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RESEARCH ARTICLE

Directional developable surfaces and their singularities in Euclidean 3-Space

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The developable surface is a surface that can be unfolded on a plane without tearing or stretching, which is widely used in many fields of engineering and manufacturing. This work presents a new version of developable ruled surfaces in Euclidean 3-space. First, we establish an adapted frame along a spatial curve, denoted by the quasi-frame. We then introduce a parametric representation of a developable ruled surface and call it a directional developable ruled surface. At the core of this paper, we investigate the existence and uniqueness of such developable surfaces, then study their classification by singularity theory and unfolding method. Some examples are given in the final.

KEYWORDS:
Quasi-frame, Directional developable surfaces, Singularity theory

1 | INTRODUCTION

Developable surface in Euclidean 3-space is a curved surface that can be developed onto a plane without tearing and stretching. The Gaussian curvature of developable surface is zero everywhere on the surface. A plane is a special surface, its Gaussian curvature at each point is constant zero, so any surface with zero curvature at each point can be unfolded into a plane by bending, that is, it has an isometric mapping to the plane. Such a surface is called a developable surface. Many applications can benefit from the use of developable surface in many areas of engineering and manufacturing, including modeling of apparel, automobile components, and ship hulls (see e.g. [19,20,23,11,12,13]). Singularity refers to a point that is different from the overall nature of things. Because of its particularity, mathematicians have paid much attention to it and formed a new branch - Singularity Theory. With the accumulation of several generations of mathematicians, it has been booming and promoted the development of other disciplines.

The developable surface can be parameterized using the Serret-Frenet frame of space curves from the viewpoint of singularity theory [2,3]. In [8], S. Izumiya et al. introduced the rectifying developable surfaces of space curves, where they showed that a regular curve is a geodesic of its rectifying developable surface and revealed the relationship between singularities of the rectifying developable surface and geometric invariant. Ishikawa investigated the relationship between the singularities of tangent developable surfaces and some types of space curves. He also gave a classification of tangent developable surfaces by using the local topological property [9]. There are several works about singularity theory of developable ruled surfaces by using the Serret-Frenet frame of space curves, for example [7,6,4]. Among them, in [6], Satoshi Hananoi and Shyuichi Izumiya also specifically discussed the ellipsoid, using a parameterized surface, the singular value of the normal developable surface of the trajectory is the focus surface of the surface of all coordinate curves.

However, the Serret-Frenet frame is undefined wherever the curvature vanishes, such as at points of inflection or along straight sections of the curve [1]. Thus, the notion of rotation minimizing frame (RMF) which is more suitable for applications was introduced by Bishop in [1,10]. But, it is well known that Bishop frame calculations are not an easy task, see [21,22]. Therefore, inspired by the work of Coquillart[5], Mustafa introduced a new adapted frame along a space curve and denoted this the quasi-frame[18]. As different disciplines are more closely related than ever, interdisciplinary subjects have drawn researchers' attention. Therefore, in the future work, we would take the advantage of singularity theory and submanifolds theory etc. presented in[14,15,16,17] to explore new results and theorems.

In this paper,we put our research content on the curve containing singular points in Euclidean space. We know that there is a big difference between the developable surface generated by regular curve and the curve containing singular points. So, we give the quasi-frame along a unit speed curve and introduce a directional developable ruled surface. Applying the unfolding theory, we classify the generic properties, and present new two invariants related to the singularities of this surface. It is demonstrated that the generic singularities are cuspidal edge and swallowtail, and the types of these singularities can be characterized by these invariants, respectively. Finally, examples are illustrated to explain the applications of the theoretical results.

2 | BASIC CONCEPTS

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a unit speed curve in Euclidean 3-space, with $\kappa(s)$ and $\tau(s)$ that denote the natural curvature and torsion of $\boldsymbol{\alpha}(s)$, respectively. Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the Serret-Frenet frame associated with the curve $\boldsymbol{\alpha}(s)$, then the Serret-Frenet formulae read:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{1}$$

where dash denotes differentiation with respect to s. Although the Serret–Frenet frame can be calculated easily, its rotation about the tangent of a general space curve often leads to undesirable twist in motion design or sweeping surface modeling. Moreover, the drawback of Serret–Frenet frame is that it is not continuously defined for a C^1-continuous space curve, and even for a C^2-continuous space curve the Serret–Frenet frame becomes undefined at an inflection point (i.e., curvature $\kappa = 0$), thus causing unacceptable discontinuity when used for surface modeling[1,10]. Therefore, Coquillart[5], and Mustafa et al.[18] gave a new frame called Quasi-frame (for short Q-frame) of a space curve as the following: Given a unit speed curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ the Q-frame is given by

$$\mathbf{e}_1(s) = \mathbf{T},\ \mathbf{e}_2(s) = \frac{\mathbf{T} \times \boldsymbol{\zeta}}{\|\mathbf{T} \times \boldsymbol{\zeta}\|},\ \mathbf{e}_3(s) = \mathbf{e}_1 \times \mathbf{e}_2, \tag{2}$$

where $\boldsymbol{\zeta}$ is called the projection vector. The relation between Serret–Frenet frame and Q-frame is given as follows:

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{3}$$

with a certain angle $\varphi(s)$. By taking the derivative of Eq. (3) with respect to s, and using the inverse transformation, we obtain:

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{4}$$

The triple $(\kappa_1; \kappa_2; \kappa_3)$ is called the Q-curvature functions of $\boldsymbol{\alpha}$. Here,

$$\left.\begin{array}{l} \kappa_1(s) = \kappa\cos\varphi = \langle \mathbf{e}_1', \mathbf{e}_2 \rangle, \\ \kappa_2(s) = -\kappa\sin\varphi = \left\langle \mathbf{e}_1', \mathbf{e}_3 \right\rangle, \\ \kappa_3(s) = \tau + \varphi' = \left\langle \mathbf{e}_2', \mathbf{e}_3 \right\rangle. \end{array}\right\} \tag{5}$$

The Q-frame have many advantages compare to other frames (Serret–Frenet, Bishop). For instance, the Q-frame can be defined even along a line (i.e., curvature $\kappa = 0$). However, the Q-frame is singular in all cases where $\mathbf{T}$ and $\boldsymbol{\zeta}$ are parallel. Thus, in these cases, where $\mathbf{T}$ and $\boldsymbol{\zeta}$ are parallel, the projection vector $\boldsymbol{\zeta}$ can be chosen as $\boldsymbol{\zeta}$=(0,1,0) or $\boldsymbol{\zeta}$=(0,0,1) (for details, see [19, 20]). From now on, we shall often not write the parameter s explicitly in our formulae.

