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Exact delay range for the stabilization of linear systems with input delays[†]

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Summary

This paper is concerned with the exact delay range making input-delay systems unstabilizable. The exact range means that the systems are unstabilizable if and only if the delay is within this range. Contributions of this paper are to characterize the exact range and to present a computation method to derive this range. It is shown that the above range is related to unstable eigenvalues of the system matrix. In the discrete-time case, if none of the eigenvalues of the system matrix is a unit root, then the above range is a finite set. If there exist some eigenvalues which are unit roots, this range may be a finite set or may be composed of several arithmetic progressions. When this range contains finite elements, the number of these elements is bounded by the geometric multiplicities of eigenvalues. When this range contains arithmetic progressions, the number of such progressions is bounded by the above multiplicities. On the other hand, our results can provide an upper bound for the well-known delay margin, which is the maximal delay value achievable by a robust controller to stabilize systems.

KEYWORDS:
Input-delay systems, stabilization, exact delay range, delay effects

1 | INTRODUCTION

Delay phenomenon exists widely in many areas, such as neuroscience[1], transportation[2], medicine[3], and communication[4]. Its appearance brings essential influences to practical processes and we have to study delay systems to deal with real situations. Various kinds of problems in control theory have been considered for delay systems. These problems contain stability[5], stabilization[6], optimal control[7], consensus control[8], formation control[9], estimation[10] and so on. On stability and stabilization, aims of the existing literature include establishment of stability and stabilization criteria and designing feedback control to stabilize systems. For examples, stability of linear systems with incommensurate delays was concerned in 11, which proposed necessary stability conditions in terms of the Lyapunov matrix; nonlinear systems with a large input delay were considered by 12 and memoryless state and/or output feedback control was presented to achieve local asymptotic stabilization; stochastic systems with input delays were studied in 13 and a necessary and sufficient stabilization condition was established via Riccati-type equations.

Different from the above-mentioned problems, delay effects or robustness problems concern influences of delay on stability or stabilization. The aim of these problems is to determine the range of delay within which systems are stable or stabilizable. Existing literature can be divided into three classes on the basis of specific problems.

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- Class I studies delay systems without control and focuses on the delay range within which the systems are stable [14,15,16,17]. For example, consider

$$\dot{x}(t) = Ax(t) + A_1 x(t-h), \tag{1}$$

where A and A_1 are constant matrices and $\tau \in [0, +\infty)$ is a delay. Suppose (1) is stable at $\tau = 0$, then it will remain stable in a neighbourhood $h \in [0, h_1)$. 17 discussed the delay margin, which is the maximal h_1.

- Class II investigates systems with control and is concerned with the maximal range of delay values within which the system can be robustly stabilized by a single controller [18,19,20,21,22]. In this class, most of the literature use frequency domain method. To compare with the other two classes, we translate the frequency-domain framework, which has been discussed in 23, to time-domain framework and replace infinite-dimensional controllers by static feedback controllers as follows:

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{2}$$

$$y(t) = Cx(t), \tag{3}$$

$$u(t) = -Ky(t). \tag{4}$$

A controller $u(t) = -Ky(t)$ is robust if it can stabilize (2)-(3) for every $h \in [0, h_1]$. The delay margin of a robust controller is

$$DM(K) \triangleq \sup\{h_1 : u(t) = -Ky(t) \text{ stabilize (2)-(3) for every } h \in [0, h_1]\}, \tag{5}$$

and the achievable delay margin for (2)-(3) is

$$DM \triangleq \sup\{DM(K) : u(t) = -Ky(t) \text{ is a robust controller}\}. \tag{6}$$

- Class III is concerned with systems with input delay and concentrates on the delay range within which systems are stabilizable [24,25]. This paper belongs to this class. In the continuous-time case, we consider

$$\dot{x}(t) = Ax(t) + B_0 u(t) + B_1 u(t-h), t \geq 0, \tag{7}$$

where $x(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^q$ are the state and control input, respectively, $x(0) \in \mathbb{R}^p$ and $u(\theta), \theta \in [-h, 0)$, are initial values, and $A \in \mathbb{R}^{p\times p}$, $B_0 \in \mathbb{R}^{p\times q}$ and $B_1 \in \mathbb{R}^{p\times q}$, are constant matrices. $CTS(h)$ is used to represent the above system. Simply speaking, if there exists a control, which stabilizes $CTS(h)$, it is said that $CTS(h)$ is stabilizable (Detailed definition of stabilization will be given in Section 3). This paper focuses on the following delay range

$$\{h \in [0, +\infty) : CTS(h) \text{ is stabilizable}\}. \tag{8}$$

Relations among the three classes are as follows.

(1) The delay margin of a robust controller $DM(K)$, i.e., (5), in class II is essentially the same as the delay margin in class I. This is because the closed-loop system consisting of (2)-(4) is

$$\dot{x}(t) = Ax(t) - BKCx(t-h),$$

which is a type of system (1).

(2) Both class II and class III consider stabilization problems. The main difference is as follows. Class II emphasizes the robustness of controllers with respect to the delay. Class III does not concern such robustness but emphasizes whether or not systems are stabilizable when delay varies.

(3) The exact delay range studied in class III provides an upper bound for the delay margin discussed in class II. For example, this paper shows that the complement set of (8) is a countable set and proposes a computation method to derive this set. Suppose this set is derived as $\{\tau_1, \cdots, \tau_n, \cdots\}$ where $\tau_1 < \cdots < \tau_n < \cdots$. This means that (7) is unstabilizable if and only if $h \in \{\tau_1, \cdots, \tau_n, \cdots\}$. If a robust controller $u(t) = Kx(t)$ stabilizes (7) for $h \in [0, h_1]$, it can be known that $h_1 < \tau_1$. Naturally it holds that $DM(K) \leq \tau_1$ and $DM \leq \tau_1$.

In summary, problems under consideration in this paper are different from the usual delay margin problems and they can provide upper bounds for the usual delay margin.

Our contribution is summarized as follows. In the discrete-time case in which delays are nonnegative integers, this paper shows that the unstable eigenvalues of the system matrix determine the exact delay range within which the systems are unstabilizable to some extent. It proves that this range is finite or is composed of several arithmetic progressions. It also proposes a method to derive the above range via computation of matrix minors and roots of polynomials. In the continuous-time case in which delays belong to the interval $[0, +\infty)$, this paper shows that if all the eigenvalues of the system matrix lie on the open right half plane, then the exact delay range within which the systems are unstabilizable is a finite set, and otherwise, this range is a finite set or a countable set. Similar to the discrete-time case, a computation method is presented to derive this range.

