Building coercive Lyapunov-Krasovskii functionals based on Razumikhin and Halanay approaches

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Abstract. In this paper, we provide a systematic and constructive way to build a Lyapunov-Krasovskii functional for time-delay systems whose stability can be established through the Razumikhin or the Halanay approaches. The constructed Lyapunov-Krasovskii functional turns out to be coercive, meaning sandwiched between functions of the state history norm, and to dissipate in terms of the whole history norm. We present these results in the framework of input-to-state stability (ISS) in order to further account for the influence of input disturbances. A special emphasis is also given on exponential stability and exponential ISS. We illustrate our findings though the study of a coupled ODE-PDE model of a chemical reactor, and show that, unlike most results in that area, our approach happens to ensure ISS in terms of the supremum norm of the state.

Keywords: time-delay systems, Halanay approach, Razumikhin approach, Lyapunov-Krasovskii functionals, Input-to-State Stability

1 Introduction

Among the Lyapunov methods available to analyze stability of nonlinear time-delay systems, we may distinguish two distinct approaches. The first one consists in studying the derivative of a functional along the solutions of the considered dynamical system. The argument of this functional is the state of the time-delay system, namely a signal encompassing all past values of the solution over a time interval whose length corresponds to largest delay involved in the dynamics. This approach finds its roots in [Krasovskii, 1963] and is referred to as the Lyapunov-Krasovskii method. The strength of this approach lies in its generality, as several stability concepts are fully characterized in terms of Lyapunov-Krasovskii functionals. For instance, it has been shown that global asymptotic stability (GAS) of the origin is equivalent to the existence of a Lyapunov-Krasovskii functional whose derivative is negative out of the origin [Pepe and Karafyllis, 2013]. A similar characterization holds for global exponential stability (GES) under additional constraints on the functional bounds and on its dissipation rate [Haidar et al., 2022].

The second approach consists in relying on a function, whose argument is a vector (not a signal) corresponding to the current value of the solution. In other words, it employs classical finite-dimensional Lyapunov functions. Along this line, we can distinguish two different methods. The first one orginates from [Razumikhin, 1960] and is referred to as the Razumikhin method. Roughly speaking, it imposes that the considered Lyapunov function decreases along solutions whenever it is larger than some function of the maximum it reached over the past delay interval. If this function is smaller than the identity, then GAS or GES is guaranteed (depending on the properties of the considered Lyapunov function): see for instance [Hale and Lunel, 1993, Karafyllis et al., 2008]. This approach thus has strong similarities with the small-gain theorem, as evidenced in [Teel, 1998]. The second finite-dimensional method to assess stability of time-delay systems relies on Halanay's inequality [Halanay, 1966]. This method requires that the derivative of the Lyapunov function along solutions is bounded by a negative linear term of itself plus a positive linear term of the maximum it reached over the past delay interval. Halanay's result then asserts that the Lyapunov function decays exponentially along solutions, provided that the gain of the negative term is greater than that of the positive term, which can then be used to derive GAS or GES depending on the function's properties [Baker and Buckwar, 2005, Bresch-Pietri et al., 2012]. It was recently shown that the gains can also be picked nonlinear, provided that their difference is a positive definite unbounded function [Pepe, 2022]. The Razumikhin and Halanay methods share a lot in common: they both rely on a Lyapunov function and both treat delayed terms as disturbances that tend to act against stability. Less general than the Lyapunov-Krasovskii approach, their main interest lies in their mathematical simplicity, as no knowledge of

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infinite-dimensional systems theory is needed to apply them.

Despite the vast literature existing on the Lyapunov-Krasovskii, Razumikhin and Halanay methods, and their ubiquitous use in the stability analysis of time-delay systems, we are not aware of any results providing a bridge between them. Since both Razumikhin and Halanay conditions ensure GAS (or GES), we know from converse theorems that the system admits a Lyapunov-Krasovskii functional. The question we address here is whether such a functional can be built explicitly based on the knowledge of the function used in the Razumikhin or Halanay analysis. We believe this question is of interest not only for the sake of unification between the two approaches, but also for practical considerations. Indeed, the explicit knowledge of a Lyapunov-Krasovskii functional is likely to provide a more complete view on the robustness of the system to exogenous inputs, parameter uncertainties or modeling imprecision.

In this paper, we show that it is indeed always possible to explicitly construct a GES Lyapunov-Krasovskii functional for systems satisfying the Razumikhin or Halanay conditions for GES. Similarly, we show that the same holds for GAS, although with slightly more demanding assumptions than the original Razumikhin and Halanay results. In both cases, the constructed Lyapunov-Krasovskii functional has a simple form and turns out to be coercive, meaning both lower and upper bounded by the full state history norm, and to dissipate according to the full state history norm too.

We present our results in the framework of input-to-state stability (ISS, [Sontag, 1989]) in order to account for possible exogenous inputs acting on the system. This property not only imposes GAS in the absence of inputs, but also guarantees bounded solutions in response to any bounded input. Although originally developed in a finite-dimensional context [Sontag, 2008, Mironchenko, 2023], the ISS framework has now become a central and mature property in the study of time-delay systems, as recently reviewed in [Chaillet et al., 2023].

The paper is organized as follows. We start in Section 2 by recalling some basics about time-delay systems and their stability properties and we recall the main Lyapunov-Krasovkii, Razumikhin and Halanay results existing for the GAS, GES and ISS properties. In Section 3, we present our main results, namely the explicit construction of a coercive Lyapunov-Krasovskii functional for both ISS and its exponential counterpart (exp-ISS), which allow to respectively consider GAS and GES by simply setting the input to zero. In Section 4, we present an illustration of how our results can be useful in practice, by studying a infinite-dimensional model (coupled PDE-ODE system) of a chemical reactor whose temperature is regulated by a cooling liquid. The proofs of our results are given in Section 5, and we conclude with some remarks and perspectives in Section 6.

2 Preliminaries and definitions

2.1 Notations

We start by introducing the notations that will be used throughout this paper. $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N}$ stands for the set of non-negative integers. Given $a \in \mathbb{R}$, $\mathbb{R}_{\geq a} := \{x \in \mathbb{R} : \ x \geq a\}$ and similarly for $\mathbb{N}_{\geq a}$. Given $\Delta \geq 0$, $\mathcal{X}^n$ denotes the space of continuous functions mapping the interval $[-\Delta, 0]$ into $\mathbb{R}^n$. Given $T \in (0, +\infty]$, $t \in [0, T)$ and a continuous function $x : [-\Delta, T) \to \mathbb{R}^n$, $x_t \in \mathcal{X}^n$ denotes the history segment at time t and is defined as $x_t(\tau) := x(t + \tau)$ for all $\tau \in [-\Delta, 0]$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector. Given a non-empty (possibly unbounded) interval $\mathcal{I} \subset \mathbb{R}$ and a Lebesgue measurable signal $u : \mathcal{I} \to \mathbb{R}^m$, we define

$$\|u\| := \sup_{t \in \mathcal{I}} |u(t)| := \inf\left\{\bar{u} \geq 0 : \ \lambda(\{t \in \mathcal{I} : |u(t)| > \bar{u}\}) = 0\right\},$$