A ruled surface in Euclidean 3-space $\mathbb{R}^3$ is a differentiable one-parameter set of straight lines L. Such a surface has a parameterization of the form

$$M\ :\ \mathbf{y}(s, v) = \boldsymbol{\alpha}(s) + v\mathbf{e}(s),\ v \in \mathbb{R}, \tag{6}$$

where $\boldsymbol{\alpha}(s)$ is its base curve and $\mathbf{e}$ is the unit vector along the ruling L of the surface. The rulings of a ruled surface are asymptotic curves. If the tangent plane of the ruled surface is constant along a fixed ruling, the ruled surface is called the developable surface[19,20,23]. Tangent planes of such surfaces depend on only one parameter. All other ruled surfaces are called the skew surfaces. The base curve is not unique, since any curve of the form:

$$\mathbf{C}(s) = \boldsymbol{\alpha}(s) - \sigma(s)\mathbf{e}(s), \tag{7}$$

may be used as its base curve, $\sigma(s)$ is a smooth function. If there exists a common perpendicular to two neighboring rulings on $\mathbf{y}(s, v)$, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the striction curve. In Eq. (7) if

$$\sigma(s) = \frac{\left\langle \boldsymbol{\alpha}'(s), \mathbf{e}'(s) \right\rangle}{\left\|\mathbf{e}'(s)\right\|^2}, \tag{8}$$

then $\mathbf{C}(s)$ is called the striction curve on the ruled surface and it is unique. In the case of $\sigma = 0$ the base curve is the striction curve. The distribution parameter of $\mathbf{y}(s, v)$ is defined by

$$\mu(s) = \frac{\det(\boldsymbol{\alpha}'(s), \mathbf{e}(s), \mathbf{e}'(s))}{\left\|\mathbf{e}'(s)\right\|^2}.$$

It is known that a ruled surface M is a developable if and only if $\mu(s) = 0$, that is,

$$\det(\boldsymbol{\alpha}'(s), \mathbf{e}(s), \mathbf{e}'(s)) = 0. \tag{9}$$

Here, we give the notions of contour generators. We suppose that M is a regular surface, and $\mathbf{n}$ is a unit normal vector field on M. For a fixed unit vector $\mathbf{x} \in \mathbb{S}^2$, the normal contour generator of the orthogonal projection with the direction $\mathbf{x}$ is defined by

$$\{\mathbf{p} \in M \mid \langle \mathbf{n}(\mathbf{p}), \mathbf{x} \rangle = 0\}.$$

Furthermore, for a fixed point $\mathbf{c} \in \mathbb{R}^3$, the normal contour generators of the central projection with the center $\mathbf{c}$ is defined by

$$\{\mathbf{p} \in M \mid \langle \mathbf{p} - \mathbf{c}, \mathbf{n}(\mathbf{p}) \rangle = 0\}.$$

For the regular surface, the concepts of contour generators plays an important role in the theory of vision[7].

3 | DIRECTIONAL DEVELOPABLE SURFACE

In this section, we introduce a new form of developable ruled surface, and call it a directional developable surface, or D-developable surface for short: Under the assumption $(\kappa_1(s), \kappa_3(s)) \neq (0, 0)$, one define the following ruled surface

$$M : \mathbf{y}(s, v) = \boldsymbol{\alpha}(s) + v\mathbf{e}(s), \ v \in \mathbb{R}, \tag{10}$$

where

$$\mathbf{e}(s) = \frac{\kappa_3\mathbf{e}_1 + \kappa_1\mathbf{e}_3}{\sqrt{\kappa_3^2 + \kappa_1^2}}. \tag{11}$$

Differentiating Eq. (11) by using formulas (4), it gives

$$\mathbf{e}'(s) = \left(\kappa_2 - \frac{\kappa_1\kappa_3' - \kappa_3\kappa_1'}{\kappa_3^2 + \kappa_1^2} \right) \frac{-\kappa_1\mathbf{e}_1 + \kappa_3\mathbf{e}_2}{\sqrt{\kappa_3^2 + \kappa_1^2}}. \tag{12}$$

So that we have $\det(\boldsymbol{\alpha}'(s), \mathbf{e}(s), \mathbf{e}'(s)) = 0$. This means that M is a developable surface. Further, we introduce two invariants $\delta(s)$, and $\sigma(s)$ of M as follows:

$$\delta(s) = \kappa_2 - \frac{\kappa_1\kappa_3' - \kappa_3\kappa_1'}{\kappa_3^2 + \kappa_1^2}, \text{ and } \sigma(s) = \frac{\kappa_3}{\sqrt{\kappa_3^2 + \kappa_1^2}} - \left(\frac{\kappa_1}{\delta(s)\sqrt{\kappa_3^2 + \kappa_1^2}} \right)', \tag{13}$$

where $\delta(s) \neq 0$. We can also calculate that

$$\mathbf{y}_s \times \mathbf{y}_v = \left(-\frac{\kappa_1}{\sqrt{\kappa_3^2 + \kappa_1^2}} + u\delta \right) \mathbf{e}_2. \tag{14}$$

Theorem 1. (Existence and uniqueness). Under the above notations there exists a unique D-developable ruled surface expreesed by Eq. (10).

Proof. For the existence, we have the D-developable along $\boldsymbol{\alpha} = \alpha(s)$ represented by Eq. (10). On the other hand, since M is a ruled surface, we assume that

$$\left.\begin{array}{l} M : \mathbf{y}(s,\upsilon) = \boldsymbol{\alpha}(s) + \upsilon\boldsymbol{\zeta}(s),\ \upsilon \in \mathbb{R},\ \text{with}\ (\kappa_1,\kappa_3) \neq (0,0), \\ \boldsymbol{\zeta}(s) = \zeta_1(s)\mathbf{e}_1 + \zeta_2(s)\mathbf{e}_2 + \zeta_3(s)\mathbf{e}_3, \\ \|\boldsymbol{\zeta}(s)\|^2 = \zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 1,\ \boldsymbol{\zeta}'(s) \neq \mathbf{0}. \end{array}\right\} \tag{15}$$

It can be immediately seen that M is developable if and only if

$$\det(\boldsymbol{\alpha}', \boldsymbol{\zeta}, \boldsymbol{\zeta}') = 0 \Leftrightarrow -\zeta_3\zeta_2' + \zeta_2\zeta_3' - \zeta_1\left(\zeta_3\kappa_1 - \zeta_2\kappa_2\right) + \kappa_3\left(\zeta_2^2 + \zeta_3^2\right) = 0. \tag{16}$$

Moreover, since M is a developable surface which is developable surface along $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, we have

$$\left(\mathbf{y}_s \times \mathbf{y}_\upsilon\right)(s,\upsilon) = \psi(s,\upsilon)\,\mathbf{e}_2, \tag{17}$$

where $\psi = \psi(s,\upsilon)$ is a differentiable function. Further, the normal vector $\mathbf{y}_s \times \mathbf{y}_\upsilon$ at the point $(s,0)$ is

$$\left(\mathbf{y}_s \times \mathbf{y}_\upsilon\right)(s,0) = -\zeta_3\mathbf{e}_2 + \zeta_2\mathbf{e}_3. \tag{18}$$

Thus, from Eqs. (17), and (18), one finds that $\zeta_2 = 0$, and $\zeta_3 = \psi(s,0)$, which follows from Eq. (16) that $\zeta_3\left(\zeta_3\kappa_3 - \zeta_1\kappa_1\right) = 0$. If $(s,0)$ is a regular point (i.e., $\psi(s,0) \neq 0$), then $\zeta_3(s) \neq 0$. Thus, we have $\zeta_1 = \frac{\kappa_3}{\kappa_1}\zeta_3$, with $\kappa_1 \neq 0$. Therefore, we obtain

$$\boldsymbol{\zeta}(s) = \frac{\kappa_3}{\kappa_1}\zeta_3\mathbf{e}_1 + \zeta_3\mathbf{e}_3 = \frac{\zeta_3}{\kappa_1}\sqrt{\kappa_3^2 + \kappa_1^2}\,\mathbf{e}(s),\ \text{with}\ \kappa_1 \neq 0. \tag{19}$$

This means that $\boldsymbol{\zeta}(s)$ has the same direction of $\mathbf{e}(s)$. If $\kappa_3 \neq 0$, we have the same result as the above case. □

Furthermore, we have the following result for $\delta(s)$, and $\sigma(s)$:

Theorem 2. Let M be the D- developable surface defined by Eq. (10). Then:
(A) The following are equivalent:
(1) M is a cylinder,
(2) $\delta(s) = 0$ for all $s \in I$,
(3) $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ is a contour generator with respect to an orthogonal projection.
(B) If $\delta(s) \neq 0$ for all $s \in I$, then the following statement are equivalent:
(1) M is a conical surface,
(2) $\sigma(s) = 0$ for all $s \in I$,
(3) $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ is a contour generator with respect to a central projection.

