Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ARTICLE TYPE

Blow-up of solutions of a non-linear wave equation with fractional damping and infinite memory

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Abstract

We consider a non-linear wave equation with an internal fractional damping, a polynomial source and an infinite memory. Using the semi-group theory, we get the existence of a local weak solution. Moreover, we show under some conditions, local solutions may blow up a in finite time; this is achieved by constructing a suitable Lyapunov functional.

KEYWORDS:
Fractional damping, Relaxation function, blow up
MSC : 35L70

1 | INTRODUCTION

We investigate the following problem:

$$(P)\begin{cases} y_{tt} - \Delta y + \displaystyle\int_0^{+\infty} g(\tau)\Delta y(t-\tau)\,d\tau + \partial_t^{\alpha,\beta} y(t) = |y|^{p-2}y, & \text{in } \Omega \times (0,\infty), \\ y = 0, & \text{on } \partial\Omega \times (0,\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & \text{in } \Omega. \end{cases}$$

where $p > 2$, Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and g is a function which will be specified later. The notation $\partial_t^{\alpha,\beta}$ stands for the modified Caputo's fractional derivative (see [1,2]) defined by:

$$\partial_t^{\alpha,\beta} u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} u_s(s)\,ds, \quad 0 < \alpha < 1, \beta \geq 0.$$

Partial differential equations with fractional derivatives arise in biology, physics, electronics and vibrations, etc. In the last years, the control of PDEs with fractional derivatives has been studied in [3,4,5,6].

It is well known that in the absence of an internal fractional damping, the polynomial source causes finite time blow up of solutions with negative initial energy (see [7,8,9,10]). Whereas, in the presence of non-linear damping, Georgiev and Todorova [11], proved under the assumption $p \leq m$, that the solution is global. However, for the opposite case, solutions may blows up in a finite time.

In the presence of fractional damping, The linear wave equation with the Riemann–Liouville fractional derivatives has been considered by Matignon et al. in [12]. The authors proved well-posedness and asymptotic stability. Later on, Kirane and Tatar [13], proved an exponential growth result. By using a new argument, Tatar[14], extended Kirane and TatarâĂŹs result to a larger class of initial data. The same author[15], proved a finite time blow up result. Recently, by writing the wave equation with a dynamic boundary dissipation of fractional derivative type as an augmented system, Aounallah et al.[16,17] proved the existence and decay properties of the sought solutions. For infinite memory problems, Appleby et al.[18], established an exponential decay of a linear integro-differential equation. Later on, Guesmia[19] investigated a class of hyperbolic problems and established a more general decay result. In[20], by describing the fractional damping by means of a suitable diffusion equation, the problem (P) was put into an augmented model which can be easily tackled. To the best of our knowledge, a non-linear wave equation with an internal fractional damping and infinite memory has not been studied yet. In addition, the finite time blow-up of the solution for this problems has not been addressed. The paper is organized as follows: In Sec. 2, we present some assumptions and tools needed to demonstrate the main results. In Sec. 3, we use the semi-group theory[22,23] to prove the existence of a local weak solution. In Sec. 4, we use a judicious Lyapunov functional to prove the finite time blow-up of a certain solution.

2 | PRELIMINARIES

In this section, we provide some material needed for the proof of our results. We need the following assumptions:

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function such that

$$g(0) > 0, \quad g_0 = \int_0^\infty g(\tau)\,d\tau = 1 - \lambda > 0;$$

(G2) there exists a positive constant θ such that:

$$g'(t) \leq -\theta g(t), \quad t \geq 0.$$

We state without proof the following claims:

Lemma 1. The following inequality holds:

$$\int_\Omega \left[\int_0^{+\infty} g(\tau)\nabla w(\tau)\,d\tau\right]^2 dx \leq (1-\lambda)\int_0^{+\infty} g(\tau)\|\nabla w(\tau)\|_2^2\,d\tau.$$

Lemma 2. [20] Let

$$b := \frac{\sin(\alpha\pi}{\pi}$$

and let η be the function:

$$\eta(\xi) := |\xi|^{\frac{(2\alpha-1)}{2}}, \quad \xi \in \mathbb{R}, \quad 0 < \alpha < 1.$$