where λ denotes the Lebesgue measure (notice that sup is intended as an esssential supremum throughout the paper). The symbol $\mathcal{U}$ denotes the set of the Lebesgue measurable and locally essentially bounded functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Given $m \in \mathbb{N}_{\geq 1}$ and $u \in \mathcal{U}^m$, $u_{\mathcal{I}}$ denotes the restriction of u to $\mathcal{I}$. Given a continuously differentiable function $V_0 : \mathbb{R}^n \to \mathbb{R}$, ∇V_0 denotes its gradient. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{N}$ if it is continuous, non-decreasing, and satisfies $\alpha(0) = 0$. It is said to be of class $\mathcal{P}$ if $\alpha \in \mathcal{N}$ and $\alpha(s) > 0$ for all $s > 0$. It is said to be of class $\mathcal{K}$ if $\alpha \in \mathcal{P}$ and it is increasing. It is said to be of class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s \to +\infty} \alpha(s) = +\infty$. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{L}$ if it is continuous, non-increasing, and satisfies $\lim_{s \to +\infty} \sigma(s) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed $s \geq 0$, $\beta(s, \cdot) \in \mathcal{L}$. Given a functional $V : \mathcal{X}^n \to \mathbb{R}^n$, its Driver's derivative $D^+V : \mathcal{X}^n \times \mathbb{R}^n \to [-\infty, +\infty]$ is defined for all $(\phi, w) \in \mathcal{X}^n \times \mathbb{R}^n$ as

$$D^+V(\phi, w) := \lim_{h \to 0^+} \frac{V(\phi_{h,w}) - V(\phi)}{h},$$

where the function $\phi_{h,w}$ is defined by

$$\phi_{h,w}(\tau) := \begin{cases} \phi(\tau+h) & \text{if } \tau \in [-\Delta, -h] \\ \phi(0) + (\tau+h)w & \text{if } \tau \in (-h, 0]. \end{cases}$$

2.2 Stability and robustness properties

This paper considers the following class of time-delay systems:

$$\dot{x}(t) := f(x_t, u(t)), \tag{1}$$

where the vector field $f : \mathcal{X}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be Lipschitz on bounded sets and to satisfy $f(0,0) = 0$. The input signal u is assumed to be in $\mathcal{U}^m$. A central property for the stability and robustness analysis of such systems is the input-to-state stability (ISS), which was originally introduced in [Sontag, 1989] for finite-dimensional systems and more recently extended to time-delay systems, as reviewed in [Chaillet et al., 2023].

Definition 1 (ISS, exp-ISS) The system (1) is said to be *input-to-state stable (ISS)* if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{N}$ such that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$, its solution satisfies

$$|x(t, x_0, u)| \leq \beta(\|x_0\|, t) + \mu(\|u_{[0,t]}\|), \quad \forall t \geq 0.$$

If, in particular, there exist $k, \lambda > 0$ such that $\beta(s,t) = kse^{-\lambda t}$ for all $s, t \geq 0$, then the system (1) is said to be *exponentially ISS (exp-ISS)*. □

Just like in finite dimension, the ISS property imposes that the solutions' norm is bounded by a vanishing term involving the initial state norm plus a term involving the amplitude of the applied input. In particular, it guarantees that the system produces only bounded solutions in response to bounded inputs, and that solutions eventually converge to a neighborhood of the origin whose size is "proportional" (through the nonlinear gain μ) to the input magnitude.

ISS (resp. exp-ISS) can be seen as a robust extension of global asymptotic stability (resp. global exponential stability), as recalled next.

Definition 2 (GAS, GES) The input-free system

$$\dot{x}(t) = f(x_t, 0) \tag{2}$$

is said to be *globally asymptotically stable (GAS)* if there exists $\beta \in \mathcal{KL}$ such that, for all $x_0 \in \mathcal{X}^n$, its solution satisfies

$$|x(t, x_0)| \leq \beta(\|x_0\|, t), \quad \forall t \geq 0.$$

It is said to be *globally exponentially stable (GES)* if there exist $k, \lambda > 0$ such that the above estimate holds with $\beta(s,t) = kse^{-\lambda t}$ for all $s, t \geq 0$. □

2.3 Lyapunov approaches for time-delay systems

A powerful tool to study stability of nonlinear systems is the Lyapunov approach. For time-delay systems, it can follow several distinct lines: the Razumikhin and Halanay approaches, which rely on a function of the current solution value, and the Krasovskii approach, which relies on a function of the state history. We briefly recall some fundamental results in these directions. To that aim, we start by recalling the notion of Lyapunov-Krasovskii functional candidate.

Definition 3 (LKF, coercive LKF) A functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ is said to be a *Lyapunov-Krasovskii functional candidate (LKF)* if it is Lipschitz on bounded sets and there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}^n.$$

The LKF V is coercive if, in addition,

$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}^n.$$

□

Coercive LKF are harder to design in practice, but offer the great advantage of allowing to seamlessly interchange the LKF V and the history norm $\|\phi\|$. It is known since [Karafyllis et al., 2008] that the ISS property is fully characterized by the existence of an ISS LKF, as we recall next.

Definition 4 (ISS LKF) A LKF V is said to be an *ISS LKF* for system (1), if there exist $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{N}$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(V(\phi)) + \gamma(|v|).$$

□

Theorem 1 (LKF characterization of ISS) The following statements are equivalent:

- (1) is ISS
- (1) admits an ISS LKF
- (1) admits a coercive ISS LKF.

□

It is worth stressing that, for time-delay systems, different notions of ISS LKF could be considered, depending on how they dissipate along the system's solutions (in terms of the current solutions norm only, in terms of the LKF itself as considered here, or in terms of the whole history norm); we refer the reader to [Chaillet et al., 2017] and [Chaillet et al., 2023] for further discussions on that matter.

Similarly, the exp-ISS property can be established through the notion of exp-ISS LKF.

Definition 5 (exp-ISS LKF) A LKF V is said to be an *exp-ISS LKF* for system (1) if there exist $\underline{a}, \overline{a}, a, p > 0$ and $\gamma \in \mathcal{N}$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$\begin{aligned} \underline{a}|\phi(0)|^p \leq V(\phi) \leq \overline{a}\|\phi\|^p \\ D^+V(\phi, f(\phi, v)) \leq -aV(\phi) + \gamma(|v|). \end{aligned} \tag{3}$$

It is said to be a coercive exp-ISS LKF if it satisfies (3) and

$$\underline{a}\|\phi\|^p \leq V(\phi) \leq \overline{a}\|\phi\|^p, \quad \forall \phi \in \mathcal{X}^n.$$

□

The following result follows from standard manipulations and was established in [Chaillet et al., 2022].

Theorem 2 (LKF condition for exp-ISS) System (1) is exp-ISS if it admits an exp-ISS LKF. □

Two alternatives to the Lyapunov-Krasovskii approach are the Razumikhin and Halanay approaches. Although less general, these methods offer the advantage of exploiting finite-dimensional reasonings, by treating delay terms as disturbances. We start by recalling the Razumikhin approach, originally presented in [Teel, 1998] and slightly refined in [Karafyllis et al., 2008].