Proof. (A) From Eq. (12), it is obvious that M is a cylinder if and only if $\mathbf{e}(s)$ is constant, i.e. $\delta(s) = 0$. Therefore, the condition (1) is equivalent to the condition (2). Suppose that the condition (3) holds. Then there exists a fixed unit vector $\mathbf{x} \in \mathbb{S}^2$ such that $\langle \mathbf{e}_2, \mathbf{x}\rangle = 0$. So there exist $a, b \in \mathbb{R}$ such that $\mathbf{x} = a\mathbf{e}_1 + b\mathbf{e}_3$. Since $\left\langle \mathbf{e}_2', \mathbf{x}\right\rangle = 0$, we have $-a\kappa_1 - b\kappa_3 = 0$, so that we have $\mathbf{x} = \frac{\mathbf{b}}{\boldsymbol{\kappa}_1}\sqrt{\kappa_3^2 + \kappa_1^2}\,\mathbf{e}(s)$, with $\kappa_1 \neq 0$. Namely, the condition (1) holds. Suppose that $\mathbf{e}(s)$ is constant. Then we choose $\mathbf{x} = \mathbf{e}(s) \in \mathbb{S}^2$. By the definition of $\mathbf{e}(s)$, we have $\langle \mathbf{x}, \mathbf{e}_2\rangle = 0$. Thus the condition (1) implies the condition (3).

(B) The condition (1) means that the singular value set of M is a constant vector. Thus, in view of Eqs. (7), (8), and Eq. (11), We can calculate that

$$\mathbf{c}'(s) = \left[\frac{\kappa_3}{\sqrt{\kappa_3^2 + \kappa_1^2}} - \left(\frac{\kappa_1}{\delta(s)\sqrt{\kappa_3^2 + \kappa_1^2}}\right)'\right]\mathbf{e}(s) = \sigma(s)\mathbf{e}(s).$$

Then M is a conical surface if and only if $\sigma(s) = 0$. It follows that (1) and (2) are equivalent. By the definition of the central projection means that there exists a fixed point $\mathbf{c} \in \mathbb{R}^3$ such that $\langle \mathbf{e}_2, \boldsymbol{\alpha} - \mathbf{c}\rangle = 0$. If (1) holds, then $\mathbf{c}(s)$ is constant. For the fixed point $\mathbf{c} = \mathbf{c}(s)$, we have

$$\langle \mathbf{e}_2, \boldsymbol{\alpha} - \mathbf{c} \rangle = \left\langle \mathbf{e}_2, \frac{\langle \boldsymbol{\alpha}', \mathbf{e}' \rangle}{\|\mathbf{e}'\|^2} \mathbf{e} \right\rangle = \frac{\langle \boldsymbol{\alpha}', \mathbf{e}' \rangle}{\|\mathbf{e}'\|^2} \langle \mathbf{e}_2, \mathbf{e} \rangle = 0.$$

This means that (3) holds. For the converse, by (3), there exists a fixed point $\mathbf{c} \in \mathbb{R}^3$ such that $\langle \mathbf{e}_2, \boldsymbol{\alpha} - \mathbf{c} \rangle = 0$. Taking the derivative of the both side, we have

$$\langle \mathbf{e}_2, \boldsymbol{\alpha} - \mathbf{c} \rangle' = \langle \kappa_1 \mathbf{e}_1 + \kappa_3 \mathbf{e}_3, \boldsymbol{\alpha} - \mathbf{c} \rangle = 0,$$

thus we may write $\boldsymbol{\alpha} - \mathbf{c} = f(s)\mathbf{e}(s)$, where $f(s)$ is a differentiable function. Taking the derivative again, we have:

$$\langle \mathbf{e}_2, \boldsymbol{\alpha} - \mathbf{c} \rangle'' = \langle \kappa_1 \mathbf{e}_1 + \kappa_3 \mathbf{e}_3, \mathbf{e}_1 \rangle + \left\langle \left(\kappa_1 \mathbf{e}_1 + \kappa_3 \mathbf{e}_3 \right)', \boldsymbol{\alpha} - \mathbf{c} \right\rangle = 0,$$

or equivalently,

$$\langle \mathbf{e}_2, \boldsymbol{\alpha} - \mathbf{c} \rangle'' = \kappa_1 - f\delta\sqrt{\kappa_3^2 + \kappa_1^2} = 0.$$

It follows that

$$\mathbf{c} = \boldsymbol{\alpha}(s) - \frac{\kappa_1}{\delta\sqrt{\kappa_3^2 + \kappa_1^2}} \mathbf{e}(s) = \boldsymbol{\alpha} - \frac{\langle \boldsymbol{\alpha}', \mathbf{e}' \rangle}{\|\mathbf{e}'\|^2} \mathbf{e}(s) = \mathbf{c}(s).$$

Therefore, $\mathbf{c}(s)$ is constant, so that (1) holds. □

As a result the following corollaries can be given.

Corollary 1. The D-developable surface M is a non-cylindrical if and only if $\delta(s) \neq 0$.

Corollary 2. The D-developable surface M is a tangential developable if and only if $\delta(s) \neq 0$, and $\sigma(s) \neq 0$.

Proof. According to the proof of Theorem 1, when $\delta(s) \neq 0$, and $\sigma(s) \neq 0$, we have $\mathbf{e}' \neq \mathbf{0}$, and $\mathbf{c}' \neq \mathbf{0}$. Since $\det(\boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\alpha}') = 0$,$\langle \mathbf{c}', \mathbf{e}' \rangle = 0$,$\langle \mathbf{e}, \mathbf{e}' \rangle = 0$, we can get $\mathbf{c}' \| \mathbf{e}$. It follows that M is a tangent surface. □

We now give relationships between the singularities of M and the two invariants $\delta(s)$ and $\sigma(s)$, as follows:

Theorem 3. Let $\boldsymbol{\alpha} : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with $\kappa_1^2 + \kappa_3^2 \neq 0$. Then we have the following:

(1) (s_0, u_0) is non-singular of the D-developable surface M if and only if

$$\frac{\kappa_1(s_0)}{\sqrt{\kappa_3^2(s_0) + \kappa_1^2(s_0)}} - u_0\delta(s_0) \neq 0.$$

(2) Suppose (s_0, v_0) is singular of M, then the D-developable surface M is locally diffeomorphic to Cuspidal edge CE at (s_0, u_0) if

(i) $\delta(s_0) \neq 0$, $\sigma(s_0) \neq 0$, and

$$u_0 = \frac{\kappa_1(s_0)}{\delta(s_0)\sqrt{\kappa_3^2(s_0) + \kappa_1^2(s_0)}},$$

or

(ii) $\delta(s_0) = \kappa_1(s_0) = 0$, $\delta'(s_0) \neq 0$, and

$$u_0 \neq -\frac{\kappa_1(s_0)}{\delta(s_0)\sqrt{\kappa_3^2(s_0) + \kappa_1^2(s_0)}},$$

or

(iii) $\delta(s_0) = \kappa_1(s_0) = 0$, $\kappa_1'(s_0) \neq 0$.