Then the relationship between the ”input” U and the ”output” O of the system

$$\begin{cases} \partial_t \phi(\xi,t) + (\xi^2+\beta)\phi(\xi,t) - U(x,t)\eta(\xi) = 0, \ \xi \in \mathbb{R}, t > 0, \beta \geq 0, \\ \phi(\xi,0) = 0, \\ O(t) := b\displaystyle\int_{-\infty}^{+\infty} \phi(\xi,t)\eta(\xi)\,d\xi \end{cases} \tag{1}$$

is given by

$$O := I^{1-\alpha,\beta}U,$$

where

$$I^{\alpha,\beta}u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\beta(t-s)}u(s)\,ds.$$

Lemma 3. [21] For all $\lambda \in D_\beta = \{\lambda \in \mathbb{C} : \Re e\lambda + \beta > 0\} \cup \{\lambda \in \mathbb{C} : \Im m\lambda \neq 0\}$,

$$A_\lambda := \int_{-\infty}^{+\infty} \frac{\eta^2(\xi)}{\lambda + \beta + \xi^2}\, d\xi = \frac{\pi}{\sin(\alpha\pi)}(\lambda+\beta)^{\alpha-1}.$$

Now, similarly to [24,25], we introduce the following new variable:

$$\mu^t(x,\tau) = y(x,t) - y(x,t-\tau), \tag{2}$$

where μ^t is the relative history of y that satisfies

$$\mu^t_t(x,\tau) - y_t(x,t) + \mu^t_\tau(x,\tau) = 0, \quad x \in \Omega,\ \mathrm{t}, \tau > 0. \tag{3}$$

Then, by using Lemma 2 and (2), system (P) takes the form :

$$(P')\begin{cases} y_{tt} - \lambda\Delta y(t) - \displaystyle\int_0^{+\infty} g(\tau)\Delta\mu^t(x,\tau)\, d\tau & \\ +b\displaystyle\int_{-\infty}^{+\infty} \phi(\xi,t)\eta(\xi)\, d\xi = |y|^{p-2}y, & x \in \Omega,\ \mathrm{t} > 0, \\ \partial_t\phi(\xi,t) + (\xi^2+\beta)\phi(\xi,t) - y_t(x,t)\eta(\xi) = 0, & \xi \in \mathbb{R},\ \mathrm{t} > 0, \beta \geq 0, \\ \mu^t_t(x,\tau) + \mu^t_\tau(x,\tau) = y_t(x,t), & x \in \Omega,\ \mathrm{t}, \tau > 0, \\ y = \mu^t(x,\tau) = 0, & x \in \partial\Omega,\ \mathrm{t}, \tau > 0, \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in \Omega, \\ \mu^t(x,0) = 0, \quad \mu^0(x,\tau) = y_0(x) - y_0(x,-\tau), & x \in \Omega,\ \mathrm{t}, \tau > 0, \\ \phi(\xi,0) = 0, & x \in \Omega, \xi \in \mathbb{R}. \end{cases}$$

Lemma 4. The energy

$$\begin{aligned} E(t) := \frac{1}{2}\|y_t(t)\|_2^2 + \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\phi(\xi,t)|^2\, d\xi dx + \frac{\lambda}{2}\|\nabla y(t)\|_2^2 \\ - \frac{1}{p}\|y(t)\|_p^p + \frac{1}{2}\int_0^{+\infty} g(\tau)\|\nabla\mu^t(\tau)\|_2^2\, d\tau \end{aligned} \tag{4}$$

satisfies

$$\begin{aligned} \frac{dE(t)}{dt} = \frac{1}{2}\int_0^{+\infty} g'(\tau)\|\nabla\mu^t(\tau)\|_2^2\, d\tau \\ - b\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)|\phi(\xi,t)|^2\, d\xi dx \leq 0. \end{aligned} \tag{5}$$

Proof. Multiplying the first equation in (P') by y_t, integrating over Ω and using integration by parts, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\left\|y_t(t)\right\|_2^2+\frac{\lambda}{2}\left\|\nabla y(t)\right\|_2^2-\frac{1}{p}\left\|y(t)\right\|_p^p\right\}\\
&+b\int_{\Omega} y_t \int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)\,d\xi dx\\
&-\int_{\Omega} y_t\int_{0}^{+\infty} g(\tau)\Delta\mu^t(\tau)\,d\tau dx=0.
\end{aligned}\tag{6}$$

