)_{\alpha_2 m}(\gamma_4)_{\beta_2 n}}{\Gamma(\delta_1+\alpha_3 m+\beta_3 n)} \frac{x^m}{\Gamma(\delta_2+\alpha_4 m)} \frac{y^n}{\Gamma(\delta_3+\beta_4 n)}, \tag{18}$$

$$D_4\left(\begin{array}{c|c} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2, \beta_2; & x \\ \delta_1, \alpha_3; \delta_2, \beta_3; \delta_3, \alpha_4; \delta_4, \beta_4; & y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m+\beta_2 n}}{\Gamma(\delta_1+\alpha_3 m)\Gamma(\delta_2+\beta_3 n)} \frac{x^m}{\Gamma(\delta_3+\alpha_4 m)} \frac{y^n}{\Gamma(\delta_4+\beta_4 n)}, \tag{19}$$

$$D_5\left(\begin{array}{c|c} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2, \beta_2; & x \\ \delta_1, \alpha_3, \beta_3; \delta_2, \alpha_4; \delta_3, \beta_4; & y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m+\beta_2 n}}{\Gamma(\delta_1+\alpha_3 m+\beta_3 n)} \frac{x^m}{\Gamma(\delta_2+\alpha_4 m)} \frac{y^n}{\Gamma(\delta_3+\beta_4 n)}, \tag{20}$$

$$E_3\left(\begin{array}{c|c} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2; & x \\ \delta_1, \alpha_3; \delta_2, \beta_2; \delta_3, \alpha_4; \delta_4, \beta_3; & y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1+\alpha_3 m)\Gamma(\delta_2+\beta_2 n)} \frac{x^m}{\Gamma(\delta_3+\alpha_4 m)} \frac{y^n}{\Gamma(\delta_4+\beta_3 n)}, \tag{21}$$

$$E_4\left(\begin{array}{c|c} \gamma_1, \alpha_1, \beta_1; & x \\ \delta_1, \alpha_2; \delta_2, \beta_2; \delta_3, \alpha_3; \delta_4, \beta_3; & y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}}{\Gamma(\delta_1+\alpha_2 m)\Gamma(\delta_2+\beta_2 n)} \frac{x^m}{\Gamma(\delta_3+\alpha_3 m)} \frac{y^n}{\Gamma(\delta_4+\beta_3 n)}, \tag{22}$$

$$E_5\left(\begin{array}{c|c} \gamma_1, \alpha_1; & x \\ \delta_1, \alpha_2, \beta_1; \delta_2, \alpha_3; \delta_3, \beta_2; & y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m}}{\Gamma(\delta_1+\alpha_2 m+\beta_1 n)} \frac{x^m}{\Gamma(\delta_2+\alpha_3 m)} \frac{y^n}{\Gamma(\delta_3+\beta_2 n)}, \tag{23}$$

$$E_6\left(\begin{array}{c}\gamma_1,\alpha_1;\gamma_2,\beta_1;\gamma_3,\alpha_2;\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3;\end{array}\middle|\begin{array}{c}x\\ y\end{array}\right)$$
$$=\sum_{m,n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1 m}(\gamma_2)_{\beta_1 n}(\gamma_3)_{\alpha_2 m}}{\Gamma(\delta_1+\alpha_3 m+\beta_2 n)}\frac{x^m}{\Gamma(\delta_2+\alpha_4 m)}\frac{y^n}{\Gamma(\delta_3+\beta_3 n)}, \tag{24}$$

$$E_7\left(\begin{array}{c}\gamma_1,\alpha_1;\gamma_2,\alpha_2;\\ \delta_1,\alpha_3,\beta_1;\delta_2,\alpha_4;\delta_3,\beta_2;\end{array}\middle|\begin{array}{c}x\\ y\end{array}\right)$$
$$=\sum_{m,n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1 m}(\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1+\alpha_3 m+\beta_1 n)}\frac{x^m}{\Gamma(\delta_2+\alpha_4 m)}\frac{y^n}{\Gamma(\delta_3+\beta_2 n)}, \tag{25}$$