Clearly, if $\delta'(s_0) \neq 0$ then

$$2\kappa_2(s_0)\kappa_3'(s_0) + \kappa_2'(s_0)\kappa_3(s_0) + \kappa_1''(s_0) \neq 0.$$

(3) Suppose (s_0, u_0) is singular of the D-developable surface M, then M is locally diffeomorphic to Swallowtail SW at (s_0, u_0) if $\delta(s_0) \neq 0$, $\sigma(s_0) = 0$, $\sigma'(s_0) \neq 0$, and

$$u_0 = -\frac{\kappa_1(s_0)}{\delta(s_0)\sqrt{\kappa_3^2(s_0) + \kappa_1^2(s_0)}}.$$

The proof will appear later.

Here,

$$CE = \left\{(x_1, x_2, x_3) | x_1 = u, x_2 = v^2, x_3 = v^3\right\},$$
$$SW = \left\{(x_1, x_2, x_3) | x_1 = u,\ x_2 = 3v^2 + uv^2,\ x_3 = 4v^3 + 2uv\right\}.$$

The pictures of CE, and SW are shown in Figure 1 and Figure 2.

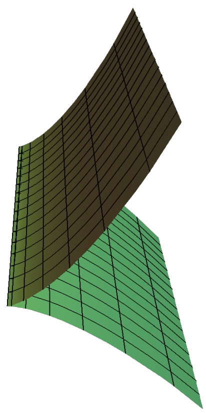

Figure 1 Cuspidal edge.

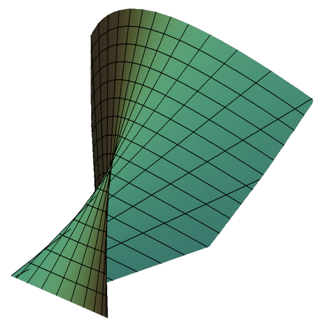

Figure 2 Swallowtail.

3.1 | Support height functions

For a unit speed space curve $\boldsymbol{\alpha}$: $I \to \mathbb{R}^3$, we introduce a height function $H : I \times \mathbb{R}^3 \to \mathbb{R}$, by $H(s, \mathbf{x}) = \langle \mathbf{e}_2(s), \mathbf{x} - \boldsymbol{\alpha}(s)\rangle$. We call it support function on $\boldsymbol{\alpha}(s)$ with respect to $\mathbf{e}_2$. We denote $h_{\mathbf{x}_0}(s) = H(s, \mathbf{x}_0)$ for any fixed $\mathbf{x}_0 \in \mathbb{R}^3$. From now on, we shall often not write the parameter s. Then, we have the following proposition:

Proposition 1. Let $\boldsymbol{\alpha}$: $I \to \mathbb{R}^3$ be a unit speed curve with $\kappa_1^2 + \kappa_3^2 \neq 0$, and $h_{\mathbf{x}_0}(s) = \langle \mathbf{e}_2(s), \mathbf{x} - \boldsymbol{\alpha}(s)\rangle$. Then, the following statements hold:

(1) $h_{\mathbf{x}_0}(s) = 0$ if and only if there exists u, $v \in \mathbb{R}$, such that $\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = u\mathbf{e}_1(s_0) + v\mathbf{e}_3(s_0)$.

(2) $h_{\mathbf{x}_0}(s_0) = h'_{x_0}(s_0) = 0$ if and only if there exists $u \in \mathbb{R}$ such that

$$\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = u\left(\frac{\kappa_3\mathbf{e}_1 + \kappa_1\mathbf{e}_3}{\sqrt{\kappa_3^2 + \kappa_1^2}}\right)(s_0).$$

(A). Suppose that $\delta(s_0) \neq 0$. Then we have the following:

(1) $h_{\mathbf{x}_0}(s_0) = h'_{x_0}(s_0) = h''_{\mathbf{x}_0}(s_0) = 0$ if and only if

$$\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = -\frac{\kappa_1}{\delta\sqrt{\kappa_3^2 + \kappa_1^2}} \frac{\kappa_3\mathbf{e}_1 + \kappa_1\mathbf{e}_3}{\sqrt{\kappa_3^2 + \kappa_1^2}}(s_0). \tag{1}$$

(2) $h_{\mathbf{x}_0}(s_0) = h'_{x_0}(s_0) = h''_{\mathbf{x}_0}(s_0) = h^{(3)}_{\mathbf{x}_0}(s_0) = 0$ if and only if $\sigma(s_0) = 0$, and (1).

(3) $h_{\mathbf{x}_0}(s_0) = h'_{x_0}(s_0) = h''_{\mathbf{x}_0}(s_0) = h^{(3)}_{\mathbf{x}_0}(s_0) = h^{(4)}_{\mathbf{x}}(s_0) = 0$ if and only if $\sigma(s) = \sigma'(s) = 0$, and (1).

(B). Suppose that $\delta(s_0) = 0$. Then we have the following:

(1) $h_{\mathbf{x}_0}(s_0) = h'_{x_0}(s_0) = h''_{\mathbf{x}_0}(s_0) = 0$ if and only if $\kappa_1(s_0) = 0$, the is, $\kappa_2(s_0) = 0$, $\kappa_1'(s_0) + \kappa_2(s_0)\kappa_3(s_0) = 0$, and there exists $u \in \mathbb{R}$ such that $\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = u\mathbf{e}_1(s_0)$.

(2) $h_{\mathbf{x}_0}(s_0) = h'_{x_0}(s_0) = h''_{\mathbf{x}_0}(s_0) = h^{(3)}_{\mathbf{x}_0}(s_0) = 0$ if and only if one of the following conditions holds

(a) $\delta'(s) \neq 0$, $\kappa_1(s_0) = 0$, that is, $\kappa_1(s_0) = 0$, $\kappa_1'(s_0) + \kappa_2(s_0)\kappa_3(s_0) = 0$,

$$2\kappa_2(s_0)\kappa_3'(s_0) + \kappa_2'(s_0)\kappa_3(s_0) + \kappa_1''(s_0) \neq 0$$

and

$$\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = -\frac{\kappa_2'(s_0)}{2\kappa_2(s_0)\kappa_3'(s_0) + \kappa_2'(s_0)\kappa_3(s_0) + \kappa_1''(s_0)}\mathbf{e}_1(s_0),$$

(b) $\delta'(s) = 0$, $\kappa_1(s_0) = \kappa_1'(s_0) = 0$, that is,

$$\kappa_1(s_0) = \kappa_1'(s_0) = \kappa_2(s_0) = 0,\ \kappa_1''(s_0) + \kappa_2'(s_0)\kappa_3(s_0) = 0,$$

and there exists $u \in \mathbb{R}$ such that $\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = u\mathbf{e}_1(s_0)$.