We use (3) to transform the last term of (6) as follows:

$$\begin{aligned}
&-\int_{\Omega} y_t\int_{0}^{+\infty} g(\tau)\Delta\mu^t(\tau)\,d\tau dx\\
&=-\int_{0}^{+\infty} g(\tau)\int_{\Omega}\left(\mu_t^t+\mu_\tau^t\right)\Delta\mu^t(\tau)\,dxd\tau\\
&=-\int_{0}^{+\infty} g(\tau)\int_{\Omega}\mu_t^t\Delta\mu^t(\tau)\,dxd\tau\\
&-\int_{0}^{+\infty} g(\tau)\int_{\Omega}\mu_\tau^t\Delta\mu^t(\tau)\,dxd\tau,
\end{aligned}$$

and integrating by parts, we get

$$\begin{aligned}
-\int_{\Omega} y_t\int_{0}^{+\infty} g(\tau)\Delta\mu^t(\tau)\,d\tau dx&=\frac{d}{dt}\left[\frac{1}{2}\int_{0}^{+\infty} g(\tau)\|\nabla\mu^t(\tau)\|_2^2\,d\tau\right]\\
&-\frac{1}{2}\int_{0}^{+\infty} g'(\tau)\|\nabla\mu^t(\tau)\|_2^2\,d\tau.
\end{aligned}\tag{7}$$

By substituting (7) in (6), we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\left\|y_t(t)\right\|_2^2+\frac{\lambda}{2}\left\|\nabla y(t)\right\|_2^2-\frac{1}{p}\left\|y(t)\right\|_p^p+\frac{1}{2}\int_{0}^{+\infty} g(\tau)\left\|\nabla\mu^t(\tau)\right\|_2^2\,d\tau\right\}\\
&-\frac{1}{2}\int_{0}^{+\infty} g'(\tau)\left\|\nabla\mu^t(\tau)\right\|_2^2 d\tau+b\int_{\Omega} y_t\int_{-\infty}^{+\infty}\eta(\xi)\phi(\xi,t)\,d\xi dx=0.
\end{aligned}\tag{8}$$

Now multiplying the second equation in (P') by $b\phi$ and integrating over $\Omega\times\mathbb{R}$, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{b}{2}\int_{\Omega}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2\,d\xi dx\right\}\\
&+b\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(\xi,t)|^2\,d\xi dx\\
&-b\int_{\Omega} y_t\int_{-\infty}^{+\infty}\eta(\xi)\phi(\xi,t)\,d\xi dx=0.
\end{aligned}\tag{9}$$

By combining (4), (8) and (9), we obtain (5). The lemma is proved. □

3 | WELL-POSEDNESS

In this section, we establish the local existence result for problem (PâĂŹ). First, we define the vector function

$$U = (y, y_t, \phi, \mu^t)^T$$

and a new dependent variable

$$u = y_t.$$

Consequently, problem (PâĂŹ) can be rewritten as follows:

$$(P'')\begin{cases} U_t(t) + AU(t) = J\left(U(t)\right), \\ U(0) = U_0, \end{cases}$$

where the operator $A : D(A) \to \mathcal{H}$ is defined by

$$AU = \begin{pmatrix} -u \\ -\lambda\Delta y - \int\limits_0^{+\infty} g(\tau)\Delta\mu^t(x,\tau)d\tau + b\int\limits_{-\infty}^{+\infty} \phi(x,\xi,t)\eta(\xi)\,d\xi \\ (\xi^2+\beta)\phi - u(x)\eta(\xi) \\ \mu^t_\tau(\tau) - u \end{pmatrix},$$

$$J(U) = (0, |y|^{p-2}y, 0, 0)^T, \tag{10}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega,\mathbb{R})\times L_g^2(\mathbb{R}_+, H_0^1(\Omega))$$

such that

$$L_g^2(\mathbb{R}_+, H_0^1(\Omega)) = \left\{ w : \mathbb{R}_+ \to H_0^1(\Omega), \int\limits_0^{+\infty} g(\tau)\,\|\nabla w(\tau)\|_2^2\,d\tau < \infty \right\};$$

the space $L_g^2(\mathbb{R}_+, H_0^1(\Omega))$ is endowed with the inner product:

$$\langle w_1, w_2\rangle_{L_g^2(\mathbb{R}_+,H_0^2(\Omega))} = \int\limits_0^{+\infty} g(\tau)\int\limits_\Omega \nabla w_1(\tau)\nabla w_2(\tau)\,dx d\tau.$$