Theorem 3 (ISS through Razumikhin) Assume that there exist $\rho \in \mathcal{K}_\infty,\ \alpha \in \mathcal{P},\ \gamma \in \mathcal{N}$, and a positive definite and radially unbounded function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ satisfying, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$V_0(\phi(0)) \geq \max\left\{\rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right), \gamma(|v|)\right\} \quad \Rightarrow \quad \nabla V_0(\phi(0))f(\phi, v) \leq -\alpha(|\phi(0)|). \tag{4}$$

Then, under the condition that

$$\rho(s) < s, \quad \forall s > 0, \tag{5}$$

the system (1) is ISS. □

Similarly, the following result, taken from [Wang and Liu, 2005], allows to ensure GES for system (2) provided adequate bounds on the considered Lyapunov function and on its dissipation rate.

Corollary 1 (GES through Razumikhin) Assume that there exist $\underline{a}, \overline{a}, p > 0$ and a function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ satisfying, for all $x \in \mathbb{R}^n$,

$$\underline{a}|x|^p \leq V_0(x) \leq \overline{a}|x|^p.$$

Assume further that there exist $\rho_0, a > 0$ such that, for all $\phi \in \mathcal{X}^n$,

$$V_0(\phi(0)) \geq \rho_0 \max_{\tau \in [-\Delta, 0]} V(\phi(\tau)) \quad \Rightarrow \quad \nabla V_0(\phi(0)) f(\phi, 0) \leq -a V_0(\phi(0)).$$

Then the input-free system (2) is GES provided that $\rho_0 < 1$. □

Another useful result to establish GES of time-delay systems is based on Halanay's inequality, originally presented in [Halanay, 1966]. In particular, the following result is an immediate consequence of [Baker and Buckwar, 2005, Theorem 7].

Theorem 4 (GES through Halanay) Assume that there exist $\underline{a}, \overline{a}, p > 0$ and a function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ satisfying, for all $x \in \mathbb{R}^n$,

$$\underline{a}|x|^p \leq V_0(x) \leq \overline{a}|x|^p.$$

Assume further that there exist $\rho_0, a > 0$ such that, for all $\phi \in \mathcal{X}^n$,

$$\nabla V_0(\phi(0)) f(\phi, 0) \leq -a V_0(\phi(0)) + \rho_0 \max_{\tau \in [-\Delta, 0]} V_0(\phi(\tau)).$$

Then, under the condition that $\rho_0 < a$, the input-free system (2) is GES. □

This result has recently been extended to ISS in [Pepe, 2022, Corollary 2].

Theorem 5 (ISS through Halanay) Assume that there exist a function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, which is positive definite and radially unbounded, $\alpha \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$ and $\gamma \in \mathcal{N}$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$\nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha(V_0(\phi(0))) + \rho\left(\max_{\tau \in [-\Delta, 0]} V_0(\phi(\tau))\right) + \gamma(|v|). \tag{6}$$

Then, under the condition that $\alpha - \rho \in \mathcal{K}_\infty$, the system (1) is ISS. □

3 Main results

Since the Razumikhin approach recalled in Theorem 3 ensures ISS, we know from the LKF characterization of ISS (Theorem 1) that the system admits a coercive ISS LKF. Similarly, the assumptions of Theorem 5, with the Halanay approach, guarantee the existence of a coercive ISS LKF. Here, we show that such coercive ISS LKF can be systematically and explicitly constructed based on the assumptions of Theorem 3 or 5. This construction relies on the following instrumental lemma, originally presented in [Karafyllis and Jiang, 2011, Lemma 6.7].

Lemma 1 Given $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ and $c > 0$, the functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ defined as

$$V(\phi) := \max_{\tau \in [-\Delta, 0]} e^{c\tau} V_0(\phi(\tau)), \quad \forall \phi \in \mathcal{X}^n,$$

is Lipschitz on bounded sets and satisfies, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$V(\phi) > V_0(\phi(0)) \quad \Rightarrow \quad D^+V(\phi, f(\phi, v)) \leq -cV(\phi) \tag{7}$$

$$V(\phi) = V_0(\phi(0)) \quad \Rightarrow \quad D^+V(\phi, f(\phi, v)) \leq \max\{-cV(\phi), \nabla V_0(\phi(0)) f(\phi, v)\}. \tag{8}$$

□

3.1 Razumikhin condition

We start by showing how to construct a coercive ISS LKF based on the assumptions of Theorem 3. Our main result in that direction is the following.

Theorem 6 (Coercive ISS LKF from Razumikhin condition) Assume that there exist $\alpha, \rho \in \mathcal{K}_\infty$, $\gamma \in \mathcal{N}$ and a positive definite and radially unbounded function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ satisfying, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$V_0(\phi(0)) \geq \max\left\{\rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right), \gamma(|v|)\right\} \quad \Rightarrow \quad \nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha(|\phi(0)|). \tag{9}$$

Under the condition that

$$\ell := \sup_{s>0} \frac{\rho(s)}{s} < 1, \tag{10}$$

the functional V defined as

$$V(\phi) := \max_{\tau\in[-\Delta,0]} \frac{1}{\ell^{\tau/\Delta}} V_0(\phi(\tau)), \quad \forall \phi \in \mathcal{X}^n, \tag{11}$$

is a coercive ISS LKF for system (1). □

It is worth stressing that condition (10) is slightly more restrictive than the main condition (5) of Theorem 3, as it prevents ρ from approaching identity at $+\infty$. For instance, the function ρ defined as $\rho(s) := \frac{s^2}{1+s}$ satisfies (5) but not (10). The proof techniques we employ here did not allow us to cover the case when $\lim_{s\to+\infty} \frac{\rho(s)}{s} = 1$. Yet, Theorem 6 covers a wide class of systems for which ISS can be guaranteed by the Razumikhin approach. In order to establish this theorem, we first provide an alternative Razumikhin-like condition for ISS, whose proof is provided in Section 5.2.

Theorem 7 (Alternative Razumikhin condition for ISS) Assume that there exist $\alpha, \rho \in \mathcal{K}_\infty$, $\gamma \in \mathcal{N}$ and a positive definite and radially unbounded function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ satisfying, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$V_0(\phi(0)) \geq \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right) \quad \Rightarrow \quad \nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha(|\phi(0)|) + \gamma(|v|). \tag{12}$$

If (10) holds, then the system (1) is ISS and the functional V defined in (11) is a coercive ISS LKF for system (1). □

Condition (12) imposes that the derivative of V_0 along the solutions of the system is negative up to a positive term involving the input norm, whenever V_0 is larger than a small function of its past values. It differs from condition (9) just by the disposition of the input gain: on the right-hand side of the implication for (12), and on the left-hand side for (9). It turns out that these two formulations are actually equivalent.