Proof. Since $h_{\mathbf{x}_0}(s) = \langle \mathbf{e}_2(s), \mathbf{x}_0 - \boldsymbol{\alpha}(s)\rangle$, we have the following:
(i) $h_{\mathbf{x}_0} = \langle \mathbf{e}_2, \mathbf{x}_0 - \boldsymbol{\alpha}\rangle$,
(ii) $h_{\mathbf{x}_0}' = \langle -\kappa_1\mathbf{e}_1 + \kappa_3\mathbf{e}_3, \mathbf{x}_0 - \boldsymbol{\alpha}\rangle$,
(iii) $h_{\mathbf{x}_0}'' = \kappa_1 + \left\langle -\left(\kappa_1' + \kappa_2\kappa_3\right)\mathbf{e}_1 - \left(\kappa_1^2 + \kappa_3^2\right)\mathbf{e}_2 - \left(\kappa_3' + \kappa_1\kappa_2\right)\mathbf{e}_3, \mathbf{x}_0 - \boldsymbol{\alpha}\right\rangle$,
(iv) $h_{\mathbf{x}_0}^{(3)} = 2\kappa_1' + \kappa_2\kappa_3 + \langle\left(\kappa_1\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) + \kappa_2'\kappa_3 + 2\kappa_2\kappa_3' - \kappa_1''\right)\mathbf{e}_1 -$
$3\left(\kappa_1\kappa_1' + \kappa_3\kappa_3'\right)\mathbf{e}_2 + \left(\kappa_3\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) - \kappa_1'\kappa_1 - 2\kappa_1\kappa_2' - \kappa_3''\right)\mathbf{e}_3, \mathbf{x}_0 - \boldsymbol{\alpha}\rangle$,
(v) $h_{\mathbf{x}_0}^{(4)} = 3\kappa_2'' - 3\kappa_1\kappa_3' + \kappa_2\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) + \langle[\kappa_2'\left(3\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) +$
$\kappa_2\left(3\kappa_1\kappa_1' + 5\kappa_2\kappa_2' + 5\kappa_3\kappa_3'\right) - \kappa_1\kappa_3\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) + \kappa_1''\kappa_3 +$
$3\kappa_1'\kappa_3' + 3\kappa_1\kappa_3'' - \kappa_2'']\mathbf{e}_1 + [\left(\kappa_1^2 + \kappa_3^2\right)\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) + 2\kappa_2\left(\kappa_3\kappa_1' - \kappa_1\kappa_3'\right) -$
$3\left(\kappa_1'^2 + \kappa_3'^2\right) - 4\left(\kappa_1\kappa_1'' + \kappa_3\kappa_3''\right)]\mathbf{e}_2 + [-\kappa_3'\left(3\kappa_2^2 + \kappa_1^2 + \kappa_3^2\right) -$
$\kappa_3\left(3\kappa_2\kappa_2' + 5\kappa_1\kappa_1' + 5\kappa_3\kappa_3'\right) + \kappa_1\kappa_3\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) - \kappa_1\kappa_2'' -$
$3\kappa_1'\kappa_2' - 3\kappa_2\kappa_1'' - \kappa_3'']\mathbf{e}_3, \mathbf{x}_0 - \boldsymbol{\alpha}\rangle$.

By definition $h_{\mathbf{x}_0}(s_0) = 0$ if and only if $\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = u\mathbf{e}_1(s_0) + w\mathbf{e}_2(s_0) + v\mathbf{e}_3(s_0)$, and

$$\langle \mathbf{x}_0 - \boldsymbol{\alpha}(s_0), \mathbf{e}_2(s_0)\rangle = 0.$$

Then, we have $\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = u\mathbf{e}_1(s_0) + v\mathbf{e}_3(s_0)$. Therefore, (1) holds.

By (ii), $h_{\mathbf{x}_0}(s_0) = h_{\mathbf{x}_0}'(s_0) = 0$ if and only if $\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = u\mathbf{e}_1(s_0) + v\mathbf{e}_3(s_0)$, and

$$-u\kappa_1(s_0) + v\kappa_3(s_0) = 0.$$

If $\kappa_1(s_0) \neq 0$, and $\kappa_3(s_0) \neq 0$, then we have

$$u = v\frac{\kappa_3(s_0)}{\kappa_1(s_0)},\ \text{and}\ v = u\frac{\kappa_1(s_0)}{\kappa_3(s_0)}.$$

Then there exists $v \in \mathbb{R}$ such that

$$\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = v\frac{\kappa_3\mathbf{e}_1 + \kappa_1\mathbf{e}_3}{\sqrt{\kappa_3^2 + \kappa_1^2}}(s_0).$$

Suppose that $\kappa_1(s_0) = 0$. Then we have $\kappa_3(s_0) \neq 0$; so that $\kappa_3(s_0)v = 0$. Therefore, we have

$$\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = u\mathbf{e}_1(s_0) = \pm u\frac{\kappa_3\mathbf{e}_1 + \kappa_1\mathbf{e}_3}{\sqrt{\kappa_3^2 + \kappa_3^2}}(s_0).$$

If $\kappa_3(s_0) = 0$, then we have $\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = v\mathbf{e}_3(s_0)$. Therefore, (2) holds.

By (iii) $h_{\mathbf{x}_0}(s_0) = h_{\mathbf{x}_0}'(s_0) = h_{\mathbf{x}_0}''(s_0) = 0$ if and only if

$$\mathbf{x}_0 - \boldsymbol{\alpha}(s_0) = v\frac{\kappa_3\mathbf{e}_1 + \kappa_3\mathbf{e}_2}{\sqrt{\kappa_3^2 + \kappa_1^2}}(s_0),$$

and

$$\kappa_1(s_0) - v\frac{\kappa_3\left(\kappa_2\kappa_3 + \kappa_1'\right) + \kappa_1\left(\kappa_1\kappa_2 + \kappa_3'\right)}{\sqrt{\kappa_3^2 + \kappa_1^2}}(s_0) = 0.$$

It follows that

$$\frac{\kappa_1}{\sqrt{\kappa_3^2+\kappa_1^2}}(s_0)+v\left(\kappa_2-\frac{\kappa_1\kappa_3'-\kappa_3\kappa_1'}{\sqrt{\kappa_3^2+\kappa_1^2}}\right)(s_0)=0.$$

Thus,

$$\delta(s_0)=\kappa_2(s_0)-\frac{\kappa_1\kappa_3'-\kappa_3\kappa_1'}{\sqrt{\kappa_3^2+\kappa_1^2}}(s_0)\neq 0,\text{ and } v=-\frac{\kappa_1}{\sqrt{\kappa_3^2+\kappa_1^2}}(s_0)$$

or $\delta(s_0)=0$, and $\kappa_1(s_0)=0$. This completes the proof of (A), (3) and (B), (1).