For any $U = (y, u, \phi, \mu^t)^T \in \mathcal{H}$ and $\bar{U} = (\bar{y}, \bar{u}, \bar{\phi}, \bar{\mu^t})^T \in \mathcal{H}$, we define the inner product

$$\left\langle U, \bar{U}\right\rangle_{\mathcal{H}} = \int\limits_\Omega \left[\lambda\nabla y.\nabla\bar{y} + u\bar{u}\right]\,dx + b\int\limits_\Omega\int\limits_{-\infty}^{+\infty} \phi\bar{\phi}\,d\xi dx$$
$$+ \int\limits_0^{+\infty} g(\tau)\int\limits_\Omega \nabla\mu^t(\tau)\nabla\bar{\mu^t}(\tau)\,dxd\tau.$$

The domain of A is given by

$$D(A)=\left\{\begin{array}{l}U=(y,u,\phi,\mu^t)^T\in\mathcal{H};\,y\in H^2(\Omega);\,u\in H_0^1(\Omega);\\ \\ (\xi^2+\beta)\phi-u\eta(\xi)\in L^2(\Omega,\mathbb{R});\\ \\ |\xi|\phi\in L^2(\Omega,\mathbb{R});\,\mu^t_\tau\in L^2_g\left(\mathbb{R}_+,H_0^1(\Omega)\right),\end{array}\right\}.$$

Now, we can present the following existence result

Theorem 1. Suppose that

$$\begin{cases}P>2, & \text{if } n=1,2.\\ \\ 2<p<\dfrac{2n}{n-2}, & \text{if } n\geq 3.\end{cases} \tag{11}$$

Assume further that

$$U_0\in\mathcal{H}, \tag{12}$$

then the problem (PâĂŹ) has a unique local solution

$$U\in C\left([0,T),\mathcal{H}\right). \tag{13}$$

Proof. The proof is based on[22]. First, we demonstrate that A is a monotone maximal operator on $\mathcal{H}$. We start by showing that the operator A is monotone. For, for any $U\in D(A)$, using (PâĂİ), we have

$$\begin{aligned}\langle AU,U\rangle_{\mathcal{H}}=&b\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi|^2\,d\xi dx\\ &-\frac{1}{2}\int_0^{+\infty}g'(\tau)\left\|\nabla\mu^t(\tau)\right\|_2^2\,d\tau\geq 0.\end{aligned} \tag{14}$$

So, A is a monotone operator. Next, we will show that the operator (I + A) is onto. For, given $F=(f_1,f_2,f_3,f_4)^T\in\mathcal{H}$, we dwill show that there exists $U\in D(A)$ such that

$$(I+A)U=F;$$

that is,

$$\begin{cases}y-u=f_1\in H_0^1(\Omega),\\ \\ u-\lambda\Delta y-\displaystyle\int_0^{+\infty}g(\tau)\Delta\mu^t(\tau)\,d\tau+b\int_{-\infty}^{+\infty}\phi(\xi)\eta(\xi)\,d\xi=f_2\in L^2(\Omega),\\ \\ \phi+(\xi^2+\beta)\phi-u\eta(\xi)=f_3(\xi)\in L^2(\Omega,\mathbb{R}),\\ \\ \mu^t+\mu^t_\tau-u=f_4(\tau)\in L^2_g(\mathbb{R}_+;H_0^1(\Omega)).\end{cases} \tag{15}$$

Using the third equation in (15), we obtain

$$\phi=\frac{f_3+u\eta(\xi)}{\xi^2+\beta+1}. \tag{16}$$

On the other hand, the fourth equation in (15) has a unique solution

$$\mu^t=\left(\int_0^\tau e^z(f_4(z)+y-f_1)\,dz\right)e^{-\tau}. \tag{17}$$

Inserting $u=y-f_1$, (16) and (17) in the second equation in (15), we obtain

$$\sigma y-\bar{\lambda}\Delta y=G, \tag{18}$$

where

$$\sigma = 1 + b \int_{-\infty}^{+\infty} \frac{\eta^2(\xi)}{\xi^2 + \beta + 1}\, d\xi > 0,$$

$$\begin{aligned}\bar{\lambda} &= \lambda + \int_0^{+\infty} g(\tau) e^{-\tau} \left(\int_0^{\tau} e^z \, dz \right) ds \\ &= 1 - \int_0^{+\infty} g(\tau) e^{-\tau} \, d\tau > 0,\end{aligned}$$

$$\begin{aligned}G &= f_2 + \sigma f_1 - b \int_{-\infty}^{+\infty} \frac{\eta(\xi) f_3(\xi)}{\xi^2 + \beta + 1}\, d\xi \\ &+ \int_0^{+\infty} g(\tau) e^{-\tau} \left(\int_0^{\tau} e^z \Delta(f_4(z) - f_1)\, dz \right) d\tau.\end{aligned}$$