$$E_8\left(\begin{array}{c}\gamma_1,\alpha_1,\beta_1;\\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3;\end{array}\middle|\begin{array}{c}x\\ y\end{array}\right)$$
$$=\sum_{m,n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}}{\Gamma(\delta_1+\alpha_2 m+\beta_2 n)}\frac{x^m}{\Gamma(\delta_2+\alpha_3 m)}\frac{y^n}{\Gamma(\delta_3+\beta_3 n)}, \tag{26}$$

$$E_9\left(\begin{array}{c}-;\\ \delta_1,\alpha_1,\beta_1;\delta_2,\alpha_2;\delta_3,\beta_2;\end{array}\middle|\begin{array}{c}x\\ y\end{array}\right)$$
$$=\sum_{m,n=0}^{\infty}\frac{1}{\Gamma(\delta_1+\alpha_1 m+\beta_1 n)}\frac{x^m}{\Gamma(\delta_2+\alpha_2 m)}\frac{y^n}{\Gamma(\delta_3+\beta_2 n)}, \tag{27}$$

$$E_{10}\left(\begin{array}{c}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\\ \delta_1,\alpha_3;\delta_2,\beta_2;\end{array}\middle|\begin{array}{c}x\\ y\end{array}\right)$$
$$=\sum_{m,n=0}^{\infty}(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m}\frac{x^m}{\Gamma(\delta_1+\alpha_3 m)}\frac{y^n}{\Gamma(\delta_2+\beta_2 n)}. \tag{28}$$

Note that the introduced generalized Mittag-Leffler functions (16) - (28) in particular values of the parameters coincide with the known hypergeometric functions. For example, if in (16) the parameters take the following values $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\beta_1=\beta_2=\beta_3=\beta_4=\delta_2=\delta_3=1$, it coincides with the Appel function, that is $D_1(x,y)=F_1(x,y)$ [2].

3. Determining the region of the convergence of a function D_1

Following Horn [2], we determine the region of convergence of the introduced hypergeometric function D_1. **Definition.** Let us call positive values r, s the associated radii of convergence of the double series

$$\sum_{m,n=0}^{\infty}A(m,n)x^m y^n, \tag{29}$$

if it converges absolutely at $|x|<r$, $|y|<s$ and diverges at $|x|>r$, $|y|>s$.

Let us also assume that the $\max(r)=R$, $\max(s)=S$. Points (r,s) lie on the curve C, which is located entirely in the rectangle $0<r<R$, $0<s<S$. This curve divides the rectangle into two parts; the part

containing the point $r=s=0$ is a two-dimensional image of the region of convergence of the double power series. Studying the convergence of the series (30), Horn introduced the functions

$$\Phi(\mu,\nu)=\lim_{t\to\infty} f(\mu t,\nu t), \quad \Psi(\mu,\nu)=\lim_{t\to\infty} g(\mu t,\nu t), \tag{30}$$

where

$$f(m,n)=\frac{A(m+1,n)}{A(m,n)}, \quad g(m,n)=\frac{A(m,n+1)}{A(m,n)}, \tag{31}$$

and showed that $R=|\Phi(1,0)|^{-1}$, $S=|\Psi(0,1)|^{-1}$ and that C has a parametric representation $r=|\Phi(\mu,\nu)|^{-1},\ s=|\Psi(\mu,\nu)|^{-1},\ \mu,\ \nu>0$. Consider the function (16). It follows from the definition of the function D_1

$$\begin{aligned} f(\mu t,\nu t) &= \frac{\Gamma(\gamma_1+\alpha_1+\alpha_1\mu t+\beta_1\nu t)\,\Gamma(\gamma_2+\alpha_2+\alpha_2\mu t)}{\Gamma(\gamma_1+\alpha_1\mu t+\beta_1\nu t)\,\Gamma(\gamma_2+\alpha_2\mu t)} \\ &\times\frac{\Gamma(\delta_1+\alpha_3\mu t+\beta_3\nu t)\,\Gamma(\delta_2+\alpha_4\mu t)}{\Gamma(\delta_1+\alpha_3+\alpha_3\mu t+\beta_3\nu t)\,\Gamma(\delta_2+\alpha_4+\alpha_4\mu t)}, \\ g(\mu t,\nu t) &= \frac{\Gamma(\gamma_1+\beta_1+\alpha_1\mu t+\beta_1\nu t)\,\Gamma(\gamma_3+\beta_2+\beta_2\nu t)}{\Gamma(\gamma_1+\alpha_1\mu t+\beta_1\nu t)\,\Gamma(\gamma_3+\beta_2\nu t)} \\ &\times\frac{\Gamma(\delta_1+\alpha_3\mu t+\beta_3\nu t)\,\Gamma(\delta_3+\beta_4\nu t)}{\Gamma(\delta_1+\beta_3+\alpha_3\mu t+\beta_3\nu t)\,\Gamma(\delta_3+\beta_4+\beta_4\nu t)}. \end{aligned} \tag{32}$$