Suppose that $\delta(s_0)\neq 0$. By (iv), $h_{\mathbf{x}_0}(s_0)=h'_{\mathbf{x}_0}(s_0)=h''_{\mathbf{x}_0}(s_0)=\overset{(3)}{h}_{\mathbf{x}_0}(s_0)=0$ if and only if

$$\begin{aligned}&2\kappa_1'+\kappa_2\kappa_3-\frac{\kappa_1}{\delta\sqrt{\kappa_3^2+\kappa_1^2}}\left(\kappa_1\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)+\kappa_1'\kappa_3+2\kappa_2\kappa_3'-\kappa_1''\right)\\&+\frac{\kappa_1}{\sqrt{\kappa_3^2+\kappa_1^2}}\left(\kappa_1\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)-\kappa_2'\kappa_1-2\kappa_2\kappa_1'-\kappa_3''\right)\\&=0\end{aligned}$$

at $s=s_0$. It follows that

$$2\kappa_1'(s_0)+\kappa_1(s_0)\kappa_3(s_0)-\frac{\kappa_2}{\delta}\left(\kappa_2'+\frac{2\kappa_2\left(\kappa_1'\kappa_1+\kappa_3'\kappa_3\right)}{\kappa_3^2+\kappa_1^2}-\frac{\kappa_3''\kappa_1-\kappa_1''\kappa_3}{\kappa_3^2+\kappa_1^2}\right)(s_0).$$

Since

$$\delta'=\kappa_2'-2\frac{\left(\kappa_1'\kappa_1+\kappa_3'\kappa_3\right)\left(\kappa_3'\kappa_1-\kappa_1'\kappa_3\right)}{\kappa_3^2+\kappa_1^2}-\frac{\kappa_3''\kappa_1-\kappa_1''\kappa_3}{\kappa_3^2+\kappa_1^2},$$

and

$$2\kappa_1'(s_0)+\kappa_2(s_0)\kappa_3(s_0)-\kappa_1(s_0)\frac{\delta'(s_0)}{\delta(s_0)}-2\kappa_1\frac{\kappa_1'\kappa_1+\kappa_3'\kappa_3}{\kappa_3^2+\kappa_1^2}(s_0)=0.$$

Further, by applying the relation

$$\left(\frac{\kappa_1}{\sqrt{\kappa_3^2+\kappa_1^2}}\right)'=\frac{\kappa_3}{\sqrt{\kappa_3^2+\kappa_1^2}}\frac{\kappa_3'\kappa_1-\kappa_1'\kappa_3}{\kappa_3^2+\kappa_1^2}=\frac{\kappa_3}{\sqrt{\kappa_3^2+\kappa_1^2}}\left(\delta-\kappa_2\right)$$

to the above. Then we have

$$\begin{aligned}&\delta(s_0)\sqrt{\kappa_3^2(s_0)+\kappa_1^2(s_0)}\left(\frac{\kappa_3}{\sqrt{\kappa_3^2(s_0)+\kappa_1^2(s_0)}}+\left(\frac{\kappa_1}{\delta\sqrt{\kappa_3^2(s_0)+\kappa_1^2(s_0)}}\right)'\right)(s_0)\\&=\delta(s_0)\sigma(s_0)\sqrt{\kappa_3^2(s_0)+\kappa_1^2(s_0)}=0,\end{aligned}$$

so that $\sigma(s_0)$. The converse assertion also holds.

Suppose that $\delta(s_0)=0$. Then by (iv), $h_{\mathbf{x}_0}(s_0)=h'_{\mathbf{x}_0}(s_0)=h''_{\mathbf{x}_0}(s_0)=\overset{(3)}{h}_{\mathbf{x}_0}(s_0)=0$ if and only if $\kappa_1(s_0)=0$, that is, $\kappa_1(s_0)=0$, $\kappa_1'(s_0)+\kappa_2(s_0)\kappa_3(s_0)=0$, there exists $u\in\mathbb{R}$ such that $\mathbf{x}_0-\boldsymbol{\alpha}(s_0)=u\mathbf{e}_1(s_0)$, and

$$2\kappa_1'(s_0)+\kappa_2(s_0)\kappa_3(s_0)-u\left(2\kappa_2(s_0)\kappa_3'(s_0)+\kappa_2'(s_0)\kappa_3(s_0)+\kappa_3''(s_0)\right)=0.$$

Since $\delta(s_0)=0$, and $\kappa_1(s_0)$, we have $\kappa_2'(s_0)\kappa_3(s_0)+\kappa_1''(s_0)=0$, so that

$$\kappa_1'(s_0)-v\left(2\kappa_2(s_0)\kappa_3'(s_0)+\kappa_2'(s_0)\kappa_3(s_0)+\kappa_3''(s_0)\right)=0.$$

It follows that $2\kappa_2(s_0)\kappa_3'(s_0)+\kappa_2'(s_0)\kappa_3(s_0)+\kappa_3''(s_0)\neq 0$, and

$$u=\frac{\kappa_1'(s_0)}{2\kappa_2(s_0)\kappa_3'(s_0)+\kappa_2'(s_0)\kappa_3(s_0)+\kappa_3''(s_0)}$$

or

$$2\kappa_2(s_0)\kappa_3'(s_0)+\kappa_2'(s_0)\kappa_3(s_0)+\kappa_3''(s_0)=0,\text{ and }\kappa_1'(s_0)=0.$$

Therefore we have (B), (2), (a) or (b). By similar arguments to the above, we have (A), (5). This completes the proof. □

3.2 | Unfolding of functions by one-variable

In this subsection, we use some general results on the singularity theory for families of function germs[8,21]. Let $F\colon \left(\mathbb{R}\times\mathbb{R}^r,(s_0,\mathbf{x}_0)\right)\to\mathbb{R}$ be a smooth function, and $f(s)=F_{\mathbf{x}_0}(s,\mathbf{x}_0)$. Then F is called an r-parameter unfolding of $f(s)$. We say that $f(s)$ has A_k-*singularity* at s_0 if $f^{(p)}(s_0)=0$ for all $1\leq p\leq k$, and $f^{(k+1)}(s_0)\neq 0$. We also say that f has $A_{\geqslant k}$-*singularity* ($k\geqslant 1$) at s_0. Let the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 be $j^{(k-1)}\left(\frac{\partial F}{\partial x_i}\left(s,\mathbf{x}_0\right)\right)\left(s_0\right)=\Sigma_{j=0}^{k-1}L_{ji}\left(s-s_0\right)^j$ (without the constant term), for $i=1,...,r$. Then $F(s)$ is called an p-versal unfolding if the $k\times r$ matrix of coefficients $\left(L_{ji}\right)$ has rank k ($k\leq r$). So, we write important set about the unfolding relative to the above notations.

We now state important set about the unfolding relative to the above notations. The discriminant set of F is the set

$$\mathfrak{D}_F=\left\{\mathbf{x}\in\mathbb{R}^r|\text{ there exists } s \text{ with } F\left(s,\mathbf{x}\right)=\frac{\partial F}{\partial s}\left(s,\mathbf{x}\right)=0\text{ at }(s,\mathbf{x})\right\}. \tag{20}$$

A well-known classification[9,7,6] follows:

Theorem 4. Let $F\colon \left(\mathbb{R}\times\mathbb{R}^r,(s_0,\mathbf{x}_0)\right)\to\mathbb{R}$ be an r-parameter unfolding of $f(s)$, which has the A_k singularity at s_0. Suppose that F is a p-versal unfolding.