To solve (18), we consider the following variational formulation:

$$B(y, w) = L(w), \quad \forall w \in H_0^1(\Omega), \tag{19}$$

where B is the bi-linear form defined by

$$B(y, w) = \sigma \int_{\Omega} y w \, dx + \bar{\lambda} \int_{\Omega} \nabla y . \nabla w \, dx, \tag{20}$$

and L is the linear functional given by

$$L(w) = \int_{\Omega} G w \, dx. \tag{21}$$

It is easy to verify that L is bounded and B is coercive and bounded. So, the Lax–Milgram theorem guarantees that for all $w \in H_0^1(\Omega)$, the linear elliptic equation (18) has a unique solution $y \in H_0^1(\Omega)$.
The substitution of y into the first equation in (15) yields $u \in H_0^1(\Omega)$.

Inserting u in (15) and bearing in mind the third equation in (15), we obtain

$$\phi \in L^2(\Omega, \mathbb{R}).$$

Similarly, we have

$$\mu^t \in L_g^2(\mathbb{R}_+; H_0^1(\Omega)).$$

Using (18), we get

$$\sigma \int_{\Omega} y w \, dx + \bar{\lambda} \int_{\Omega} \nabla y . \nabla w \, dx = \int_{\Omega} G w \, dx. \tag{22}$$

The elliptic regularity theory, then, implies that $y \in H^2(\Omega)$. So, I+A is onto.
Now, we prove that the operator defined in (10) is locally Lipschitzian in $\mathcal{H}$. For $U, \tilde{U} \in \mathcal{H}$, we get

$$\begin{aligned}
\left\| J(U) - J(\bar{U}) \right\|_{\mathcal{H}} &= \left\| \left(0, u|u|^{p-2} - \bar{u}|\bar{u}|^{p-2}, 0\right) \right\|_{\mathcal{H}} \\
&= \left\| u|u|^{p-2} - \bar{u}|\bar{u}|^{p-2} \right\|_{L^2(\Omega)} \\
&= \left\| |u|^p - |\bar{u}|^p \right\|_{L^2(\Omega)} \\
&= \left\| (u-\bar{u})(|u|^{p-1} + u^{p-2}\bar{u} + \dots + \bar{u}^{p-1}) \right\|_{L^2(\Omega)} \\
&= C \left\| (u-\bar{u})(u^{p-1} + \bar{u}^{p-1}) \right\|_{L^2(\Omega)} \\
&\le C \left(\int_\Omega (|u-\bar{u}|^2)(|u|^{p-1} + |\bar{u}|^{p-1})^2 \, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Using Hölder's inequality, we have

$$\left\| J(U) - J(\bar{U}) \right\|_{\mathcal{H}} \le C \left(\int_\Omega |u-\bar{u}|^{2\gamma} \, dx \right)^{\frac{1}{2\gamma}} \left(\int_\Omega (|u|^{p-1} + |\bar{u}|^{p-1})^{2\delta} \, dx \right)^{\frac{1}{2\delta}}, \; \frac{1}{\gamma} + \frac{1}{\delta} = 1,$$

with $\gamma = \dfrac{n}{n-2}$ and $\delta = \dfrac{n}{2}$. So we have

$$\begin{aligned}
\left\| J(U) - J(\bar{U}) \right\|_{\mathcal{H}} &\le C \left(\int_\Omega (|u-\bar{u}|^{\frac{2n}{n-2}\gamma}) \right)^{\frac{n-2}{2n}} \left(\int_\Omega (|u|^{p-1} + |\bar{u}|^{p-1})^n \, dx \right)^{\frac{1}{n}} \\
&\le C \left(\int_\Omega (|u-\bar{u}|^{\frac{2n}{n-2}}\gamma) \right)^{\frac{n-2}{2n}} \left(\int_\Omega (|u|^{n(p-1)} + |\bar{u}|^{n(p-1)}) \, dx \right)^{\frac{1}{n}} \\
&\le C \left\| u-\bar{u} \right\|_{L^{\frac{2n}{n-2}}(\Omega)} \left[\left(\int_\Omega |u|^{n(p-1)} dx \right)^{\frac{1}{n}} + \left(\int_\Omega |\bar{u}|^{n(p-1)} \, dx \right)^{\frac{1}{n}} \right] \\
&\le C \left\| u-\bar{u} \right\|_{L^{\frac{2n}{n-2}}(\Omega)} \left[\|u\|^{p-1}_{L^{n(p-1)}(\Omega)} + \|\bar{u}\|^{p-1}_{L^{n(p-1)}(\Omega)} \right].
\end{aligned} \tag{23}$$