Due to the asymptotics of the Gamma function for large arguments [20]

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\sim z^{\alpha-\beta}\left[1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+O\left(z^{-2}\right)\right], \ |\arg(z)|\le\pi,$$

we have

$$\begin{aligned} & f(\mu t,\nu t)\sim\frac{1}{E}t^{-\Delta}, \ \ \Delta=\alpha_3+\alpha_4-\alpha_1-\alpha_2, \\ & E=\mu^{\alpha_4-\alpha_2}\frac{(\alpha_3\mu+\beta_3\nu)^{\alpha_3}(\alpha_4)^{\alpha_4}}{(\alpha_1\mu+\beta_1\nu)^{\alpha_1}(\alpha_2)^{\alpha_2}}, \quad G=\frac{(\alpha_3)^{\alpha_3}(\alpha_4)^{\alpha_4}}{(\alpha_1)^{\alpha_1}(\alpha_2)^{\alpha_2}}, \end{aligned} \tag{33}$$

Similarly, we define

$$\begin{aligned} & g(\mu t,\nu t)\sim\frac{1}{E'}\cdot t^{-\Delta'}, \ \ \Delta'=\beta_3+\beta_4-\beta_1-\beta_2, \\ & E'=\nu^{\beta_4-\beta_2}\frac{(\alpha_3\mu+\beta_3\nu)^{\beta_3}(\beta_4)^{\beta_4}}{(\alpha_1\mu+\beta_1\nu)^{\beta_1}(\beta_2)^{\beta_2}}, \ G'=\frac{(\beta_3)^{\beta_3}(\beta_4)^{\beta_4}}{(\beta_1)^{\beta_1}(\beta_2)^{\beta_2}}. \end{aligned} \tag{34}$$

Now consider some cases:
Case 1. Let $\Delta>0,\ \ \Delta'>0$. Then from (33) and (34) it follows

$$\Phi(\mu,\nu)=\lim_{t\to\infty} f(\mu t,\nu t)=0, \ \Psi(\mu,\nu)=\lim_{t\to\infty} g(\mu t,\nu t)=0. \tag{35}$$

Positive numbers r and s large numbers. The series converges for any value of the argument.

Case 2. Let $\Delta = 0$, $\Delta' = 0$. Then from (33) and (34) it follows

$$\Phi(\mu,\nu) = \frac{1}{E}, \qquad \Psi(\mu,\nu) = \frac{1}{E'}, \tag{36}$$

which immediately led us to the parametric representation of the curve C on the plane (r,s) in the form $r = G$, $s = G'$. Therefore, the series converges absolutely for the values of $|x| < \rho$ and $|y| < \rho'$ where $\rho = \min\limits_{\mu,\nu>0}(E)$, $\rho' = \min\limits_{\mu,\nu>0}(E')$.

Case 3. Let $\Delta < 0$, $\Delta' < 0$. The series diverges $r = s = 0$. The series converges only at the point $x = y = 0$.

Case 4. Let $\Delta = 0$, $\Delta' > 0$. Then the series converges absolutely in the region $|x| < \rho$ and $|y| < \infty$.

Case 5. Let $\Delta > 0$, $\Delta' = 0$. Then the series converges absolutely in the region $|x| < \infty$ and $|y| < \rho'$.