(a). If $k=2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\mathbf{C}\times\mathbb{R}^{r-1}$;

(b) If $k=3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW\times\mathbb{R}^{r-2}$.

Hence, for the proof of Theorem 3, we have the following proposition:

Proposition 2. Let $\boldsymbol{\alpha}\colon I\to\mathbb{R}^3$ be a unit speed curve with $\kappa_2^2+\kappa_3^2\neq 0$, and $h_{\mathbf{x}_0}(s)=\langle\mathbf{e}_2(s),\mathbf{x}-\boldsymbol{\alpha}(s)\rangle$. If $h_{\mathbf{x}_0}$ has an A_k-singularity ($k=2,\ 3$) at $s_0\in\mathbb{R}$, then H is a $p-$versal unfolding of $h_{\mathbf{x}_0}(s_0)$.

Proof. Let $\mathbf{x}=\left(x_1,x_2,x_3\right)$, $\boldsymbol{\alpha}=\left(\alpha_1,\alpha_2,\alpha_3\right)$ and $\mathbf{e}_2=\left(l_1,l_2,l_3\right)$. Then, we have

$$H(s,\mathbf{x})=\left(x_1-\alpha_1(s)\right)l_1(s)+\left(x_2-\alpha_2(s)\right)l_2(s)+\left(x_3-\alpha_3(s)\right)l_3(s) \tag{21}$$

and

$$\frac{\partial H}{\partial x_i}(s,\mathbf{x})=l_i(s),\ \text{(i=1, 2, 3)}.$$

Therefore, the 2-jets of $\frac{\partial H}{\partial x_i}$ at s_0 is as follows:

$$j^2\frac{\partial H}{\partial x_0}(s_0,\mathbf{x}_0)=l_i(s_0)+l_i'(s_0)(s-s_0)+\frac{1}{2}l_i''(s_0)\left(s-s_0\right)^2.$$

We consider the following matrix:

$$A=\begin{pmatrix} l_1(s_0) & l_2(s_0) & l_3(s_0)\\ l_1'(s_0) & l_2'(s_0) & l_3'(s_0)\\ l_1''(s_0) & l_2''(s_0) & l_3''(s_0)\end{pmatrix}=\begin{pmatrix}\mathbf{e}_2(s_0)\\ \mathbf{e}_2'(s_0)\\ \mathbf{e}_2''(s_0)\end{pmatrix}. \tag{22}$$

By the formula in Eq. (4), we have

$$A(s_0)=\begin{pmatrix}\mathbf{e}_2\\ -\kappa_1\mathbf{e}_1+\kappa_3\mathbf{e}_3\\ -\left(\kappa_2\kappa_3+\kappa_1'\right)\mathbf{e}_1-\left(\kappa_2^2+\kappa_3^2\right)\mathbf{e}_2+\left(\kappa_3'-\kappa_1\kappa_2\right)\mathbf{e}_3\end{pmatrix}(s_0). \tag{23}$$

Since the orthonormal frame $\{\mathbf{e}_1(s),\mathbf{e}_2(s),\ \mathbf{e}_3(s)\}$ is a basis of $\mathbb{R}^3$, then the rank of $A(s_0)$ is equal to the rank of

$$\begin{pmatrix} 0 & 1 & 0\\ -\kappa_1(s_0) & 0 & -\kappa_3(s_0)\\ -\left(\kappa_2\kappa_3+\kappa_1'\right)(s_0) & -\left(\kappa_2^2+\kappa_3^2\right)(s_0) & \left(\kappa_3'-\kappa_1\kappa_2\right)(s_0)\end{pmatrix}. \tag{24}$$

This means rank $A = 3$, if and only if

$$-\kappa_1\left(\kappa_3' - \kappa_1\kappa_2\right) + \kappa_3\left(\kappa_2\kappa_3 + \kappa_1'\right) = \kappa_2\left(\kappa_1^2 + \kappa_3^2\right) - \left(\kappa_1\kappa_3' - \kappa_1'\kappa_3\right) \neq 0.$$

The last condition is equivalent to the condition $\delta(s_0) \neq 0$.
Also, the rank of

$$\begin{pmatrix} \mathbf{e}_2(s_0) \\ \mathbf{e}_2'(s_0) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_2(s_0) \\ -\kappa_1'(s_0)\mathbf{e}_1(s_0) + \kappa_3'(s_0)\mathbf{e}_3(s_0) \end{pmatrix}$$

is always two. If $h_{\mathbf{x}_0}$ has an A_k-singularity (k = 2, 3) at s_0, then H is p-versal unfolding of $h_{\mathbf{x}_0}$. This completes the proof. □

3.3 | Proof of Theorem 3

By direct calculation, we have

$$\mathbf{y}_s \times \mathbf{y}_v = \left(\frac{\kappa_1}{\sqrt{\kappa_3^2 + \kappa_1^2}} + u\delta\right)\mathbf{e}_2.$$

Therefore, (s_0, v_0) is non-singular if and only if $\mathbf{y}_s \times \mathbf{y}_v \neq \mathbf{0}$. This condition is equivalent to

$$\frac{\kappa_1(s_0)}{\sqrt{\kappa_3^2(s_0) + \kappa_1^2(s_0)}} + u_0\delta(s_0) \neq 0.$$

This completes the proof of the assertion (1).

By Proposition 1-(2), $\mathfrak{D}_H$ is the image of the D-developable surface. Suppose $\delta(s_0) \neq 0$. By Proposition 1-(A)-(1), (2), and (3), $h_{\mathbf{x}_0}(s_0)$ has an A_2-type singularity (respectively, an A_3-type singularity) at $s = s_0$ if and only if

$$u_0 = \frac{\kappa_1(s_0)}{\delta(s_0)\sqrt{\kappa_3^2(s_0) + \kappa_1^2(s_0)}}$$

and $\sigma(s_0) \neq 0$ (respectively, $\sigma(s_0) = 0$ and $\sigma'(s_0) \neq 0$). By Theorem 4 and Proposition 1, we have (2)-(i) and (3). Suppose $\delta(s_0) = 0$. By Proposition 1-(B)-(1) and (2), $h_{\mathbf{x}_0}(s_0)$ has an A_2-type singularity if and only if $\kappa_1(s_0) = 0$, and

$$\kappa_1'(s_0) \neq 0 \text{ or } \kappa_1'(s_0) + v_0\left(2\kappa_1(s_0)\kappa_3'(s_0) + \kappa_1'(s_0)\kappa_3(s_0) - \kappa_3''(s_0)\right) \neq 0.$$

Following from Theorem 4 and Proposition 2, we obtain (2)-(iii). This completes the proof.

4 | EXAMPLES

In this section, we give some examples.