The Sobolev embedding theorem gives

$$\|u-\bar{u}\|_{L^{\frac{2n}{n-2}}(\Omega)} \le C \, \|u-\bar{u}\|_{L^2(\Omega)} \le C \left\| U - \bar{U} \right\|_{\mathcal{H}}. \tag{24}$$

The necessity to estimate $\|u\|_{n(p-1)}$ by the energy norm $\|U\|_{\mathcal{H}}$ requires to consider different ranges of p. Namely, we need $n(p-1) \le \dfrac{2n}{n-2}$ and this coincides with the cut in our assumption $p \le \dfrac{n}{n-2}$. Thus, the Sobolev embedding theorem

$$L_{\frac{n}{n-2}}(\Omega) \subset H^1(\Omega)$$

, it holds

$$\|u\|^{p-1}_{L^{n(p-1)}(\Omega)} \le C \, \|u\|^{p-1}_{H^1}(\Omega). \tag{25}$$

Therefore, by combining (24)âĂŞ(25), we obtain

$$\left\| U - \bar{U} \right\|_{\mathcal{H}} \le C(\|u\|^{p-1}_{H^1}(\Omega), \|\bar{u}\|^{p-1}_{H^1}(\Omega)) \left\| U - \bar{U} \right\|_{\mathcal{H}}.$$

So, J is locally Lipschitzian. Therefore, the well-posedness result follows from the theorem of Sigal. □

4 | BLOW UP RESULT

In this section, we use a judicious Lyapunov functional to prove that some solutions can experience blow-up in a finite time. To achieve our goal, we need the following lemma.

Lemma 5. Suppose that $p \geq 2$. Then, there exists a positive constant $C > 1$ such that

$$\|y\|_p^s \leq C_2 \left(\|y\|_p^p + \|\nabla y\|_2^2 \right) \tag{26}$$

for any $y \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Proof. If $\|y\|_p \geq 1$ then $\|y\|_p^s \leq \|y\|_p^p$.
If $\|y\|_p \leq 1$ then $\|y\|_p^s \leq \|y\|_p^2 \leq C_* \|\nabla y\|_2^2$ by the Sobolev embedding theorem. □

Let

$$H(t) = -E(t). \tag{27}$$

Theorem 2. Suppose that $p > 4$ satisfies (11). Assume further that (G1)

$$g_0 = \int_0^\infty g(\tau)\, d\tau < \frac{p-4}{p} \tag{28}$$

and

$$E(0) < 0. \tag{29}$$

Then, the solution of system (PâĂŹ) blows-up in a finite time.

Proof. Using (5), we have

$$E(t) \leq E(0) < 0. \tag{30}$$

Thus, we get

$$\begin{aligned} H'(t) = -E'(t) = &-\frac{1}{2} \int_0^{+\infty} g'(\tau) \left\|\nabla \mu^t(\tau)\right\|_2^2 d\tau \\ &+ b \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta)|\phi(\xi,t)|^2\, d\xi dx \geq 0. \end{aligned} \tag{31}$$

Furthermore, we have

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|y\|_p^p. \tag{32}$$

Let

$$A(t) = H^{1-\gamma}(t) + \epsilon \int_\Omega u u_t\, dx, \tag{33}$$

where $\epsilon > 0$ to be specified later and

$$0 < \gamma < \frac{p-2}{2p}. \tag{34}$$

Differentiating (33) and using (PâĂŹ), we obtain

$$\begin{aligned} A'(t) =&(1-\gamma)H^{-\gamma}(t)H'(t) + \epsilon\|y_t\|_2^2 - \epsilon\lambda\|\nabla y\|_2^2 \\ &- b\epsilon \int_\Omega y \int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t)\, d\xi dx + \epsilon\|y\|_p^p \\ &- \epsilon \int_\Omega \nabla y \int_0^\infty g(\tau)\nabla\mu^t(\tau) d\tau\, dx. \end{aligned} \tag{35}$$