Notation: $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ are the sets of integers, nonnegative integers, real numbers, and complex numbers, respectively. $\mathbb{R}^n(\mathbb{C}^n)$ is the set of $n-$dimensional column vectors with elements in $\mathbb{R}(\mathbb{C})$. $\mathbb{R}^{n\times m}(\mathbb{C}^{n\times m})$ is the set of $n \times m$ matrices with elements in $\mathbb{R}(\mathbb{C})$. For $z \in \mathbb{C}$, $\overline{z}$, $|z|$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the conjugate, the molule, the real part and the imaginary part of z, respectively. $\sqrt{-1}$ denotes the imaginary unit. $\mathbb{U}$ is the set of unit roots, i.e., $\mathbb{U} \triangleq \{\beta \in \mathbb{C}$: there exists a positive integer k, such that $\beta^k = 1\}$. For any $\beta \in \mathbb{U}$, define $o(\beta)$ to be the smallest positive integer k such that $\beta^k = 1$ and call $o(\beta)$ the order of β. For a vector $\xi = \begin{pmatrix} \xi_1 & \cdots & \xi_n \end{pmatrix}' \in \mathbb{C}^n$, $\|\xi\|$ denotes its 2-norm, i.e., $\|\xi\| = \sqrt{\sum_{i=1}^{n} |\xi_i|^2}$. For a matrix $X = [X_{ij}] \in \mathbb{C}^{n\times m}$, $\mathrm{rank}(X)$, X', and $(X)_i$ stand for the rank of X, the transpose of X, and the i-th row vector of X, respectively. $\overline{X}$ represents the matrix $[\overline{X_{ij}}]_{n\times m}$. For a square matrix X, $\rho(X)$ denotes its spectral radius. I stands for a unit matrix with suitable dimension. L^2_{loc} is the set of functions which are locally square-integrable. $L^1(0, +\infty)$ stands for $\{f(t) \in \mathbb{R}^n, t \in (0, +\infty) : \int_0^{+\infty} \|f(t)\|dt < +\infty\}$. For a set S, $|S|$ represents the number of elements in S.

2 | EXACT DELAY RANGE FOR DISCRETE-TIME SYSTEMS WITH INPUT DELAYS

2.1 | Characterization of the exact delay range

In this section, we consider the following discrete-time systems with input delays

$$x_{k+1} = Ax_k + B_0u_k + B_1u_{k-d}, k \geq 0, \tag{9}$$

where $x_k \in \mathbb{R}^p$ and $u_k \in \mathbb{R}^q$ are the state and input control, respectively, $A \in \mathbb{R}^{p\times p}$, $B_0 \in \mathbb{R}^{p\times q}$, and $B_1 \in \mathbb{R}^{p\times q}$, are constant matrices, $d \in \mathbb{N}$ is the delay, and $x_0, u_{-d}, \cdots, u_{-1}$ are initial values. $DTS(d)$ will be used to represent system (9). Two kinds of stabilization definitions are given.

Definition 1. $DTS(d)$ is open-loop stabilizable if for any initial values $x_0, u_{-d}, \cdots, u_{-1}$, there exists $u_k, k \geq 0$, such that $\sum_{k=0}^{+\infty} \|u_k\| < +\infty$ and $\sum_{k=0}^{+\infty} \|x_k\| < +\infty$.

Definition 2. $DTS(d)$ is feedback stabilizable if there exists a feedback control

$$u_k = Kx_k + \sum_{i=1}^{d} K_i u_{k-i}, k \geq 0,$$

where K and $K_i, i = 1, \cdots, d$, are constant matrices, such that for any initial values $x_0, u_{-d}, \cdots, u_{-1}$, it holds that $\lim_{k\to+\infty} u_k = 0$ and $\lim_{k\to+\infty} x_k = 0$.

It is known that the above two definitions are equivalent. In the rest of the paper, it will be said that $DTS(d)$ is stabilizable if it is open-loop stabilizable or feedback stabilizable. Otherwise, it will be said that $DTS(d)$ is unstabilizable. Now the exact delay range is explained.

Definition 3. The exact delay range rendering $DTS(d)$ unstabilizable is defined to be

$$DR_D \triangleq \{d \in \mathbb{N} : DTS(d) \text{ is unstabilizable}\}.$$

The complementary set of DR_D is

$$\overline{DR}_D = \{d \in \mathbb{N} : DTS(d) \text{ is stabilizable}\}.$$

Obviously, determining $\overline{DR}_D$ is equivalent to determining DR_D.The reason for discussing DR_D but not $\overline{DR}_D$ is that DR_D is much more convenient to characterize (This point will be seen in the future results). A necessary and sufficient condition for $DTS(d)$ to be unstabilizable is presented below.

Lemma 1. $DTS(d)$ is unstabilizable if and only if there exists a $\beta \in \mathbb{C}$, which is an unstable eigenvalue of A, such that

$$\mathrm{rank}\begin{pmatrix} \beta I - A & B_0 + \beta^{-d}B_1 \end{pmatrix} < p. \tag{10}$$

Proof. Detailed proof is omitted due to length limit. □

Lemma 1 shows that the stabilization of $DTS(d)$ only depends on unstable eigenvalues of A. Naturally, the following assumption is made.

Assumption 1. Each eigenvalue of A lies on or outside the unit circle.

In this section,

$$\beta_1, \cdots, \beta_r, \beta_{r+1}, \cdots, \beta_{r+m}, \beta_{r+m+1}, \cdots, \beta_{r+2m}, \tag{11}$$

which all lie on or outside the unit circle, denote all the distinct eigenvalues of A. Here $\beta_1, \cdots, \beta_r$ are real and $\beta_{r+1}, \cdots, \beta_{r+m}$, $\beta_{r+m+1}, \cdots, \beta_{r+2m}$ are pairs of complex conjugate eigenvalues with $\beta_{r+j} = \overline{\beta_{r+m+j}}, j = 1, \cdots, m$. For $i = 1, \cdots, r+m$, denote

$$s_i \triangleq \text{the geometric multiplicity of } \beta_i, \tag{12}$$

and $t \triangleq r + 2m$. Then there are s_i Jordan blocks associated with β_i in a Jordan canonical form of A. Let P be a nonsingular matrix making A to be the Jordan canonical form Λ, i.e.,

$$P^{-1}AP = \Lambda, \tag{13}$$

where

$$\Lambda = \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_t \end{pmatrix}, \tag{14}$$

$$Q_i = \begin{pmatrix} J_1(\beta_i) & & \\ & \ddots & \\ & & J_{s_i}(\beta_i) \end{pmatrix}, i = 1, \cdots, t, \tag{15}$$

$$J_j(\beta_i) = \begin{pmatrix} \beta_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \beta_i \end{pmatrix}, j = 1, \cdots, s_i. \tag{16}$$

Define

$$C_0 \triangleq P^{-1}B_0, C_1 \triangleq P^{-1}B_1, \tag{17}$$

$$n_{i,j} \triangleq \text{the row number of } J_j(\beta_i)\text{'s last row in } \Lambda, i = 1, \cdots, t, j = 1, \cdots, s_i, \tag{18}$$

$$\Gamma(\beta_i, d) \triangleq \begin{pmatrix} (C_0)_{n_{i,1}} \\ \vdots \\ (C_0)_{n_{i,s_i}} \end{pmatrix} + \beta_i^{-d} \begin{pmatrix} (C_1)_{n_{i,1}} \\ \vdots \\ (C_1)_{n_{i,s_i}} \end{pmatrix}, \tag{19}$$

where $(C_0)_{n_{i,j}}$ and $(C_1)_{n_{i,j}}$ represent the $n_{i,j}-$row of the matrix C_0 and that of C_1, respectively.