Example 1. We consider a curve $\gamma : I \to \mathbb{R}^3, I \subset \mathbb{R}$, defined by

$$\gamma(t) = (t, \frac{1}{2}t^2, \frac{1}{3}t^3).$$

It is located in a surface which is globally diffeomorphic to standard cuspidal edge. We take (0,0,1) is the projection vector. Then the Q-frames are given by

$$e_1(t) = \frac{1}{\sqrt{1 + t^2 + t^4}}(1, t, t^2);$$
$$e_2(t) = \frac{1}{\sqrt{1 + t^2}}(t, -1, 0);$$
$$e_3(t) = \frac{1}{\sqrt{1 + t^2 + t^4}\sqrt{1 + t^2}}(t^2, t^3, -1 - t^2).$$

We obtained that

$$e_1'(t) = \left(\frac{-t-2t^3}{(1+t^2+t^4)^{\frac{3}{2}}}, \frac{1-t^4}{(1+t^2+t^4)^{\frac{3}{2}}}, \frac{2t+t^3}{(1+t^2+t^4)^{\frac{3}{2}}} \right);$$

$$e_2'(t) = \left(\frac{1}{(1+s^2)^{\frac{3}{2}}}, \frac{s}{(1+s^2)^{\frac{3}{2}}}, 0 \right).$$

The Q-curvature functions of γ are given by

$$\kappa_1(t) = \left\langle e_1', e_2 \right\rangle = -\frac{1}{\sqrt{1+t^2+t^4}\sqrt{1+t^2}};$$

$$\kappa_3(t) = \left\langle e_2', e_3 \right\rangle = \frac{t^2}{\sqrt{1+t^2+t^4}(1+t^2)}.$$

The directional developable surface is

$$\begin{aligned} \phi(t,v) &= \gamma(t) + ve(t) \\ &= (t, \frac{1}{2}t^2, \frac{1}{3}t^3 + v), v \in \mathbb{R}, \end{aligned}$$

where

$$e(t) = \frac{\kappa_3 e_1 + \kappa_1 e_3}{\sqrt{\kappa_3^2 + \kappa_1^2}} = (0, 0, 1).$$

The pictures of the curve γ and the D-developable surface see Figure 3 and Figure 4.

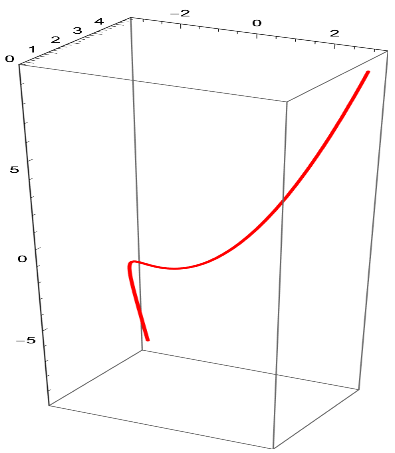

Figure 3 The curve γ.

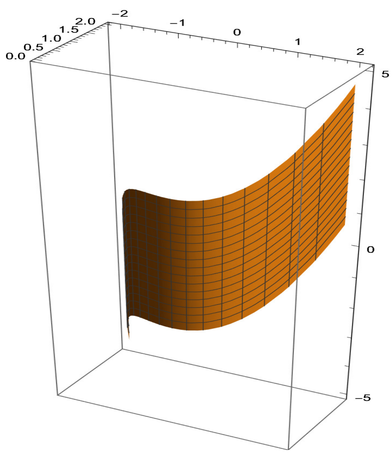

Figure 4 D-developable surface of γ.

We consider an other example.

Example 2. Let $\gamma : I \to \mathbb{R}^3, I \subset \mathbb{R}$ be a curve on the standard swallowtail defined by

$$\gamma(t) = (3t^4, 4t^3, 0),$$

Similarly, we take (a,b,1) is the projection vector (one will see that $\kappa_1^2+\kappa_3^2=0$ if the third component is zero). Then the Q-frames are given by

$$e_1(t)=\frac{1}{\sqrt{1+t^2}}(t,1,0)\,;$$
$$e_2(t)=\frac{1}{\sqrt{1+t^2+(bt-a)^2}}(1,-t,bt-a);$$
$$e_3(t)=\frac{1}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2+(bt-a)^2}}(bt-a,-bt^2+at,-t^2-1).$$

We obtained that

$$e_1'(t)=\frac{1}{(1+t^2)^{\frac{3}{2}}}(1,-t,0)\,;$$
$$e_2'(t)=\frac{1}{(1+t^2+(bt-a)^2)^{\frac{3}{2}}}\left(-b^2t-t+ab,abt-a^2-1,at+b\right).$$

The Q-curvature functions of γ are given by

$$\kappa_1(t)=\left\langle e_1',e_2\right\rangle=\frac{1+t^2}{(1+t^2)^{\frac{3}{2}}(1+t^2+(bt-a)^2)^{\frac{1}{2}}};$$
$$\kappa_3(t)=\left\langle e_2',e_3\right\rangle=\frac{-a(b^2+1)t^3+b(2a^2-b^2-1)t^2+a(-a^2+2b^2-1)t-b(a^2+1)}{(1+t^2)^{\frac{1}{2}}(1+t^2+(bt-a)^2)^2}.$$

It can be directly calculate that the ruling vector $e(t)$ is paralleled with the projection vector, which is

$$e(t)=\frac{\kappa_3e_1+\kappa_1e_3}{\sqrt{\kappa_3^2+\kappa_1^2}}\|(a,b,1).$$

Then The directional developable surface is

$$\phi(t,v)=\gamma(t)+ve(t)=(3t^4,4t^3,0)+v(a,b,1),v\in\mathbb{R}.$$

It shows the intrinsic beauty of geometry that the D-developable surface of γ is exactly the projection surface of γ alongside the projection vector. The pictures of the curve γ and the D-developable surface see Figure 5 and Figure 6.

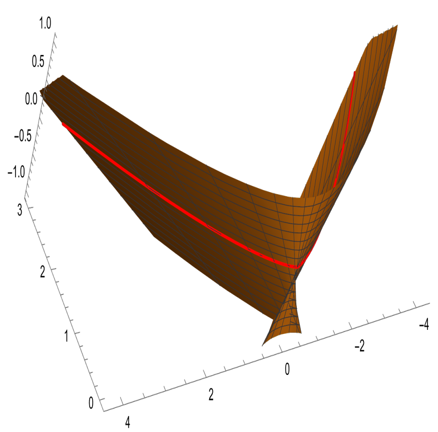

Figure 5 The red curve γ located in a standard swallowtail.

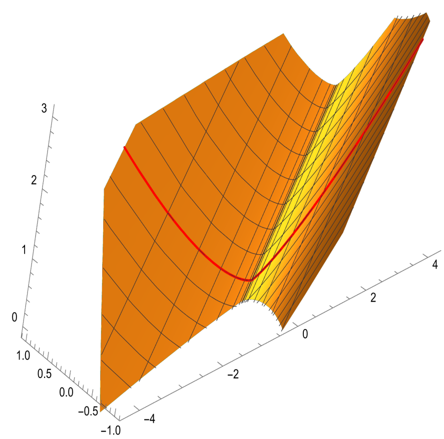

Figure 6 D-developable surface of γ.

5 | CONCLUSION

In this paper, we shown a new version of developable ruled surfaces in Euclidean 3-space. We establish the quasi-frame along a unit speed curve and introduce a directional developable ruled surface. Applying the unfolding theory, we classify the generic properties, and present new two invariants related to the singularities of this surface. It is demonstrated that the generic singularities are cuspidal edge and swallowtail, and the types of these singularities can be characterized by these invariants, respectively. Finally, examples are illustrated to explain the applications of the theoretical results.

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Author contributions

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Conflict of interest

The authors declare no potential conflict of interests.

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