Using Young's inequality and Lemma 1, we find

$$\begin{aligned}&\int_{\Omega} \nabla y(t) \int_{0}^{+\infty} g(\tau)\nabla \mu^{t}(\tau)\, d\tau dx \\ &\leq \frac{1}{4}\int_{0}^{+\infty} g(\tau)\, \|\nabla \mu^{t}(\tau)\|_{2}^{2}\, d\tau + (1-\lambda)\, \|\nabla y(t)\|_{2}^{2}.\end{aligned} \tag{36}$$

Substituting (36) in (35), we get

$$\begin{aligned}A'(t) \geq &(1-\gamma)H^{-\gamma}(t)H'(t) + \epsilon\|y_t\|_2^2 - \epsilon\|\nabla y\|_2^2 \\ &- b\epsilon \int_{\Omega} y \int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t)\, d\xi dx \\ &+ \epsilon\|y\|_p^p - \frac{\epsilon}{4}\int_{0}^{+\infty} g(\tau)\, \|\nabla \mu^{t}(\tau)\|_{2}^{2}\, d\tau.\end{aligned} \tag{37}$$

Using Young's inequality and (31), we find

$$\begin{aligned}&b\int_{\Omega} y \int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t)\, d\xi dx \\ &\leq \delta C_1\|y\|_2^2 + \frac{b}{4\delta}\int_{\Omega}\int_{-\infty}^{+\infty} (\xi^2+\beta)|\phi(x,\xi,t)|^2\, d\xi dx \\ &\leq \delta C_1\|y\|_2^2 + \frac{1}{4\delta}H'(t),\end{aligned} \tag{38}$$

for $C_1 := b\displaystyle\int_{-\infty}^{+\infty} \frac{\eta^2(\xi)}{\xi^2+\beta}\, d\xi$ and $\delta > 0$, which may depend on t.
Substituting (38) in (37), we have

$$\begin{aligned}A'(t) \geq &\left((1-\gamma)H^{-\gamma}(t) - \frac{\epsilon}{4\delta}\right)H'(t) \\ &+ \epsilon\|y_t\|_2^2 - \epsilon\|\nabla y\|_2^2 - \epsilon\delta C_1\|y\|_2^2 \\ &+ \epsilon\|y\|_p^p - \frac{\epsilon}{4}\int_{0}^{+\infty} g(\tau)\, \|\nabla \mu^{t}(\tau)\|_{2}^{2}\, d\tau.\end{aligned} \tag{39}$$

Next, we choose an appropriate δ as follows:

$$\frac{1}{4\delta} = kH^{-\gamma}(t), \tag{40}$$

where k is some positive constant to be determined later. Substituting (40) into (39), we get

$$\begin{aligned}A'(t) \geq &\left[(1-\gamma) - \epsilon k\right]H^{-\gamma}(t)H'(t) + \epsilon\|y_t\|_2^2 \\ &- \epsilon\|\nabla y\|_2^2 - \frac{\epsilon C_1}{4k}H^{\gamma}(t)\|y\|_2^2 \\ &+ \epsilon\|y\|_p^p - \frac{\epsilon}{4}\int_{0}^{+\infty} g(\tau)\, \|\nabla \mu^{t}(\tau)\|_{2}^{2}\, d\tau.\end{aligned} \tag{41}$$

Using (32), we have

$$H^{\gamma}(t) \leq \frac{1}{p^{\gamma}}\|y\|_p^{p\gamma}. \tag{42}$$

Thus, we have

$$C_1 H^{\gamma}(t)\|y\|_2^2 \leq C_2\|y\|_p^{p\gamma+2}, \tag{43}$$

for some $C_2 > 0$. Combining (41) and (43), we obtain

$$\begin{aligned} A'(t) \geq & \left[(1-\gamma) - \epsilon k\right] H^{-\gamma}(t) H'(t) + \epsilon \left(\frac{p}{4}+1\right) \|y_t\|_2^2 \\ & + \frac{\epsilon}{2}\|y\|_p^p + \epsilon\left[\frac{\lambda p}{4} - 1\right] \|\nabla y\|_2^2 \\ & + \frac{\epsilon b p}{4} \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 \, d\xi dx \\ & + \epsilon\left(\frac{p}{2} H(t) - \frac{C_2}{4k}(t)\|y\|_2^{p\gamma+2}\right) \\ & + \epsilon\left(\frac{p-1}{4}\right)\int_0^{+\infty} g(\tau)\left\|\nabla \mu^t(\tau)\right\|_2^2 \, d\tau. \end{aligned} \tag{44}$$