4. Integral representations of a function D_1

For a generalized hypergeometric function of the Mittag-Leffler type D_1, following integral representations of the Euler type are valid

$$D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}x\\ y\end{matrix}\right.\right) = \frac{\Gamma(\mu)}{\Gamma(\gamma_2)\Gamma(\mu-\gamma_2)}$$
$$\times\int\limits_0^1 \xi^{\gamma_2-1}(1-\xi)^{\mu-\gamma_2-1} D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\mu,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}x\xi^{\alpha_2}\\ y\end{matrix}\right.\right)d\xi, \tag{37}$$
$$\operatorname{Re}\mu > \operatorname{Re}\gamma_2 > 0,$$

$$D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}x\\ y\end{matrix}\right.\right) = \frac{\Gamma(\mu)}{\Gamma(\gamma_3)\Gamma(\mu-\gamma_3)}$$
$$\times\int\limits_0^1 \xi^{\gamma_3-1}(1-\xi)^{\mu-\gamma_3-1} D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\mu,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}x\\ y\xi^{\beta_2}\end{matrix}\right.\right)d\xi, \tag{38}$$
$$\operatorname{Re}\mu > \operatorname{Re}\gamma_3 > 0,$$

$$D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}x\\ y\end{matrix}\right.\right) = \frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\gamma_2)\Gamma(\gamma_3)\Gamma(\mu_1-\gamma_2)\Gamma(\mu_2-\gamma_3)}$$
$$\times\int\limits_0^1\int\limits_0^1 \xi^{\gamma_2-1}\eta^{\gamma_3-1}(1-\xi)^{\mu_1-\gamma_2-1}(1-\eta)^{\mu_2-\gamma_3-1}$$
$$\times D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\mu_1,\alpha_2;\mu_2,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}x\xi^{\alpha_2}\\ y\eta^{\beta_2}\end{matrix}\right.\right)d\xi d\eta, \operatorname{Re}\mu_1 > \operatorname{Re}\gamma_2 > 0,\ \operatorname{Re}\mu_2 > \operatorname{Re}\gamma_3 > 0, \tag{39}$$

$$\int\limits_0^1\int\limits_0^1 \xi^{\delta_2-1}\eta^{\delta_3-1}(1-\xi)^{\sigma_1-1}(1-\eta)^{\sigma_2-1} D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}x\xi^{\alpha_4}\\ y\eta^{\beta_4}\end{matrix}\right.\right)d\xi d\eta =$$
$$= \Gamma(\sigma_1)\Gamma(\sigma_2) D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2+\sigma_1,\alpha_4;\delta_3+\sigma_2,\beta_4;\end{matrix}\left|\begin{matrix}x\\ y\end{matrix}\right.\right),\ \operatorname{Re}\sigma_1 > 0,\ \operatorname{Re}\sigma_2 > 0, \tag{40}$$

$$D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=\frac{\Gamma(\mu)}{\Gamma(\gamma_1)\Gamma(\mu-\gamma_1)}$$
$$\times\int\limits_0^1\xi^{\gamma_1-1}(1-\xi)^{\mu-\gamma_1-1}D_1\left(\begin{matrix}\mu,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\xi^{\alpha_1}\\ y\xi^{\beta_1}\end{matrix}\right)d\xi,\ \operatorname{Re}\mu>\operatorname{Re}\gamma_1>0, \tag{41}$$

$$D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \mu_1+\mu_2,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)$$
$$=\int\limits_0^1\xi^{\mu_1-1}(1-\xi)^{\mu_2-1}D_2\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \mu_1,\alpha_3;\mu_2,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\xi^{\alpha_3}\\ y(1-\xi)^{\beta_3}\end{matrix}\right)d\xi,$$
$$\operatorname{Re}\mu_1>0,\operatorname{Re}\mu_2>0, \tag{42}$$

$$D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_1,\beta_1;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=\frac{1}{\Gamma(\gamma_1)\Gamma(\delta_1-\gamma_1)}$$
$$\times\int\limits_0^1\xi^{\gamma_1-1}(1-\xi)^{\delta_1-\gamma_1-1}E\left(\begin{matrix}\gamma_2,\alpha_2;\\ \delta_2,\alpha_4;\end{matrix}x\xi^{\alpha_1}\right)E\left(\begin{matrix}\gamma_3,\beta_2;\\ \delta_3,\beta_4;\end{matrix}y\xi^{\beta_1}\right)d\xi,$$
$$\operatorname{Re}\delta_1>\operatorname{Re}\gamma_1>0, \tag{43}$$