Theorem 1. (1) The exact delay range rendering $DTS(d)$ unstabilizable is given by

$$DR_D = \cup_{i=1}^{r+m} S(\beta_i), \tag{20}$$

where for $i = 1, \cdots, r+m$,

$$S(\beta_i) \triangleq \{d \in \mathbb{N} : \text{rank} \begin{pmatrix} \beta_i I - A & B_0 + \beta_i^{-d} B_1 \end{pmatrix} < p\}. \tag{21}$$

(2) If $q < s_i$ (q and s_i are the dimension of control input and the geometric multiplicity of β_i, respectively), then $S(\beta_i) = \mathbb{N}$. If $q \geq s_i$, then

$$S(\beta_i) = \{d \in \mathbb{N} : \text{ all the } s_i - \text{order minors of } \Gamma(\beta_i, d) \text{ are zero}\}, \tag{22}$$

where $\Gamma(\beta_i, d)$ is defined by (19).

(3) After excluding trivial cases of $S(\beta_i) = \mathbb{N}$ and $S(\beta_i) = \emptyset$ (for example, if $\beta_i = 1$, then $S(\beta_i) = \mathbb{N}$ or $S(\beta_i) = \emptyset$), one and only one of the following two cases will happen.

- If $\beta_i \notin \mathbb{U}$ ($\mathbb{U}$ is the set of unit roots, see Notation), then $S(\beta_i)$ has finite elements and $|S(\beta_i)| \leq s_i$ ($|S(\beta_i)|$ is the number of elements of $S(\beta_i)$).

- If $\beta_i \in \mathbb{U}$, then

$$S(\beta_i) = \cup_{j=1}^{\tau}\{d_j + o(\beta_i)k \,:\, k \in \mathbb{Z}, k \geq -\frac{d_j}{o(\beta_i)}\}. \tag{23}$$

Here $d_j \in \mathbb{N}$ is a solution to some equation $\beta_i^{-d} = z_j, j = 1, \cdots, \tau$, and $o(\beta_i)$ is the order of β_i, which has been defined in Notation. Also, τ satisfies $\tau \leq s_i$.

Proof. See Appendix A. □

Theorem 1 uncovers the following laws for the exact delay range making $DST(d)$ unstabilizable. An eigenvalue of A, which is not a unit root, will render $DST(d)$ unstabilizable at finite delays. An eigenvalue of A, which is a unit root, may render $DST(d)$ unstabilizable at several sequences of delays. Each sequence is an arithmetic progression and the common difference of this progression is equal to the order of this eigenvalue (see (23)). In addition, Theorem 1 yields the following corollary.

Corollary 1. If there exists an eigenvalue of A, such that the geometric multiplicity of this eigenvalue exceeds the dimension of the control input, then $DR_D = \mathbb{N}$.

2.2 | Procedures for computing DR_D

The proof of Theorem 1 gives a hand computation method to derive DR_D. This method is formulated as follows.

Step 1: Find a nonsingular matrix P to transform A to its canonical form Λ (see (13)-(16)). Compute C_0 and C_1 by (17).

Step 2: Based on Λ, derive $n_{1,j}, j = 1, \cdots, s_1$ for the eigenvalue β_1 via (18). Obtain $\Gamma(\beta_1, d)$ according to (19).

Step 3: In $\Gamma(\beta_1, d)$, do the variable transformation $z = \beta_1^{-d}$. Calculate all the s_1−order minors of $\Gamma(\beta_1, d)$. Denote these minors by $f_i(z), i = 1, \cdots, \upsilon$. Solve polynomial equations $0 = f_i(z)$ and derive the set $\Omega = \{z \in \mathbb{C} \,:\, 0 = f_i(z), i = 1, \cdots, \upsilon\}$. Denote elements of Ω by $z_1, \cdots, z_\tau$.

Step 4: Compute $S(\beta_1)$ as $S(\beta_1) = \cup_{i=1}^{\tau}\{d \in \mathbb{N} \,:\, \beta_1^{-d} = z_i\}$.

Step 5: In a similar line to steps 2-4, derive $S(\beta_i)$ for $i = 2, \cdots, r + m$.

Step 6: Compute DR_D as $DR_D = \cup_{i=1}^{r+m} S(\beta_i)$.

Examples will be given in the next subsection to illustrate the above procedures.

2.3 | Examples

Example 1: Consider $DTS(d)$ with

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, B_0 = \begin{pmatrix} 1 & 0 & -2 \\ 11 & -7 & 3 \\ 1 & 1 & -3 \\ 1.5 & -4 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 3 \\ 2 & 5 & -9 \\ 0 & 1 & 2 \\ 1 & -4.5 & 6 \end{pmatrix}.$$

The matrix A is a Jordan canonical form, so $P = I, \Lambda = A, C_0 = B_0$, and $C_1 = B_1$. The two eigenvalues are -1 and 2. For -1, there are two Jordan blocks, $J_1(-1) = -1$ and $J_2(-1) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. The row number of the last row of $J_1(-1)$ in A is $n_{1,1} = 1$. The row number of the last row of $J_2(-1)$ in A is $n_{1,2} = 3$. Thus

$$\begin{aligned} \Gamma(-1, d) &= \begin{pmatrix} (B_0)_1 \\ (B_0)_3 \end{pmatrix} + (-1)^{-d} \begin{pmatrix} (B_1)_1 \\ (B_1)_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -2 + 3z \\ 1 & 1 + z & -3 + 2z \end{pmatrix}, \end{aligned}$$

where $z \triangleq (-1)^{-d}$. The 2-order minors of $\Gamma(-1, d)$ are

$$f_1(z) = \begin{vmatrix} 1 & 0 \\ 1 & 1 + z \end{vmatrix} = 1 + z, f_2(z) = \begin{vmatrix} 1 & -2 + 3z \\ 1 & -3 + 2z \end{vmatrix} = -z - 1,$$

$$f_3(z) = \begin{vmatrix} 0 & -2 + 3z \\ 1 + z & -3 + 2z \end{vmatrix} = -(1 + z)(-2 + 3z).$$