By Lemma 5 and (34), for $s = p\gamma + 2 \leq p$, we find

$$\begin{aligned} A'(t) \geq & \left((1-\gamma) - \epsilon k\right) H^{-\gamma}(t) H'(t) + \epsilon \left(\frac{p}{4}+1\right) \|y_t\|_2^2 \\ & + \frac{\epsilon}{2}\left(1 - \frac{C_3}{2k}\right)\|y\|_p^p + \frac{\epsilon}{4}\left[(\lambda p - 4) - \frac{C_3}{k}\right] \|\nabla y\|_2^2 \\ & + \frac{\epsilon b p}{4} \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 \, d\xi dx + \frac{p\epsilon}{2} H(t) \\ & + \epsilon\left(\frac{p-1}{4}\right)\int_0^{+\infty} g(\tau)\left\|\nabla \mu^t(\tau)\right\|_2^2 \, d\tau, \end{aligned} \tag{45}$$

where $C_3 = C\,C_2$. Using (28) and (G1), we get $p\lambda - 4 > 0$.

At this point, we choose k large enough such that

$$1 - \frac{C_3}{2k} > 0, \quad p\lambda - 4 - \frac{C_3}{k} > 0.$$

When k is fixed, we pick ϵ small enough such that

$$(1-\gamma) - \epsilon k > 0, \quad H(0) + \epsilon \int_\Omega y_0 y_1 \, dx > 0.$$

Therefore, there exists a positive constant C_4 such that

$$A'(t) \geq C_4 \left(H(t) + \|y_t\|_2^2 + \|y\|_p^p + \|\nabla y\|_2^2\right). \tag{46}$$

Furthermore, we get

$$A(t) \geq A(0) > 0, \; t > 0. \tag{47}$$

By HölderâĂŹs inequality and the embedding inequalities, we have

$$\int_\Omega y y_t \, dx \leq \|y\|_2 \, \|y_t\|_2 \leq d\, \|y\|_p \, \|y_t\|_2 ,$$

where $d > 0$ is the best embedding constant. Using Young's inequality, we find

$$\left|\int_\Omega y y_t \, dx\right|^{\frac{1}{1-\gamma}} \leq d_1 \left(\|y_t\|_2^{\frac{\theta'}{1-\gamma}} + \|y\|_p^{\frac{\theta}{1-\gamma}}\right), \tag{48}$$

where d_1 is a constant and $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. Using Lemma 5, for $\theta' = 2(1-\gamma)$, we obtain

$$\frac{\theta}{1-\gamma} = \frac{2}{1-2\gamma} \leq p.$$

Thus, for $s = \dfrac{2}{1-2\gamma}$, we obtain

$$\left| \int_{\Omega} y y_t \, dx \right|^{\frac{1}{1-\gamma}} \leq d_2 \left(\|y_t\|_2^2 + \|y\|_p^p + \|\nabla y\|_2^2 \right), \tag{49}$$

where $d_2 > 0$ is a constant. Consequently, by (49), we have

$$\begin{aligned} A^{\frac{1}{1-\gamma}}(t) &\leq \left(H^{1-\gamma}(t) + \int_{\Omega} y y_t \, dx \right)^{\frac{1}{1-\gamma}} \\ &\leq d_3 \left(H(t) + \left(\int_{\Omega} y y_t \, dx \right)^{\frac{1}{1-\gamma}} \right) \\ &\leq d_3 \left(H(t) + \| y_t \|_2^2 + \| \nabla y \|_2^2 + \| y \|_p^p \right), \quad t \geq 0, \end{aligned} \tag{50}$$

where d_3 is a positive constant. Combining (46) and (50), we obtain

$$A'(t) \geq d_4 A^{\frac{1}{1-\gamma}}(t), \quad t \geq 0, \tag{51}$$

where d_4 is a positive constant. Integrating (51) over (0, t), we get

$$A(t) \geq \frac{1}{A^{\frac{-\gamma}{1-\gamma}}(t) - \dfrac{\gamma d_4 t}{1-\gamma}}. \tag{52}$$

So, $A(t)$ blows up in a finite time

$$T \leq T^* = \frac{1-\gamma}{d_4 \gamma A^{\frac{\gamma}{1-\gamma}}(0)}.$$

□

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

ACKNOWLEDGEMENTS

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How to cite this article: M. Kirane, A. Radhouane, Blow-up of solutions of a non-linear wave equation with fractional damping and infinite memory, *Math Methods Appl Sci*, submitted.