$$D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=\frac{\Gamma(\gamma_2+\gamma_3)}{\Gamma(\gamma_2)\Gamma(\gamma_3)}$$
$$\times\int\limits_0^1\xi^{\gamma_2-1}(1-\xi)^{\gamma_3-1}D_5\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2+\gamma_3,\alpha_2,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\xi^{\alpha_2}\\ y(1-\xi)^{\beta_2}\end{matrix}\right)d\xi$$
$$\operatorname{Re}\gamma_2>0,\ \operatorname{Re}\gamma_3>0, \tag{44}$$

$$D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=\frac{1}{\Gamma(\gamma_3)}\int\limits_0^1\xi^{\gamma_3-1}(1-\xi)^{\delta_1-\gamma_3-1}$$
$$\times E_2\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\\ \delta_1-\gamma_3,\alpha_3,\beta_3-\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3;\end{matrix}\middle|\begin{matrix}x(1-\xi)^{\alpha_3}\\ y\xi^{\beta_2}(1-\xi)^{\beta_3-\beta_2}\end{matrix}\right)d\xi,$$
$$\operatorname{Re}\delta_1>\operatorname{Re}\gamma_3>0, \tag{45}$$

$$D_1\left(\begin{matrix}\gamma_1,\alpha_3-\alpha_2,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_1;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=\frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\delta_1-\gamma_1-\gamma_2)}$$
$$\times\int\limits_0^1\int\limits_0^1\xi^{\gamma_2-1}\eta^{\gamma_1-1}(1-\xi)^{\delta_1-\gamma_2-1}(1-\eta)^{\delta_1-\gamma_1-\gamma_2-1}$$
$$\times E_{\delta_2,\alpha_4}^{\gamma_3,\beta_2}\left(x\xi^{\alpha_2}[(1-\xi)\eta]^{(\alpha_3-\alpha_2)}\right)E_{\delta_3,\beta_4}\left(y(1-\xi)^{\beta_1}\eta^{\beta_1}\right)d\xi d\eta,$$
$$\operatorname{Re}\gamma_1>0,\ \operatorname{Re}\gamma_2>0,\ \operatorname{Re}(\delta_1-\gamma_1-\gamma_2)>0,\ \ \alpha_3-\alpha_2>0. \tag{46}$$

Let us prove the validity of the integral representation (46). We expand the integrands on the right-hand side into a series, then we have

$$I\left(x,y\right)=\frac{1}{\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\Gamma\left(\delta_1-\gamma_1-\gamma_2\right)}\times$$
$$\int\limits_0^1\int\limits_0^1 \xi^{\gamma_2-1}\eta^{\gamma_1-1}(1-\xi)^{\delta_1-\gamma_2-1}(1-\eta)^{\delta_1-\gamma_1-\gamma_2-1}\times$$
$$\sum_{m=0}^{\infty}\frac{(\gamma_3)_{\beta_2 n}\left(x\xi^{\alpha_2}\left[(1-\xi)\,\eta\right]^{(\alpha_3-\alpha_2)}\right)^m}{\Gamma\left(\delta_2+\alpha_4 m\right)}\sum_{n=0}^{\infty}\frac{\left(y(1-\xi)^{\beta_1}\eta^{\beta_1}\right)^n}{\Gamma\left(\delta_3+\beta_4 n\right)}d\xi d\eta$$
$$=\frac{1}{\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\Gamma\left(\delta_1-\gamma_1-\gamma_2\right)}\sum_{m,n=0}^{\infty}\frac{(\gamma_3)_{\beta_2 n}}{\Gamma\left(\delta_2+\alpha_4 m\right)\Gamma\left(\delta_3+\beta_4 n\right)}x^m y^n$$
$$\times\int\limits_0^1 \xi^{\gamma_2+\alpha_2 m-1}(1-\xi)^{\delta_1-\gamma_2+(\alpha_3-\alpha_2)m+\beta_1 n-1}d\xi$$
$$\times\int\limits_0^1 \eta^{\gamma_1+(\alpha_3-\alpha_2)m+\beta_1 n-1}(1-\eta)^{\delta_1-\gamma_1-\gamma_2-1}d\eta=$$
$$=\frac{1}{\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\Gamma\left(\delta_1-\gamma_1-\gamma_2\right)}\sum_{m,n=0}^{\infty}\frac{(\gamma_3)_{\beta_2 n}}{\Gamma\left(\delta_2+\alpha_4 m\right)\Gamma\left(\delta_3+\beta_4 n\right)}x^m y^n$$
$$\times B\left(\gamma_2+\alpha_2 m,\delta_1-\gamma_2+\left(\alpha_3-\alpha_2\right)m+\beta_1 n\right)\times$$
$$B\left(\gamma_1+\left(\alpha_3-\alpha_2\right)m+\beta_1 n,\delta_1-\gamma_1-\gamma_2\right),$$