Proposition 1 (Equivalence between the two Razumikhin conditions) Let $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ be a positive definite and radially unbounded function and $\rho \in \mathcal{K}_\infty$. Then the following statements are equivalent:

(i) there exist $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{N}$ such that

$$V_0(\phi(0)) \geq \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right) \quad \Rightarrow \quad \nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha(|\phi(0)|) + \gamma(|v|). \tag{13}$$

(ii) there exist $\tilde{\alpha} \in \mathcal{K}_\infty$ and $\tilde{\gamma} \in \mathcal{N}$ such that

$$V_0(\phi(0)) \geq \max\left\{\rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right), \tilde{\gamma}(|v|)\right\} \quad \Rightarrow \quad \nabla V_0(\phi(0)) f(\phi, v) \leq -\tilde{\alpha}(|\phi(0)|). \tag{14}$$

Moreover, if (i) holds, then $\tilde{\alpha}$ can be picked as $\tilde{\alpha} = \alpha/2$ in (14). Conversely, if (ii) holds, then α can be picked as $\alpha = \tilde{\alpha}$ in (13). □

This result is established in Section 5.1. Relying on this proposition, conditions (9) and (12) turn out to be equivalent with the same function ρ. From this equivalence, Theorem 6 results directly from Theorem 7.

Similarly, it is possible to explicitely construct a coercive LKF for exponential ISS, by imposing more constraints on the bounds of V_0 and on its dissipation rate.

Corollary 2 (Coercive exp-ISS LKF from Razumikhin condition) Assume that there exist $\underline{a}, \overline{a}, p > 0$ and a function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ such that, for all $x \in \mathbb{R}^n$,

$$\underline{a}|x|^p \leq V_0(x) \leq \overline{a}|x|^p. \tag{15}$$

Assume further that there exist $a, \rho_0 > 0$ and $\gamma \in \mathcal{N}$, such that any of the following implications hold for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$:

$$V_0(\phi(0)) \geq \max\left\{\rho_0 \max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)), \gamma(|v|)\right\} \quad \Rightarrow \quad \nabla V_0(\phi(0))f(\phi,v) \leq -a|\phi(0)|^p \tag{16}$$

$$V_0(\phi(0)) \geq \rho_0 \max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)) \quad \Rightarrow \quad \nabla V_0(\phi(0))f(\phi,v) \leq -a|\phi(0)|^p + \gamma(|v|). \tag{17}$$

Then, provided that $\rho_0 < 1$, the functional V defined as

$$V(\phi) := \max_{\tau\in[-\Delta,0]} \frac{1}{\rho_0^{\tau/\Delta}} V_0(\phi(\tau)), \quad \forall \phi \in \mathcal{X}^n, \tag{18}$$

is a coercive exp-ISS LKF for system (1). □

The proof of this result is provided in Section 5.3. For input-free systems (namely $v = 0$), the conditions of Corollary 2 coincide exactly with those of Corollary 1. In other words, Corollary 2 allows to construct a coercive LKF for GES for any systems satisfying the GES Razumikhin conditions. Nevertheless, Corollary 2 goes beyond this result, as it also allows to generate an explicit ISS LKF when a non-zero input term acts on the system.

3.2 Halanay condition

We now provide an explicit coercive LKF construction under the Halanay assumptions. Our main result in that direction is the following, which is established in Section 5.4.

Theorem 8 (Coercive ISS LKF from Halanay condition) Assume that there exist $\alpha \in \mathcal{K}_\infty$, $\rho \in \mathcal{P}$, $\gamma \in \mathcal{N}$, and a positive definite and radially unbounded function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ satisfying, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$\nabla V_0(\phi(0))f(\phi,v) \leq -\alpha(V_0(\phi(0))) + \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right) + \gamma(|v|). \tag{19}$$

If there exists $q > 1$ such that the function $s \mapsto \alpha(s) - \rho(qs)$ is of class $\mathcal{K}_\infty$, then the functional V defined as

$$V(\phi) := \max_{\tau\in[-\Delta,0]} q^{\tau/\Delta} V_0(\phi(\tau)), \quad \forall \phi \in \mathcal{X}^n,$$

is a coercive ISS LKF for system (1). □

As compared to Theorem 5, the advantage of this result is that it not only ensures ISS but also provides an explicit ISS LKF, which turns out to be coercive. However, the price to pay is the more restrictive condition that $s \mapsto \alpha(s) - \rho(qs) \in \mathcal{K}_\infty$ for some $q > 1$, while Theorem 5 requires merely $\alpha - \rho \in \mathcal{K}_\infty$. Although these two conditions are equivalent for specific classes of functions (for instance, if both α and ρ are monomials of the same degree), the condition of Theorem 8 remains, in general, slightly more conservative than that of Theorem 5.

The following result, proved in Section 5.5, shows that a similar result can be derived for exp-ISS.

Corollary 3 (Coercive exp-ISS LKF from Halanay condition) Assume that there exist $\underline{a}, \overline{a}, p > 0$ and a function $V_0 \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ satisfying, for all $x \in \mathbb{R}^n$,

$$\underline{a}|x|^p \leq V_0(x) \leq \overline{a}|x|^p. \tag{20}$$

Assume further that there exist $a, \rho_0 > 0$ and $\gamma \in \mathcal{N}$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$\nabla V_0(\phi(0))f(\phi,v) \leq -aV_0(\phi(0)) + \rho_0 \max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)) + \gamma(|v|).$$

If $\rho_0 < a$, then any constant $q \in (1, a/\rho_0)$ is such that the functional V defined as

$$V(\phi) := \max_{\tau\in[-\Delta,0]} q^{\tau/\Delta} V_0(\phi(\tau)), \quad \forall \phi \in \mathcal{X}^n, \tag{21}$$

is a coercive exp-ISS LKF for the system (1). □

In the special case of input-free systems, the conditions of Corollary 3 coincide exactly with those of Theorem 4.

4 Application: chemical reactor model

In this section, we provide an illustration of the interest to explicitly know a coercive LKF. To that aim, we consider a mathematical model of a chemical reactor in which an exothermic chemical reaction takes place. A cooling jacket with negligible axial heat conduction surrounds the reactor. The corresponding model takes the form of the following coupled PDE-ODE system:

$$\partial_t x_c(t,z) + c\partial_z x_c(t,z) = -\xi x_c(t,z) + \xi x(t) \tag{22a}$$

$$\dot{x}(t) = g(x(t)) - (\mu+1)x(t) + \mu \int_0^1 x_c(t,z)dz + u(t). \tag{22b}$$

In this model, $z \in [0,1]$ denotes the position within the cooling jacket (which is here assumed to be of length 1). $x(t) \in \mathbb{R}$ and $x_c(t,z) \in \mathbb{R}$, refer to the temperatures of the reactor mixture and of the cooling medium respectively. Given any $t \geq 0$, we let $x_c[t] : [0,1] \to \mathbb{R}$ be defined as $x_c[t](z) = x_c(t,z)$ for all $z \in [0,1]$ and we embed the two state variables into $X(t) := (x_c[t], x(t))^\top \in \mathcal{C}$, where $\mathcal{C} := C^1([0,1],\mathbb{R}) \times \mathbb{R}$ is the state space of this model. The cooling medium enters the jacket at a fixed temperature that we pick equal to zero, namely:

$$x_c(t,0) = 0, \quad \forall t \geq 0. \tag{23}$$

The constants c, ξ and μ are positive and the function $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz, non-decreasing, with $g(0) = 0$ and $g(x) = -g_0$ for all $x \leq -x^*$, for some constants $g_0, x^* > 0$. $u \in \mathcal{U}$ is an input signal that accounts for possible exogenous disturbances in the evolution of the reactor temperature as well as modelling uncertainties. The derivation of the PDE-ODE model (22) is described in [Karafyllis and Krstic, 2019, Chapter 2], where existence and uniqueness of solutions are also proved.