Then one obtains

$$\Omega_1 = \{z : 1+z=0, -z-1=0, -(1+z)(-2+3z)=0\} = \{-1\},$$
$$S(-1) = \{d \in \mathbb{N} : (-1)^{-d} = -1\} = \{2k+1 : k \in \mathbb{Z}, k \geq 0\}.$$

The eigenvalue 3 corresponds a 1−order Jordan block, which is in the fourth row of A. Thus one has $n_{2,1} = 4$ and

$$\Gamma(3,d) = (B_0)_4 + (B_1)_4 3^{-d} = \begin{pmatrix} 1.5+z & -4-4.5z & 1+6z \end{pmatrix},$$

where $z \triangleq 3^{-d}$. The 1-order minors of $\Gamma(3,d)$ are $1.5+z$, $-4-4.5z$, and $1+6z$. So

$$\Omega_2 = \{z : 1.5+z=0, -4-4.5z=0, 1+6z=0\} = \emptyset,$$

and $S(3) = \emptyset$. Finally, one has $DR_D = S(-1) \cup S(3) = \{2k+1 : k \in \mathbb{Z}, k \geq 0\}$.

Example 2: Consider $DTS(d)$ with

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{4}\sqrt{3} \\ 0 & 0 & \sqrt{3} & -\frac{1}{2} \end{pmatrix}, B_0 = \begin{pmatrix} 1 & 0 \\ -5 & 7 \\ \sqrt{3} & 2\sqrt{3} \\ 2 & 4 \end{pmatrix}, B_1 = \begin{pmatrix} -32 & 0 \\ 6 & 8 \\ 0 & 0 \\ 4 & 8 \end{pmatrix}.$$

Choose $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & -\sqrt{-1} \\ 0 & 0 & 2 & 2 \end{pmatrix}$, then one has

$$\Lambda = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1} \end{pmatrix}, C_0 = \begin{pmatrix} 1 & 0 \\ -5 & 7 \\ \frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1} & 1-\sqrt{3}\sqrt{-1} \\ \frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1} & 1+\sqrt{3}\sqrt{-1} \end{pmatrix}, C_1 = \begin{pmatrix} -32 & 0 \\ 6 & 8 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

The eigenvalue 2 corresponds to a 1−order Jordan block, which is in the first row of Λ. So one obtains $n_{1,1} = 1$, and $\Gamma(2,d) = (C_0)_1 + 2^{-d}(C_1)_1 = \begin{pmatrix} 1-2^{-d}32 & 0 \end{pmatrix}$. Let $z = 2^{-d}$. Note that $\Gamma(2,d)$ has two 1-order minors, $1-32z$ and 0. Thus

$$\Omega_1 = \{z : 1-32z=0\} = \{\frac{1}{32}\},$$
$$S(2) = \{d \in \mathbb{N} : 2^{-d} = \frac{1}{32}\} = \{5\}.$$

For the eigenvalue -3, one has $n_{2,1} = 2$, and

$$\Gamma(-3,d) = \begin{pmatrix} -5+6(-3)^{-d} & 7+8(-3)^{-d} \end{pmatrix}.$$

Set $z = (-3)^{-d}$. $\Gamma(-3,d)$ has two 1-order minors, $-5+6z$ and $7+8z$. Then $\Omega_2 = \{z : -5+6z=0, 7+8z=0\} = \emptyset$, and $S(-3) = \emptyset$. For the eigenvalue $\beta \triangleq -\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}$, one has $n_{3,1} = 3$, and

$$\Gamma(\beta,d) = (C_0)_3 + \beta^{-d}(C_1)_3 = \begin{pmatrix} \frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1}+\beta^{-d} & 1-\sqrt{3}\sqrt{-1}+2\beta^{-d} \end{pmatrix}.$$

Let $z = \beta^{-d}$. $\Gamma(\beta,d)$ has two 1-order minors, $\frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1}+z$ and $1+\sqrt{3}\sqrt{-1}+2z$. Then

$$\Omega_3 = \{z : \frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1}+z=0, 1-\sqrt{3}\sqrt{-1}+2z=0\} = \{\beta\},$$
$$S(\beta) = \{d \in \mathbb{N} : \beta^{-d} = \beta\}.$$

Note that $\beta \in \mathbb{U}$ and $o(\beta) = 3$, so $S(\beta) = \{2+3k : k \in \mathbb{N}\}$. Finally, one has $DR_D = S(2) \cup S(-3) \cup S(\beta) = \{2+3k : k \in \mathbb{N}\}$.

Example 3: Consider $DTS(d)$ with

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, B_0 = \begin{pmatrix} 11 & -7 \\ 1 & 0 \\ 1 & 1 \\ 1.5 & -4 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 \\ 2 & 5 \\ 0 & 1 \\ -3 & 8 \end{pmatrix}.$$

The matrix A has been a Jordan canonical form. For the eigenvalue -1, one has $n_{1,1} = 1$, $n_{1,2} = 3$, and

$$\Gamma(-1, d) = \begin{pmatrix} (B_0)_1 \\ (B_0)_3 \end{pmatrix} + (-1)^{-d} \begin{pmatrix} (B_1)_1 \\ (B_1)_3 \end{pmatrix} = \begin{pmatrix} 11 & -7 \\ 1 & 1+z \end{pmatrix},$$

where $z \triangleq (-1)^{-d}$. The unique 2-order minor of $\Gamma(-1, d)$ is $f(z) = \begin{vmatrix} 11 & -7 \\ 1 & 1+z \end{vmatrix} = 11z + 18$. Then one obtains $\Omega_1 = \{z : 11z + 18 = 0\} = \{-\frac{18}{11}\}$, and $S(-1) = \{d \in \mathbb{N} : (-1)^{-d} = -\frac{18}{11}\} = \emptyset$. For the eigenvalue $\sqrt{2}$, one has $n_{2,1} = 4$, and

$$\Gamma(\sqrt{2}, d) = (B_0)_4 + (B_1)_4 \sqrt{2}^{-d} = \begin{pmatrix} 1.5 - 3z & -4 + 8z \end{pmatrix},$$

where $z \triangleq \sqrt{2}^{-d}$. The 1-order minors of $\Gamma(\sqrt{2}, d)$ are $1.5 - 3z$ and $-4 + 8z$. So

$$\Omega_2 = \{z : 1.5 - 3z = 0, -4 + 8z = 0\} = \{0.5\},$$

and $S(\sqrt{2}) = \{d \in \mathbb{N} : \sqrt{2}^{-d} = 0.5\} = \{2\}$. Finally, one has $DR_D = S(-1) \cup S(\sqrt{2}) = \{2\}$.

3 | EXACT DELAY RANGE FOR CONTINUOUS-TIME SYSTEMS WITH INPUT DELAYS

3.1 | Characterization of the exact delay range

Consider the continuous-time input-delay system (7). We use $CTS(h)$ to represent this system.