or

$$I\left(x,y\right)=\frac{1}{\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\Gamma\left(\delta_1-\gamma_1-\gamma_2\right)}\sum_{m,n=0}^{\infty}\frac{(\gamma_3)_{\beta_2 n}}{\Gamma\left(\delta_2+\alpha_4 m\right)\Gamma\left(\delta_3+\beta_4 n\right)}x^m y^n$$
$$\times\frac{\Gamma\left(\gamma_2+\alpha_2 m\right)\Gamma\left(\delta_1-\gamma_2+\left(\alpha_3-\alpha_2\right)m+\beta_1 n\right)}{\Gamma\left(\delta_1+\alpha_3 m+\beta_3 n\right)}$$
$$\times\frac{\Gamma\left(\gamma_1+\left(\alpha_3-\alpha_2\right)m+\beta_1 n\right)\Gamma\left(\delta_1-\gamma_1-\gamma_2\right)}{\Gamma\left(\delta_1-\gamma_2+\left(\alpha_3-\alpha_2\right)m+\beta_1 n\right)}=$$
$$=\frac{1}{\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)}\sum_{m,n=0}^{\infty}\frac{\Gamma\left(\gamma_1+\left(\alpha_3-\alpha_2\right)m+\beta_1 n\right)\Gamma\left(\gamma_2+\alpha_2 m\right)(\gamma_3)_{\beta_2 n}}{\Gamma\left(\delta_1+\alpha_3 m+\beta_3 n\right)}$$
$$\times\frac{x^m}{\Gamma\left(\delta_2+\alpha_4 m\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_4 n\right)}=D_1\left(\begin{matrix}\gamma_1,\alpha_3-\alpha_2,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_1;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right).$$

The integral representation (46) is proved.

5. Integral Laplace transforms

Let L_1 and L_2 denote the one-dimensional and two-dimensional Laplace transforms:

$$L_1\left\{f\left(t\right);p\right\}=\int\limits_0^{\infty}f\left(t\right)e^{-pt}dt,\quad \mathrm{Re}\,p>0,\tag{47}$$

$$L_2\left\{f\left(t_1,t_2\right);p,q\right\}=\int\limits_0^\infty\int\limits_0^\infty f\left(t_1,t_2\right)e^{-t_1p-t_2q}dt_1dt_2,\ \ \operatorname{Re}p>0,\ \ \operatorname{Re}q>0. \tag{48}$$

The following Laplace transformations are valid

$$\begin{gathered} L_1\left\{t^{\delta_1-1}D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3,\beta_3;\delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}xt^{\alpha_3}\\ yt^{\beta_3}\end{matrix}\right.\right):p\right\}\\ =\frac{1}{p^{\delta_1}}E_{11}\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_2,\alpha_4;\delta_3,\beta_4;\end{matrix}\left|\begin{matrix}\dfrac{x}{p^{\alpha_3}}\\ \dfrac{y}{p^{\beta_3}}\end{matrix}\right.\right),\operatorname{Re}p>0, \end{gathered} \tag{49}$$

$$\begin{gathered} L_1\left\{t^{\rho-1}E_1\left(\begin{matrix}\gamma_1,\alpha_1;\gamma_2,\beta_1;\\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3;\end{matrix}\left|\begin{matrix}xt^{\mu_1}\\ yt^{\mu_2}\end{matrix}\right.\right);p\right\}\\ =\frac{\Gamma\left(\rho\right)}{p^{\rho}}D_1\left(\begin{matrix}\rho,\mu_1,\mu_2;\gamma_1,\alpha_1;\gamma_2,\beta_1;\\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3;\end{matrix}\left|\begin{matrix}\dfrac{x}{p^{\mu_1}}\\ \dfrac{y}{p^{\mu_2}}\end{matrix}\right.\right),\\ \operatorname{Re}\rho>0,\ \operatorname{Re}\mu_1>0,\ \operatorname{Re}\mu_2>0, \end{gathered} \tag{50}$$