The stability of model (22) is analyzed in [Karafyllis and Krstic, 2019, Chapter 8] in the absence of an input (namely, $u \equiv 0$). By means of small-gain arguments, it is shown there that the equilibrium point $0 \in \mathcal{C}$ is GES in the state norm $|x(t)| + \|x_c[t]\|$, provided that g is differentiable and the following condition is satisfied:

$$\sup_{x\in\mathbb{R}} g'(x) < 1 + \mu e^{-\xi/c}. \tag{24}$$

System (22) was also studied in [Ahmed-Ali et al., 2021], where a sampled-data observer was designed with measured output $y(t) = x_c(t,1)$ and measured input $u(t)$.

Here, we study the impact of the exogenous disturbance u on the stability of the origin. Our next result provides a condition on the function g under which (22) is exp-ISS in the state norm $\|X(t)\|_{\mathcal{C}} := |x(t)| + \|x_c[t]\| + \|\partial_z x_c[t]\|$.

Proposition 2 (exp-ISS of the chemical reactor) Assume that

$$\ell_g := \sup_{x\neq 0} \frac{g(x)}{x} < 1 + \frac{\mu c}{\xi}\left(1 - e^{-\xi/c}\right). \tag{25}$$

Then the system (22) with boundary condition (23) is exp-ISS in the state norm $\|\cdot\|_{\mathcal{C}}$. More precisely, there exist $K, \lambda, \gamma_0 > 0$ such that, for all $X(0) \in \mathcal{C}$ and all $u \in \mathcal{U}$, its solution satisfies

$$\|X(t)\|_{\mathcal{C}} \leq K\|X(0)\|_{\mathcal{C}} e^{-\lambda t} + \gamma_0 \|u_{[0,t]}\|, \quad \forall t \geq 0.$$

□

It is worth stressing that the constant ℓ_g in (25) may be smaller than the Lipschitz constant of g. The present result extends the stability result in [Karafyllis and Krstic, 2019] in the following ways:

1. The restriction (25) on the Lipschitz constant of g is less demanding than condition (24) as it does not require that g is differentiable and, even in the case when it is, $1 + \frac{\mu c}{\xi}\left(1 - e^{-\xi/c}\right) > 1 + \mu e^{-\xi/c}$ and $\ell_g \leq \sup_{x\in\mathbb{R}} g'(x)$.
2. We do not simply prove exponential stability in the state norm $|x(t)| + \|x_c[t]\|$ but the much stronger exp-ISS property with respect to the input u in the stronger state norm $|x(t)| + \|x_c[t]\| + \|\partial_z x_c[t]\|$.

3. Based on Corollary 2, an explicit ISS Lyapunov functional is constructed, which can allow the study of (22) under different perturbations and not only under the perturbation considered here; for example, one can study the ISS property with respect to boundary disturbances or disturbances acting in the PDE.

The proof of Proposition 2 is detailed in the upcoming subsections.

4.1 A preliminary transformation

Let $C_0^1([0,1],\mathbb{R}) := \{w \in C^1([0,1],\mathbb{R}) : w(0) = 0\}$. To simplify the analysis, we consider the invertible transformation $T : C^0([0,1],\mathbb{R}) \to C_0^1([0,1],\mathbb{R})$ defined for all $w \in C^0([0,1],\mathbb{R})$ as

$$Tw(z) = x_c(z) = \frac{\xi}{c}\int_0^z e^{-\xi s/c}w(s)ds, \quad \forall z \in [0,1]. \tag{26}$$

Its inverse is then given by

$$w(z) = \frac{c}{\xi}e^{\xi z/c}x_c'(z), \quad \forall z \in [0,1]. \tag{27}$$

With this change of variable, the model (22) becomes

$$\partial_t w(t,z) + c\partial_z w(t,z) = 0 \tag{28a}$$

$$\dot{x}(t) = g(x(t)) - (\mu+1)x(t) + \frac{\mu\xi}{c}\int_0^1\int_0^z e^{-\xi s/c}w(t,s)dsdz + u(t), \tag{28b}$$

whereas the boundary condition (23) reads

$$w(t,0) = x(t), \quad \forall t \geq 0. \tag{29}$$

Notice that every solution of (28a)-(29) satisfies, for all $t \geq 0$,

$$w(t,z) = \begin{cases} x(t - z/c) & \text{if } z \in [0, ct] \\ w(0, z - ct) & \text{if } z \in (ct, 1]. \end{cases} \tag{30}$$

Combining (28b) and (30), we conclude that the following delay equation holds for all $t \geq 1/c$:

$$\dot{x}(t) = g(x(t)) - (\mu+1)x(t) + \mu\xi\int_0^1\int_0^{z/c} e^{-\xi s}x(t-s)dsdz + u(t). \tag{31}$$

4.2 Construction of the exp-ISS Lyapunov functional

Consider the function $V_0(x) := x^2/2$, define $\Delta := 1/c$ and let f denote the vector field involved in (31), namely:

$$f(\phi, v) := g(\phi(0)) - (\mu+1)\phi(0) + \xi\mu\int_0^1\int_0^{\Delta z} e^{-\xi s}\phi(-s)dsdz + v, \quad \forall \phi \in \mathcal{X},\ v \in \mathbb{R}.$$

Then, observing that $\phi(0)g(\phi(0)) \leq \ell_g|\phi(0)|^2$, it holds for all $\phi \in \mathcal{X}$ and all $v \in \mathbb{R}$, that

$$\begin{aligned}\nabla V_0(\phi(0))f(\phi,v) &= \phi(0)g(\phi(0)) - (\mu+1)\phi(0)^2 + \xi\mu\phi(0)\int_0^1\int_0^{\Delta z} e^{-\xi s}\phi(-s)dsdz + v\phi(0) \\ &\leq -\left(\mu+1-\ell_g\right)\phi(0)^2 + \xi\mu|\phi(0)|\int_0^1\int_0^{\Delta z} e^{-\xi s}|\phi(-s)|dsdz + |v||\phi(0)|.\end{aligned}$$

Let ρ and γ be positive constants to be selected. When $V_0(\phi(0)) \geq \rho \max\limits_{\tau\in[-\Delta,0]} V_0(\phi(\tau))$ (which gives $|\phi(\tau)| \leq |\phi(0)|/\sqrt{\rho}$ for all $\tau \in [-\Delta, 0]$) and $V_0(\phi(0)) \geq \gamma|v|^2/2$ (which gives $|v| \leq |\phi(0)|/\sqrt{\gamma}$), the above inequality gives

$$\nabla V_0(\phi(0))f(\phi,v) \leq -\left(\mu + 1 - \ell_g - \frac{1}{\sqrt{\gamma}} - \frac{\mu}{\sqrt{\rho}}\left(1 - \frac{1}{\xi\Delta}\left(1 - e^{-\xi\Delta}\right)\right)\right)\phi(0)^2 \tag{32}$$