Definition 4. [26] $CTS(h)$ is open-loop stabilizable if for any $x(0)$ and $u(\theta), \theta \in [-h, 0)$ satisfying $u(\theta) \in L^2_{loc}$, there exists a control $u(t), t \geq 0$, such that the functions $x(t)$ and $u(t)$ are in $L^1(0, +\infty)$.

Definition 5. [27] $CTS(h)$ is feedback stabilizable if there exists a feedback control

$$u(t) = Kx(t) + \int_{-h}^{0} K(\theta)u(t+\theta)d\theta, \tag{24}$$

where $K \in \mathbb{R}^{p\times q}$ is a constant matrix and $K(\theta) : [-h, 0] \to \mathbb{R}^{q\times q}$ is a continuous function, such that for any initial values $x(0)$ and $u(\theta), \theta \in [-h, 0)$ satisfying $u(\theta) \in L^2_{loc}$, $x(t)$ and $u(t)$ are in $L^1(0, +\infty)$.

The above two definions are equivalent. In the rest of our paper, it will be said that $CTS(h)$ is stabilizable if it is open-loop stabilizable or feedback stabilizable. Otherwise, it will be said that $CTS(h)$ is unstabilizable.

Definition 6. The exact delay range rendering $CTS(h)$ unstabilizable is defined to be

$$DR_C \triangleq \{h \in [0, +\infty) : CTS(h) \text{ is ustabilizable}\}.$$

Lemma 2. $CTS(h)$ is unstabilizable if and only if there exists a $\beta \in \mathbb{C}$, which is an unstable eigenvalue of A, such that

$$\text{rank} \begin{pmatrix} \beta I - A & B_0 + e^{-\beta h} B_1 \end{pmatrix} < p. \tag{25}$$

Proof. Details are omitted here. □

From Lemma 2, it can be seen that the stabilization of $CTS(h)$ is independent of stable eigenvalues of A. Hence, the following assumption is made throughout this section.

Assumption 2. All the eigenvalues of A are on the closed right half plane.

The setting (11)-(19), will still be used in this section. Two differences are as follows. The first one is that each $\beta_i, i = 1, \cdots, t$, is on the closed right half plane. The second one is the definition of $\Gamma(\beta_i, d)$. In this section, $\Gamma(\beta_i, h)$ is defined to be

$$\Gamma(\beta_i, h) \triangleq \begin{pmatrix} (C_0)_{n_{i,1}} \\ \vdots \\ (C_0)_{n_{i,s_i}} \end{pmatrix} + e^{-\beta_i h} \begin{pmatrix} (C_1)_{n_{i,1}} \\ \vdots \\ (C_1)_{n_{i,s_i}} \end{pmatrix}. \tag{26}$$

Theorem 2. (1) The exact delay range rendering $CTS(h)$ unstabilizable is given by

$$DR_C = \cup_{i=1}^{r+m} S(\beta_i), \tag{27}$$

where

$$S(\beta_i) \triangleq \{h \in [0, +\infty) : \text{rank} \begin{pmatrix} \beta_i I - A & B_0 + e^{-\beta_i h} B_1 \end{pmatrix} < p\}, i = 1, \cdots, r + m.$$

(2) If $q < s_i$ (q and s_i are the dimension of control input and the geometric multiplicity of β_i, respectively), then $S(\beta_i) = [0, +\infty)$. If $q \geq s_i$, then

$$S(\beta_i) = \{h \in [0, +\infty) : \text{ all the } s_i - \text{order minors of } \Gamma(\beta_i, h) \text{ are zero}\}, \tag{28}$$

where $\Gamma(\beta_i, h)$ is given by (26).

(3) By excluding trivial cases of $S(\beta_i) = [0, +\infty)$ and $S(\beta_i) = \emptyset$ (for example, if $\beta_i = 0$, then $S(\beta_i) = [0, +\infty)$ or $S(\beta_i) = \emptyset$), one and only one of the following two cases will happen.

- If $\text{Re}(\beta_i) \neq 0$, then $S(\beta_i)$ has finite elements and $|S(\beta_i)| \leq s_i$. Here, $|S(\beta_i)|$ is the number of elements in $S(\beta_i)$.
- If $\text{Re}(\beta_i) = 0$, then

$$S(\beta_i) = \cup_{j=1}^{\tau} \{h_j + \frac{2k\pi}{Im(\beta_i)} : k \in \mathbb{Z}, k \geq -\frac{h_j \text{Im}(\beta_i)}{2\pi}\}, \tag{29}$$

where $h_j \in [0, +\infty)$ is a solution to some equation $e^{-\beta_i h} = z_j, j = 1, \cdots, \tau$, and $\tau \leq s_i$.

Proof. See Appendix B. □

From Theorem 2, it is known that DR_C has at most countable elements (excluding the trivial case of $DR_C = [0, +\infty)$). Specifically, an eigenvalue of A, which is not on the imaginary axis, will make $CTS(h)$ unstabilizable at finite delays. While an eigenvalue of A, which is on the imaginary axis, may make $CTS(h)$ unstabilizable at some sequences of delays. Each sequence is an arithmetic progression and the common difference of this progression is $\frac{2\pi}{\text{Im}(\beta_i)}$ (see (29)). In addition, the following corollary can be obtained directly.

Corollary 2. If there exists an eigenvalue of A, such that the geometric multiplicity of this eigenvalue exceeds the dimension of the control input, then $DR_C = [0, +\infty)$.

3.2 | Procedures for computing DR_C

A way of obtaining DR_C is presented as follows.

Step 1: Find a nonsingular matrix P to convert A to its canonical form Λ (see (13)-(16)). Calculate C_0 and C_1 via (17).

Step 2: Based on Λ, derive $n_{1,j}, j = 1, \cdots, s_1$ for the eigenvalue β_1 via (18). Obtain $\Gamma(\beta_1, h)$ according to (26).

Step 3: In $\Gamma(\beta_1, h)$, do the variable transformation $z = e^{-\beta_1 h}$. Calculate all the s_1−order minors of $\Gamma(\beta_1, h)$. Denote these minors by $f_i(z), i = 1, \cdots, v$. By solving polynomial equations $0 = f_i(z)$, derive the solution set $\Omega = \{z \in \mathbb{C} : 0 = f_i(z), i = 1, \cdots, v\}$. Suppose all the elements of Ω are $z_1, \cdots, z_\tau$.

Step 4: Compute $S(\beta_1)$ as $S(\beta_1) = \cup_{i=1}^{\tau} \{h \in [0, +\infty) : e^{-\beta_1 h} = z_i\}$.

Step 5: In a similar line to steps 2-4, calculate $S(\beta_i)$, for $i = 2, \cdots, r + m$.