$$\begin{gathered} L_2\left\{t_1^{\rho_1-1}t_2^{\rho_2-1}E_8\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3;\end{matrix}\left|\begin{matrix}xt_1^{\mu_1}\\ yt_2^{\mu_2}\end{matrix}\right.\right);p,q\right\}=\\ =\frac{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}{p^{\rho_1}q^{\rho_2}}D_1\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\rho_1,\mu_1;\rho_2,\mu_2;\\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3;\end{matrix}\left|\begin{matrix}x\\ y\end{matrix}\right.\right),\ \ \operatorname{Re}\rho_1>0,\ \ \operatorname{Re}\rho_2>0, \end{gathered} \tag{51}$$

$$\begin{gathered} L_2\left\{x^{\delta_1-1}y^{\delta_2-1}D_2\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3;\delta_2,\beta_3;\delta_3,\alpha_4;\delta_4,\beta_4;\end{matrix}\left|\begin{matrix}x^{\alpha_3}\\ y^{\beta_3}\end{matrix}\right.\right);p,q\right\}\\ =\frac{1}{p^{\delta_1}q^{\delta_2}}E_{11}\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_3,\alpha_4;\delta_4,\beta_4;\end{matrix}\left|\begin{matrix}\dfrac{1}{p^{\alpha_3}}\\ \dfrac{1}{q^{\beta_3}}\end{matrix}\right.\right),\operatorname{Re}p>0,\ \ \operatorname{Re}q>0, \end{gathered} \tag{52}$$

$$\begin{gathered} E_{11}\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\gamma_3,\beta_2;\\ \delta_1,\alpha_3;\delta_2,\beta_3;\end{matrix}\left|\begin{matrix}x\\ y\end{matrix}\right.\right)\\ =\sum_{m,n=0}^{\infty}(\gamma_1)_{\alpha_1m+\beta_1n}(\gamma_2)_{\alpha_2m}(\gamma_3)_{\beta_2n}\frac{x^m}{\Gamma\left(\delta_1+\alpha_3m\right)}\frac{y^n}{\Gamma\left(\delta_2+\beta_3n\right)}. \end{gathered} \tag{53}$$

Equalities (49)-(53) are verified by direct calculations.

6. System of partial differential equations

Theorem. Let $\theta\equiv x\left(\partial/\partial x\right)$, $\phi\equiv y\left(\partial/\partial y\right)$, then for $\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3,x,y\in\mathbb{C}$, $\alpha_j,\beta_j\in\mathbb{N}$, $(j=1,2,3,4)$ the Mittag-Leffler-type function of two variables D_1, defined by (16), satisfies the following partial differential equations:

$$\prod_{i=1}^{\alpha_3}\left(\delta_1+\alpha_3-i+\alpha_3\theta+\beta_3\ \phi\right)\prod_{i=1}^{\alpha_4}\left(\delta_2+\alpha_4-i+\alpha_4\theta\right)x^{-1}$$
$$-\prod_{i=1}^{\alpha_1}\left(\gamma_1+\alpha_1-i+\alpha_1\theta+\beta_1\phi\right)\prod_{i=1}^{\alpha_2}\left(\gamma_2+\alpha_2-i+\alpha_2\theta\right)D_1(x,y)=0,$$
$$\prod_{i=1}^{\beta_3}\left(\delta_1+\beta_3-i+\alpha_3\theta+\beta_3\ \phi\right)\prod_{i=1}^{\beta_4}\left(\delta_2+\beta_4-i+\beta_4\ \phi\right)y^{-1}$$
$$-\prod_{i=1}^{\beta_1}\left(\gamma_1+\beta_1-i+\alpha_1\theta+\beta_1\ \phi\right)\prod_{i=1}^{\beta_2}\left(\gamma_2+\beta_2-i+\beta_2\ \phi\right)D_1(x,y)=0. \tag{54}$$