The function $\kappa : \mathbb{R}_{>0} \to \mathbb{R}$ defined as $\kappa(\rho) := \mu + 1 - \ell_g - \frac{\mu}{\sqrt{\rho}}\left(1 - \frac{1}{\xi\Delta}\left(1 - e^{-\xi\Delta}\right)\right)$ is continuous at $\rho = 1$ and satisfies, by virtue of (25), $\kappa(1) = 1 - \ell_g + \frac{\mu}{\xi\Delta}\left(1 - e^{-\xi\Delta}\right) = 1 - \ell_g + \frac{\mu c}{\xi}\left(1 - e^{-\xi/c}\right) > 0$. Consequently, there exists a neighborhood of 1 on which $\kappa(\rho) > 0$. Pick any $\rho_0 \in (0,1)$ for which $\kappa(\rho_0) > 0$ and let $\gamma_0 := 4/\kappa(\rho_0)^2$. Then we get from (32) that

$$V_0(\phi(0)) \geq \max\left\{\rho_0 \max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)),\, \frac{\gamma_0}{2}|v|^2\right\} \quad\Rightarrow\quad \nabla V_0(\phi(0))f(\phi,v) \leq -\frac{\kappa(\rho_0)}{2}\phi(0)^2.$$

Therefore, all the assumptions of Corollary 2 are satisfied with $a := \kappa(\rho_0)/2$ and $\gamma(s) := 2s^2/\kappa(\rho_0)^2$. We conclude that the functional defined as

$$\tilde{V}(\phi) := \frac{1}{2}\max_{\tau\in[-\Delta,0]} \frac{1}{\rho_0^{c\tau}}\phi(\tau)^2, \quad \forall \phi \in \mathcal{X},$$

is a coercive exp-ISS LKF for the delay system (31). Observe that, for $t \geq \Delta = 1/c$,

$$\tilde{V}(x_t) = \frac{1}{2}\max_{\tau\in[-\Delta,0]} \frac{1}{\rho_0^{c\tau}}x(t+\tau)^2 = \frac{1}{2}\max_{z\in[0,1]} \rho_0^z x(t-z/c)^2.$$

Therefore, by means of (30), we get that for all $t \geq 1/c$

$$\tilde{V}(x_t) = \frac{1}{2}\max_{z\in[0,1]} \rho_0^z w(t,z)^2.$$

This suggests that the functional $V : C^0([0,1],\mathbb{R}) \to \mathbb{R}_{\geq 0}$ defined as

$$V(w) = \frac{1}{2}\max_{z\in[0,1]} \rho_0^z w(z)^2, \qquad \forall w \in C^0([0,1],\mathbb{R}), \tag{33}$$

for some $\rho_0 > 0$, contains interesting information for the ISS analysis of (28). The functional defined by (33) is very different from Lyapunov functionals that have been used for interconnections of hyperbolic PDEs with ODEs (see for instance [Tang and Mazanti, 2017] and [Dos Santos et al., 2008]) which involve weighted L^2 norms of the distributed states. It is a functional that is based on a weighted sup-norm. It should be noted J.-M. Coron in [Coron, 1999] was the first to use a functional which is based on a weighted sup-norm.

4.3 ISS analysis for the PDE-ODE system

Having constructed a candidate ISS Lyapunov functional for the PDE-ODE system (28), we next show that the functional defined by (33) is indeed an exp-ISS Lyapunov functional for (28) and we derive the corresponding exp-ISS estimate for the original system (22). For every solution of (28) and any $h \in [0,1/c)$, we get from (30) that

$$\begin{aligned}
V(w[t+h]) &= \frac{1}{2}\max_{z\in[0,1]} \rho_0^z w(t+h,z)^2 \\
&= \frac{1}{2}\max\left\{\max_{z\in[0,ch]} \rho_0^z w(t+h,z)^2,\, \max_{z\in[ch,1]} \rho_0^z w(t+h,z)^2\right\} \\
&= \frac{1}{2}\max\left\{\max_{z\in[0,ch]} \rho_0^z x(t+h-z/c)^2,\, \max_{z\in[ch,1]} \rho_0^z w(t,z-ch)^2\right\} \\
&= \frac{1}{2}\max\left\{\max_{z\in[0,ch]} \rho_0^z x(t+h-z/c)^2,\, \rho_0^{ch}\max_{z\in[0,1-ch]} \rho_0^z w(t,z)^2\right\} \\
&\leq \max\left\{\frac{1}{2}\max_{z\in[0,h]} \rho_0^{c(h-z)} x(t+z)^2,\, \rho_0^{ch}V(w[t])\right\},
\end{aligned}$$

where we successively used the change of variable $z \leftarrow z - ch$ and $z \leftarrow h - z/c$. Defining, with an abuse of notation, $\dot{V}(t) := \limsup_{h\to 0^+} \frac{V(w[t+h])-V(w[t])}{h}$, this implies that

$$\begin{aligned}
\dot{V}(t) &\leq \limsup_{h\to 0^+} \frac{1}{h}\max\left\{\frac{1}{2}\max_{z\in[0,h]} \rho_0^{c(h-z)} x(t+z)^2 - V(w[t]),\, \left(\rho_0^{ch}-1\right)V(w[t])\right\} \\
&\leq \limsup_{h\to 0^+} \frac{1}{h}\max\left\{\frac{1}{2}\max_{z\in[0,h]} \rho_0^{c(h-z)}\left(x(t)^2 + 2\int_t^{t+z} x(s)\dot{x}(s)ds\right) - V(w[t]),\, \left(\rho_0^{ch}-1\right)V(w[t])\right\} \\
&\leq \limsup_{h\to 0^+} \frac{1}{h}\max\left\{\max_{z\in[0,h]}\left(\frac{x(t)^2-2V(w[t])}{2} + \rho_0^{c(h-z)}\int_t^{t+z} x(s)\dot{x}(s)ds\right),\, \left(\rho_0^{ch}-1\right)V(w[t])\right\}.
\end{aligned} \tag{34}$$

- If $x(t)^2/2 < V(w[t])$, then it follows that

$$\begin{aligned}\dot{V}(t) &\leq \limsup_{h\to 0^+} \max\left\{\max_{z\in[0,h]}\left(\frac{x(t)^2-2V(w[t])}{2h}+\frac{z}{h}\sup_{s\in[t,t+h]}|x(s)\dot{x}(s)|\right), \frac{\rho_0^{ch}-1}{h}V(w[t])\right\}\\ &\leq \limsup_{h\to 0^+} \max\left\{\max_{z\in[0,h]}\left(\frac{x(t)^2-2V(w[t])}{2h}+\sup_{s\in[t,t+h]}|x(s)\dot{x}(s)|\right), \frac{\rho_0^{ch}-1}{h}V(w[t])\right\}\end{aligned} \tag{35}$$