Step 6: Derive DR_C as $DR_C = \cup_{i=1}^{r+m} S(\beta_i)$.

Examples will be presented below to illustrate the above procedures.

3.3 | Examples

Example 1: Consider $CTS(h)$ with

$$A=\frac{1}{7}\begin{pmatrix}27 & 11 & -8 & -15\\ 2 & 20 & 2 & 16\\ -4 & 2 & 31 & -4\\ 6 & -17 & 6 & 55\end{pmatrix}, B_0=\begin{pmatrix}3 & 8\\ 8 & 10\\ 2.5 & 9.5\\ 5.5 & 1.5\end{pmatrix}, B_1=\begin{pmatrix}-5 & -12\\ -11 & -18\\ -4 & -11\\ -7 & -7\end{pmatrix}.$$

Select $P=\begin{pmatrix}1 & 0 & -1 & 2\\ -2 & 0 & 0 & 3\\ 0 & 1 & 1 & 2\\ -2 & -1 & 0 & 1\end{pmatrix}$, then $\Lambda=\begin{pmatrix}5 & 1 & 0 & 0\\ 0 & 5 & 0 & 0\\ 0 & 0 & 5 & 0\\ 0 & 0 & 0 & 4\end{pmatrix}$,$C_0=\begin{pmatrix}-1 & 1\\ -1.5 & 0.5\\ 0 & 1\\ 2 & 4\end{pmatrix}$, and $C_1=\begin{pmatrix}1 & 0\\ 2 & 1\\ 0 & 0\\ -3 & -6\end{pmatrix}$. For the eigenvalue 5, there are two Jordan blocks. The last row of the first Jordan block is in the second row of Λ and the last row of the second Jordan block is in the third row of Λ. So $s_1=2, n_{1,1}=2, n_{1,2}=3$, and

$$\begin{aligned}\Gamma(5,h) &= \begin{pmatrix}(C_0)_2\\ (C_0)_3\end{pmatrix}+e^{-5h}\begin{pmatrix}(C_1)_2\\ (C_1)_3\end{pmatrix}\\ &= \begin{pmatrix}-1.5+2e^{-5h} & 0.5+e^{-5h}\\ 0 & 1\end{pmatrix}.\end{aligned}$$

The unique 2−order minor of $\Gamma(5,h)$ is $det(\Gamma(5,h))=-1.5+2z$, where $z=e^{-5h}$. Then one has that $\Omega_1=\{z:-1.5+2z=0\}=\{0.75\}$, and

$$S(5)=\{h\in[0,+\infty): e^{-5h}=0.75\}=\{0.0575\}.$$

For the eigenvalue 4, there is a Jordan block and the row number of the last row of this block in Λ is 4, so $s_2=1, n_{2,1}=4$, and $\Gamma(4,h)=(C_0+e^{-4h}C_1)_4=\begin{pmatrix}2-3e^{-4h} & 4-6e^{-4h}\end{pmatrix}$. The 1−order minors of $\Gamma(4,h)$ are $2-3z$, and $4-6z$, where $z=e^{-4h}$. Thus

$$\Omega_2=\{z: 2-3z=0, 4-6z=0\}=\{\frac{2}{3}\},$$
$$S(4)=\{h\in[0,+\infty): e^{-4h}=\frac{2}{3}\}=\{0.1014\}.$$

Finally, one has $DR_C=S(5)\cup S(4)=\{0.0575, 0.1014\}$.

Example 2: Consider $CTS(h)$ with $A=\beta I$, $B_0=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$, and $B_1=\begin{pmatrix}1 & 0\\ 16 & 5\end{pmatrix}$. Set $z\triangleq e^{-\beta h}$. Direct computation leads to

$$\begin{aligned}&det(B_0+zB_1)=5z^2-5z+1,\\ &\Omega=\{z: 0=5z^2-5z+1\}=\{\frac{5+\sqrt{5}}{10},\frac{5-\sqrt{5}}{10}\},\\ &DR_C=\{h\in[0,+\infty): e^{-\beta h}\in\Omega\}=\{\frac{1}{\beta}\ln(\frac{10}{5+\sqrt{5}}),\frac{1}{\beta}\ln(\frac{10}{5-\sqrt{5}})\}.\end{aligned}$$

Example 3: Consider $CTS(h)$ with

$$A=\begin{pmatrix}1 & 5\\ -1/4 & -1\end{pmatrix}, B_0=\begin{pmatrix}4\sqrt{3}+8\\ -2\end{pmatrix}, B_1=\begin{pmatrix}16\\ -4\end{pmatrix}.$$

Choose $P=\begin{pmatrix}4+2\sqrt{-1} & 4-2\sqrt{-1}\\ -1 & -1\end{pmatrix}$, then $\Lambda=\begin{pmatrix}\frac{1}{2}\sqrt{-1} & 0\\ 0 & -\frac{1}{2}\sqrt{-1}\end{pmatrix}$, $C_0=\begin{pmatrix}-\sqrt{3}\sqrt{-1}+1\\ \sqrt{3}\sqrt{-1}+1\end{pmatrix}$, and $C_1=\begin{pmatrix}2\\ 2\end{pmatrix}$. Since A has two conjugate eigenvalues, one has that $DR_C=S(\beta)$ where $\beta=\frac{1}{2}\sqrt{-1}$. Then

$$\begin{aligned}&\Gamma(\beta,h)=(C_0)_1+e^{-\beta h}(C_1)_1=-\sqrt{3}\sqrt{-1}+1+2e^{-\beta h},\\ &\Omega=\{z\in\mathbb{C}: -\sqrt{3}\sqrt{-1}+1+2z\}=\{-\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}\},\\ &DR_C=\{h\in[0,+\infty): e^{-\beta h}=-\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}\}=\{\frac{8\pi}{3}+4k\pi, k\in\mathbb{Z}, k\geq 0\}.\end{aligned}$$

4 | CONCLUSIONS

This paper presents the exact delay range for input-delay systems to be unstabilizable. Both discrete-time and continuous-time systems are investigated. The results show that this range is closely connected with the unstable eigenvalues of the system matrix. Computation methods to derive the exact range are presented. On the other hand, the range can provide a upper bound for the well-known delay margin achievable by a robust stabilizing controller.

□

APPENDIX

A PROOF OF THEOREM 1

Before showing Theorem 1, a useful lemma will be stated.

Lemma 3. Let $\beta \in \mathbb{C}, |\beta| \geq 1$, and $\delta \in \mathbb{C}$ be fixed. Consider the solution set

$$\Phi(\beta, \delta) \triangleq \{d \in \mathbb{N} : \beta^{-d} = \delta\}. \tag{A1}$$

(1) Suppose $\beta = 1$. If $\delta = 1$, then $\Phi(\beta, \delta) = \mathbb{N}$. If $\delta \neq 1$, then $\Phi(\beta, \delta) = \emptyset$.