Proof. Let us show the validity of the first equation in (54). The right side of the first equation (54) will be denoted by $J(x,y)$. Substituting the function D_1 into the first equation (54) and taking into account the equalities $\theta\left(x^m\right)=mx^m,\ \phi\left(y^n\right)=ny^n$, we obtain

$$\prod_{i=1}^{\alpha_4}\left(\delta_2+\alpha_4-i+\alpha_4\theta\right)x^{-1}D_1(x,y)$$
$$=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1m+\beta_1n}(\gamma_2)_{\alpha_2m}(\gamma_3)_{\beta_2n}}{\Gamma\left(\delta_1+\alpha_3m+\beta_3n\right)}\frac{x^{m-1}}{\Gamma\left(\delta_2+\alpha_4\left(m-1\right)\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_4n\right)},$$
$$\prod_{i=1}^{\alpha_2}\left(\gamma_2+\alpha_2-i+\alpha_2\theta\right)D_1(x,y)$$
$$=\sum_{m,n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1m+\beta_1n}(\gamma_2)_{\alpha_2(m+1)}(\gamma_3)_{\beta_2n}}{\Gamma\left(\delta_1+\alpha_3m+\beta_3n\right)}\frac{x^{m}}{\Gamma\left(\delta_2+\alpha_4m\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_4n\right)}.$$

Hence we get

$$\prod_{i=1}^{\alpha_3}\left(\delta_1+\alpha_3-i+\alpha_3\theta+\beta_3\ \phi\right)\prod_{i=1}^{\alpha_4}\left(\delta_2+\alpha_4-i+\alpha_4\theta\right)x^{-1}D_1(x,y)$$
$$=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1m+\beta_1n}(\gamma_2)_{\alpha_2m}(\gamma_3)_{\beta_2n}}{\Gamma\left(\delta_1+\alpha_3\left(m-1\right)+\beta_3n\right)}\frac{x^{m-1}}{\Gamma\left(\delta_2+\alpha_4\left(m-1\right)\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_4n\right)}, \tag{55}$$

$$\prod_{i=1}^{\alpha_1}\left(\gamma_1+\alpha_1-i+\alpha_1\theta+\beta_1\phi\right)\prod_{i=1}^{\alpha_2}\left(\gamma_2+\alpha_2-i+\alpha_2\theta\right)D_1(x,y)$$
$$=\sum_{m,n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1(m+1)+\beta_1n}(\gamma_2)_{\alpha_2(m+1)}(\gamma_3)_{\beta_2n}}{\Gamma\left(\delta_1+\alpha_3m+\beta_3n\right)}\frac{x^{m}}{\Gamma\left(\delta_2+\alpha_4m\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_4n\right)}. \tag{56}$$

Substituting (55) - (56) into the first equation of system (54), we determine

$$J(x,y)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1m+\beta_1n}(\gamma_2)_{\alpha_2m}(\gamma_3)_{\beta_2n}}{\Gamma\left(\delta_1+\alpha_3\left(m-1\right)+\beta_3n\right)}\frac{x^{m-1}}{\Gamma\left(\delta_2+\alpha_4\left(m-1\right)\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_4n\right)}-$$
$$-\sum_{m,n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1(m+1)+\beta_1n}(\gamma_2)_{\alpha_2(m+1)}(\gamma_3)_{\beta_2n}}{\Gamma\left(\delta_1+\alpha_3m+\beta_3n\right)}\frac{x^{m}}{\Gamma\left(\delta_2+\alpha_4m\right)}\frac{y^n}{\Gamma\left(\delta_3+\beta_4n\right)}.$$

In the first term, changing the summation index m to $m+1$, we determine that $J(x,y)=0$. The validity of the second equation of (54) is proved in a similar way. $\square$

7. Conclusion

We have introduced the series of two-variable Mittag-Leffler-type functions and studied certain properties of these functions. Namely, we have determined the region of the convergence, Euler-type integral representation, and one and two-dimensional Laplace transform and determined the system of partial differential equations linked with these functions. We note that we did these for function $D_1(x,y)$, but results can be easily obtained for other introduced functions. We believe that obtained result will be applied in near future since similar functions $E_1(x,y)$ and $E_2(x,y)$ already found their applications in [8],[9].

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