Observing that the first term in the above max tends to $-\infty$ as h tends to 0^+, we conclude that

$$\begin{aligned}\dot{V}(t) &\leq \limsup_{h\to 0^+}\frac{\rho_0^{ch}-1}{h}V(w[t])\\ &\leq -c\,\ln(1/\rho_0)V(w[t]).\end{aligned} \tag{36}$$

- If $x(t)^2/2 = V(w[t])$, then we have from (34) that

$$\begin{aligned}\dot{V}(t) &\leq \limsup_{h\to 0^+}\frac{1}{h}\max\left\{\max_{z\in[0,h]}\rho_0^{c(h-z)}\int_t^{t+z}x(s)\dot{x}(s)ds\,,\,\left(\rho_0^{ch}-1\right)V(w[t])\right\}\\ &\leq \limsup_{h\to 0^+}\max\left\{\max_{z\in[0,h]}\rho_0^{c(h-z)}\frac{x(t+z)^2-x(t)^2}{2h}\,,\,\frac{\rho_0^{ch}-1}{h}V(w[t])\right\}.\end{aligned}$$

Proceeding as in the proof of Lemma 1 (see [Karafyllis and Jiang, 2011, Lemma 6.7]), we get that

$$\dot{V}(t) \leq \max\left\{\limsup_{h\to 0^+}\frac{x(t+h)^2-x(t)^2}{2h}\,,\,-c\,\ln(1/\rho_0)V(w[t])\right\}. \tag{37}$$

We now proceed to estimating the lim sup term appearing in this expression. Using (28b) and recalling that $|g(x(t))| \leq \ell_g|x(t)|$, it holds for almost all $t \geq 0$ that

$$x(t)\dot{x}(t) \leq -\left(\mu+1-\ell_g\right)x(t)^2+\frac{\xi\mu}{c}|x(t)|\int_0^1\int_0^z e^{-\xi s/c}|w(t,s)|dsdz+|x(t)||u(t)|.$$

In view of (33), we have that $|w(t,z)| \leq \rho_0^{-z/2}\sqrt{2V(w[t])}$ for all $t \geq 0$ and all $z \in [0,1]$. Consequently, using Young's inequality $ab \leq \frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2$ that holds for all $\varepsilon > 0$ and all $a, b \in \mathbb{R}$, we get that, for almost all $t \geq 0$, all $\rho_0 \in (e^{-\frac{2\xi}{c}}, 1)$ and all $\varepsilon > 0$,

$$x(t)\dot{x}(t) \leq -\left(\mu+1-\ell_g-\frac{\varepsilon}{2}\right)x(t)^2+\frac{\xi\mu}{ck}\left(1-\frac{1}{k}\left(1-e^{-k}\right)\right)|x(t)|\sqrt{2V(w[t])}+\frac{1}{2\varepsilon}\|u\|^2,$$

where $k := \xi/c + \ln(\rho_0)/2$. Using again Young's inequality, it follows that

$$\begin{aligned}x(t)\dot{x}(t) &\leq -\left(\mu+1-\ell_g-\frac{\varepsilon+\sigma}{2}\right)x(t)^2+\sigma V(w[t])+\frac{1}{2\varepsilon}\|u\|^2\\ &\leq -2\left(\mu+1-\ell_g-\frac{\varepsilon}{2}-\sigma\right)V(w[t])+\frac{1}{2\varepsilon}\|u\|^2,\end{aligned}$$

where $\sigma := \frac{\xi\mu}{ck}\left(1-\frac{1}{k}\left(1-e^{-k}\right)\right)$. We conclude that

$$\limsup_{h\to 0^+}\frac{x(t+h)^2-x(t)^2}{2h} \leq -2\left(\mu+1-\ell_g-\frac{\varepsilon}{2}-\sigma\right)V(w[t])+\frac{1}{2\varepsilon}\|u\|^2. \tag{38}$$

Plugging (38) into (37), we obtain that

$$\dot{V}(t) \leq \max\left\{-2\left(\mu+1-\ell_g-\frac{\varepsilon}{2}-\sigma\right)V(w[t])+\frac{1}{2\varepsilon}\|u\|^2\,,\,-c\,\ln(1/\rho_0)V(w[t])\right\}. \tag{39}$$

Combining (36) and (39), we conclude that the following inequality holds for all $t \geq 0$, all $\rho_0 \in (e^{-\frac{2\xi}{c}}, 1)$ and all $\varepsilon > 0$:

$$\dot{V}(t) \leq -\min\left\{c\,\ln\left(1/\rho_0\right), 2\left(\mu+1-\ell_g-\frac{\varepsilon}{2}-\sigma\right)\right\}V(w[t])+\frac{1}{2\varepsilon}\|u\|^2. \tag{40}$$

The expression $\Gamma(k) := \mu + 1 - \ell_g - \sigma = \mu + 1 - \ell_g - \frac{\xi\mu}{ck}\left(1 - \frac{1}{k}\left(1 - e^{-k}\right)\right)$ is continuous at $k_0 := \xi/c$ and satisfies, by virtue of (25), $\Gamma(k_0) = 1 - \ell_g + \frac{\mu c}{\xi}\left(1 - e^{-\frac{\xi}{c}}\right) > 0$. Therefore, there exists a neighborhood of k_0 on which $\Gamma(k) > 0$. Selecting any arbitrary $k \in (0, k_0)$ (meaning $\rho_0 \in (e^{-\frac{2\xi}{c}}, 1)$) for which $\Gamma(k) > 0$ and picking $\varepsilon = \Gamma(k)$, we get from (40) that, for all $t \geq 0$,

$$\dot{V}(t) \leq -2\lambda V(w[t]) + \frac{1}{2\Gamma(k)}\|u\|^2,$$

where $\lambda := \frac{1}{2}\min\{c\ln(1/\rho_0), \Gamma(k)\} > 0$. Using the comparison lemma [Khalil, 2002, Lemma 3.4], we obtain for all $t \geq 0$:

$$V(w[t]) \leq V(w[0])e^{-2\lambda t} + \frac{\|u\|^2}{4\lambda\Gamma(k)}, \tag{41}$$

Definition (33) implies the following estimates:

$$\frac{\rho_0}{2}\|w\|^2 \leq V(w) \leq \frac{1}{2}\|w\|^2, \quad \forall w \in C^0([0,1], \mathbb{R}). \tag{42}$$

Combining (41) and (42), we obtain that

$$\|w[t]\| \leq \frac{\|w[0]\|}{\sqrt{\rho_0}}e^{-\lambda t} + \frac{\|u\|}{\sqrt{2\rho_0\lambda\Gamma(k)}}. \tag{43}$$

In view of (26) and (27), we also have that

$$\max\left\{\frac{c}{\xi}\|x_c'\|, \frac{1}{1 - e^{-\xi/c}}\|x_c\|\right\} \leq \|w\| \leq \frac{c}{\xi}e^{\xi/c}\|x_c'\|, \quad \forall x_c \in C_0^1([0,1], \mathbb{R}). \tag{44}$$