(2) Suppose $\beta \neq 1$. Assume that $\Phi(\beta, \delta)$ is nonempty and denote one element in $\Phi(\beta, \delta)$ by d_0. Then one and only one of the two cases will happen.

- If $\beta \in \mathbb{U}$, then

$$\Phi(\beta, \delta) = \{d_0 + o(\beta)k : k \in \mathbb{Z}, d_0 + o(\beta)k \geq 0\}.$$

- If $\beta \notin \mathbb{U}$, then $\Phi(\beta, \delta) = \{d_0\}$.

For definitions of $\mathbb{U}$ and $o(\beta)$, see Notation.

Proof. Detailed proof is omitted here. □

Now the proof of Theorem 1 is presented.

Proof. (1) According to Lemma 1, $d \in DR_D$, i.e., $DTS(d)$ is unstabilizable, if and only if there exists an unstable eigenvalue of A, β, such that (10) holds. Recall that all the unstable eigenvalues of A are $\beta_1, \cdots, \beta_t$. Therefore, $d \in DR_D$ if and only if for some β_i, it holds that rank $\left(\beta_i I - A \;\; B_0 + \beta_i^{-d} B_1 \right) < p$. This means that

$$DR_D = \cup_{i=1}^{t} \{d \in \mathbb{N} : \text{rank} \left(\beta_i I - A \;\; B_0 + \beta_i^{-d} B_1 \right) < p\}. \tag{A2}$$

For $i = r+1, \cdots, r+m$, one has that $\left(\beta_i I - A \;\; B_0 + \beta_i^{-d} B_1 \right) = \overline{\left(\beta_{i+m} I - A \;\; B_0 + \beta_{i+m}^{-d} B_1 \right)}$, which implies that rank $\left(\beta_i I - A \;\; B_0 + \beta_i^{-d} B_1 \right) =$ rank $\left(\beta_{i+m} I - A \;\; B_0 + \beta_{i+m}^{-d} B_1 \right)$. Therefore, it holds that

$$\{d \in \mathbb{N} : \text{rank} \left(\beta_i I - A \;\; B_0 + \beta_i^{-d} B_1 \right) < p\} = \{d \in \mathbb{N} : \text{rank} \left(\beta_{i+m} I - A \;\; B_0 + \beta_{i+m}^{-d} B_1 \right) < p\}.$$

So (A2) leads to (20).

(2) By the elementary transformation

$$P^{-1} \left(\beta_i I - A \;\; B_0 + \beta_i^{-d} B_1 \right) \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} = \left(\beta_i I - \Lambda \;\; C_0 + \beta_i^{-d} C_1 \right),$$

it is known that rank $\left(\beta_i I - A \;\; B_0 + \beta_i^{-d} B_1 \right) =$ rank $\left(\beta_i I - \Lambda \;\; C_0 + \beta_i^{-d} C_1 \right)$. Thus (21) becomes

$$S(\beta_i) = \{d \in \mathbb{N} : \text{rank} \left(\beta_i I - \Lambda \;\; C_0 + \beta_i^{-d} C_1 \right) < p\}. \tag{A3}$$

For simplicity, denote $F \triangleq C_0 + \beta_1^{-d} C_1$. From (19), it is seen that $\Gamma(\beta_1, d)$ is

$$\Gamma(\beta_1, d) = \begin{pmatrix} (F)_{n_{1,1}} \\ \vdots \\ (F)_{n_{1,s_1}} \end{pmatrix}.$$

Here s_1 is the geometric multiplicity of the eigenvalue β_1, $n_{1,j}$ is the row number of the last row of $J_j(\beta_1)$ in the matrix Λ, and $(F)_i$ denotes the $i-$th row vector of F. It will be shown that rank $\left(\, \beta_1 I - \Lambda \;\; F \,\right) < p$ if and only if $\Gamma(\beta_1, d)$ does not have full row rank. Without loss of generality, the following proof will concern the case of $s_1 = 2$. Partition the rows of F according to those of Λ (see (14)) as $F = \begin{pmatrix} F_1 \\ \vdots \\ F_t \end{pmatrix}$, then

$$\left(\, \beta_1 I{-}\Lambda \;\; F \,\right) = \begin{pmatrix} \beta_1 I{-}Q_1 & & & & F_1 \\ & \beta_1 I{-}Q_2 & & & F_2 \\ & & \ddots & & \vdots \\ & & & \beta_1 I{-}Q_t & F_t \end{pmatrix}.$$

For $i \neq 1$, all the diagonal elements of the upper triangular matrix $\beta_1 I - Q_i$ are all nonzero, so $\beta_1 I - Q_i$ is nonsingular. Hence, by elementary column transformations, $\left(\, \beta_1 I - \Lambda \;\; F \,\right)$ can be converted to

$$\begin{pmatrix} \beta_1 I - Q_1 & & & & F_1 \\ & I & & & 0 \\ & & \ddots & & \vdots \\ & & & I & 0 \end{pmatrix}.$$

Thus $\left(\, \beta_1 I - \Lambda \;\; F \,\right)$ does not have full row rank if and only if $\left(\, \beta_1 I - Q_1 \;\; F_1 \,\right)$ does not have full row rank. Recalling $s_1 = 2$, $\left(\, \beta_1 I - Q_1 \;\; F_1 \,\right)$ has the following form as

$$\left(\, \beta_1 I - Q_1 \;\; F_1 \,\right) = \left(\begin{array}{ccccc|ccccc|c} 0 & -1 & & & & & & & & & (F)_1 \\ & 0 & -1 & & & & & & & & (F)_2 \\ & & \ddots & \ddots & & & & & & & \vdots \\ & & & 0 & -1 & & & & & & (F)_{n_{1,1}-1} \\ & & & & 0 & & & & & & (F)_{n_{1,1}} \\ \hline & & & & & 0 & -1 & & & & (F)_{n_{1,1}+1} \\ & & & & & & 0 & -1 & & & (F)_{n_{1,1}+2} \\ & & & & & & & \ddots & \ddots & & \vdots \\ & & & & & & & & 0 & -1 & (F)_{n_{1,2}-1} \\ & & & & & & & & & 0 & (F)_{n_{1,2}} \end{array}\right),$$

where the fact that the m-th row of F_1 is just the m-th row of F has been applied. By elementary row transformations, $\left(\, \beta_1 I - Q_1 \;\; F_1 \,\right)$ becomes

$$\left(\begin{array}{ccccc|ccccc|c} 0 & 1 & & & & & & & & & 0 \\ & 0 & 1 & & & & & & & & 0 \\ & & \ddots & \ddots & & & & & & & \vdots \\ & & & 0 & 1 & & & & & & 0 \\ & & & & 0 & & & & & & (F)_{n_{1,1}} \\ \hline & & & & & 0 & 1 & & & & 0 \\ & & & & & & 0 & 1 & & & 0 \\ & & & & & & & \ddots & \ddots & & \vdots \\ & & & & & & & & 0 & 1 & 0 \\ & & & & & & & & & 0 & (F)_{n_{1,2}} \end{array}\right).$$

The above matrix does not have full row rank if and only if the matrix $\begin{pmatrix} (F)_{n_{1,1}} \\ (F)_{n_{1,2}} \end{pmatrix}$ does not have full row rank.

In a similar line to the above proof, it can be shown that rank $\left(\, \beta_i I - \Lambda \;\; C_0 + \beta_i^{-d} C_1 \,\right) < p$ if and only if $\Gamma(\beta_i, d)$ does not have full row rank. Note that the order of $\Gamma(\beta_i, d)$ is $s_i \times q$. Hence, $\Gamma(\beta_i, d)$ does not have full row rank if and only if one and only one of the following two cases happens.

- $q \geq s_i$ and all the s_i-order minors of $\Gamma(\beta_i, d)$ are zero.

- $q < s_i$.

If $q \geq s_i$, then (A3) implies (22). If $q < s_i$, then $\Gamma(\beta_i, d)$ does not have full row rank for any d and $S(\beta_i) = \mathbb{N}$, which means that $DR_D = \mathbb{N}$.

(3) To show the third statement in Theorem 1, it is enough to consider general $S(\beta)$ as

$$S(\beta) = \{d \in \mathbb{N} : \quad \text{all the } \gamma - \text{order minors of } \Gamma(\beta, d) \text{ are zero}\}, \tag{A4}$$

where $\Gamma(\beta, d)$ is given by

$$\Gamma(\beta, d) = \begin{pmatrix} g_{11} + \beta^{-d} h_{11} & \cdots & g_{1q} + \beta^{-d} h_{1q} \\ \vdots & \vdots & \vdots \\ g_{\gamma 1} + \beta^{-d} h_{\gamma 1} & \cdots & g_{\gamma q} + \beta^{-d} h_{\gamma q} \end{pmatrix}.$$

Here $g_{ij}, h_{ij} \in \mathbb{C}$. Since d appears in $\Gamma(\beta, d)$ in the form of $y(d) \triangleq \beta^{-d}$, $\Gamma(\beta, d)$ can be viewed as a function of $y(d)$ and all the $\gamma-$order minors of $\Gamma(\beta, d)$ are functions of $y(d)$. Denote these $\gamma-$order minors by $f_1(y(d)), \cdots, f_\upsilon(y(d))$, where $\upsilon \triangleq \frac{q!}{\gamma!(q-\gamma)!}$ is the number of $\gamma-$order minors. Obviously, each $f_j(y(d))$ is like

$$\begin{vmatrix} g_{1i_1} + y(d)h_{1i_1} & \cdots & g_{1i_\gamma} + y(d)h_{1i_\gamma} \\ \vdots & \vdots & \vdots \\ g_{\gamma i_1} + y(d)h_{\gamma i_1} & \cdots & g_{\gamma i_\gamma} + y(d)h_{\gamma i_\gamma} \end{vmatrix},$$

with $\{i_1, \cdots, i_\gamma\}$ being chosen from $\{1, \cdots, q\}$, and $f_i(y(d))$ is a polynomial of $y(d)$, with degree less than or equal to γ. Then (A4) can be written as $S(\beta) = \{d \in \mathbb{N} : f_i(y(d)) = 0, i = 1, \cdots, \upsilon\}$. To obtain $S(\beta)$, the first step is to derive solution sets $\Omega_i \triangleq \{z \in \mathbb{C} : f_i(z) = 0\}, i = 1, \cdots, \upsilon$, and $\Omega = \cap_{i=1}^{\upsilon} \Omega_i$. The second step is to compute $S(\beta) = \{d \in \mathbb{N} : \beta^{-d} \in \Omega\}$. Obviously, if $\Omega = \emptyset(\mathbb{C})$, then $S(\beta) = \emptyset(\mathbb{N})$. By excluding $S(\beta) = \emptyset$ and $S(\beta) = \mathbb{N}$ and observing that the degree of polynomial $f_i(z)$ does not exceed γ, it is known that $|\Omega| \leq |\Omega_i| \leq \gamma$. Let $\Omega = \{z_1, \cdots, z_\tau\}$ with $\tau \leq \gamma$. Then $S(\beta) = \cup_{j=1}^{\tau} \Phi(\beta, z_j)$, where $\Phi(\beta, z_j)$ is defined via (A1). If $\Phi(\beta, z_{j_0}) = \mathbb{N}$, then $S(\beta) = \mathbb{N}$. Also, note that $S(\beta) = \cup_{j, \Phi(\beta, z_j) \neq \emptyset} \Phi(\beta, z_j)$. Consequently, it is reasonable to assume that for any j, $\Phi(\beta, z_j) \neq \mathbb{N}$ and $\Phi(\beta, z_j) \neq \emptyset$. By Lemma 3, if $\beta \notin \mathbb{U}$, then each $\Phi(\beta, z_j)$ has exact one element and then $S(\beta)$ has at most τ elements with $\tau \leq \gamma$. If $\beta \in \mathbb{U}$, then $\Phi(\beta, z_j) = \{d_j + o(\beta)k : k \in \mathbb{Z}, d_j + o(\beta)k \geq 0\}$, where d_j satisfies $\beta^{-d_j} = z_j$. Therefore, $S(\beta)$ is as (23). This ends the proof. □

B PROOF OF THEOREM 2

The proof of Theorem 2 is similar to that of Theorem 1. Main differences lie in the following lemma, which corresponds to Lemma 3. Detailed proof will be omitted.

Lemma 4. Let $\beta \in \mathbb{C}$ and $\delta \in \mathbb{C}$ be fixed. Consider $\Phi(\beta, \delta) \triangleq \{h \in [0, +\infty) : e^{-\beta h} = \delta\}$.

(1) Suppose $\beta = 0$. If $\delta = 1$, then $\Phi(\beta, \delta) = [0, +\infty)$. If $\delta \neq 1$, then $\Phi(\beta, \delta) = \emptyset$.

(2) Suppose $\beta \neq 0$. Assume that $\Phi(\beta, \delta)$ is nonempty and denote one element in $\Phi(\beta, \delta)$ by h_0. Then one and only of the two cases will happen.

- If $\mathrm{Re}(\beta) = 0$, then

$$\Phi(\beta, \delta) = \{h_0 + \frac{2k\pi}{Im(\beta)} : k \in \mathbb{Z}, h_0 + \frac{2k\pi}{Im(\beta)} \geq 0\}.$$

- If $\mathrm{Re}(\beta) \neq 0$, then $\Phi(\beta, \delta) = \{h_0\}$.

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