Combining (43) and (44) and using (29), we obtain that, for all $t \geq 0$,

$$\max\left\{|x(t)|, \frac{c}{\xi}\|\partial_z x_c[t]\|, \frac{1}{1 - e^{-\xi/c}}\|x_c[t]\|\right\} \leq \frac{c}{\xi\sqrt{\rho_0}}e^{\xi/c}\|\partial_z x_c[0]\|e^{-\lambda t} + \frac{\|u\|}{\sqrt{2\rho_0\lambda\Gamma(k)}}.$$

Therefore by means of causality, we conclude that every solution of (22)-(23) with $u \in \mathcal{U}$ satisfies the following estimate for all $t \geq 0$:

$$\max\left\{|x(t)|, \frac{c}{\xi}\|\partial_z x_c[t]\|, \frac{1}{1 - e^{-\xi/c}}\|x_c[t]\|\right\} \leq \frac{c}{\xi\sqrt{\rho_0}}e^{\xi/c}\|\partial_z x_c[0]\|e^{-\lambda t} + \frac{\|u_{[0,t]}\|}{\sqrt{2\rho_0\lambda\Gamma(k)}}.$$

By letting $\eta := \min\left\{1, c/\xi, \frac{1}{1-e^{-\xi/c}}\right\}$, the conclusion follows with

$$K := \frac{3ce^{\xi/c}}{\xi\eta\sqrt{\rho_0}}, \qquad \gamma_0 := \frac{3}{\eta\sqrt{2\rho_0\lambda\Gamma(k)}}.$$

5 Proofs

In this section, we provide the proofs of our main results. We start by recalling the following fact that will be used several times: given any function $V_0 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous, positive definite and radially unbounded, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \leq V_0(x) \leq \overline{\alpha}(|x|), \quad \forall x \in \mathbb{R}^n. \tag{45}$$

5.1 Proof of Proposition 1

For the sake of completeness, we first state the following technical lemma.

Lemma 2 (Bound on vector fields) Given any mapping $F : \mathcal{X}^n \times \mathbb{R}^m \to \mathbb{R}^n$, which is Lipschitz on bounded sets, there exist $\mu \in \mathcal{K}_\infty$ and $c \geq 0$ such that for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$|F(\phi, v)| \leq \mu(\|\phi\| + |v|) + c. \tag{46}$$

Moreover, if $F(0, 0) = 0$, then we can pick $c = 0$. □

Proof. Consider the function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as:

$$\rho(s) := \sup_{\|\phi\|+|v|\leq s} |F(\phi, v)|, \quad \forall s \geq 0.$$

The function ρ is well defined and locally bounded as F is Lipschitz on bounded sets. Moreover, ρ is non-negative and non-decreasing. Hence $c := \lim_{s\to 0^+} \rho(s)$ exists and is finite. Consider the function $\tilde{\rho} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$\tilde{\rho}(s) := \begin{cases} 0 & \text{if } s = 0 \\ \rho(s) - c & \text{if } s > 0. \end{cases}$$

Then $\tilde{\rho}$ is non-decreasing and continuous at 0. Applying [Karafyllis and Jiang, 2011, Lemma 3.4], there exists $\mu \in \mathcal{K}_\infty$ such that $\tilde{\rho}(s) \leq \mu(s)$ for all $s \geq 0$. It follows that

$$\begin{aligned} |F(\phi, v)| &\leq \rho(\|\phi\| + |v|) \\ &\leq \tilde{\rho}(\|\phi\| + |v|) + c \\ &\leq \mu(\|\phi\| + |v|) + c. \end{aligned}$$

Moreover, if F is 0 at 0 then $\rho(0) = 0$ (since F is continuous) and consequently $c = 0$, which completes the proof. ■

We start by showing that condition (13) implies the Razumikhin condition (14). To that aim, recall that (13) means that there exist $\alpha, \rho \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{N}$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$V_0(\phi(0)) \geq \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right) \quad \Rightarrow \quad \nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha(|\phi(0)|) + \gamma(|v|). \tag{47}$$

We claim that (14) then holds with the functions $\tilde{\alpha} \in \mathcal{K}_\infty$ and $\tilde{\gamma} \in \mathcal{N}$ defined as

$$\tilde{\alpha} := \alpha/2, \qquad \tilde{\gamma} := \overline{\alpha} \circ \alpha^{-1} \circ 2\gamma, \tag{48}$$

where $\overline{\alpha}$ is defined in (45). To that aim, consider all $\phi \in \mathcal{X}^n$ satisfying the left-hand side of (14), namely

$$V_0(\phi(0)) \geq \max\left\{\rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right), \tilde{\gamma}(|v|)\right\}. \tag{49}$$

Then we have in particular that $V_0(\phi(0)) \geq \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right)$ and it follows from (47) that

$$\nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha(|\phi(0)|) + \gamma(|v|). \tag{50}$$

Due to (45) and (49), we have that

$$\tilde{\gamma}(|v|) \leq V_0(\phi(0)) \leq \overline{\alpha}(|\phi(0)|).$$

It follows from (48) that $\gamma(|v|) = \frac{1}{2}\alpha \circ \overline{\alpha}^{-1} \circ \tilde{\gamma}(|v|) \leq \frac{1}{2}\alpha(|\phi(0)|)$. So, (50) ensures that

$$\nabla V_0(\phi(0)) f(\phi, v) \leq -\tilde{\alpha}(|\phi(0)|),$$

which corresponds to the right-hand side of (14) and thus proves that (13) implies (14).

Let us now prove the second part of equivalence, that is (14) implies (13). To that aim, let us first apply Proposition 2 to the vector field f of system (1) to get that there exists $\alpha_1 \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$|f(\phi, v)| \leq \alpha_1(\|\phi\| + |v|). \tag{51}$$

Also, since V_0 is a continuously differentiable positive definite function, it reaches a global minimum at 0. Hence, $\nabla V_0(0) = 0$. Using continuity of ∇V_0 and [Karafyllis and Jiang, 2011, Lemma 2.4], we conclude that there exists $\alpha_2 \in \mathcal{K}_\infty$ such that:

$$|\nabla V_0(\phi(0))| \leq \alpha_2(|\phi(0)|), \quad \forall \phi \in \mathcal{X}^n. \tag{52}$$

Now, assume that (14) holds, meaning that there exist $\tilde{\alpha}, \rho \in \mathcal{K}_\infty$ and $\tilde{\gamma} \in \mathcal{N}$ such that:

$$V_0(\phi(0)) \geq \max\left\{\rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right), \tilde{\gamma}(|v|)\right\} \quad \Rightarrow \quad \nabla V_0(\phi(0))f(\phi, v) \leq -\tilde{\alpha}(|\phi(0)|). \tag{53}$$

Notice that if (53) holds then it also holds for any $\overline{\gamma}$ satisfying $\overline{\gamma}(s) \geq \gamma(s)$ for all $s \geq 0$. In particular, γ can be assumed to be of class $\mathcal{K}_\infty$ with no loss of generality. Consider all $\phi \in \mathcal{X}^n$ satisfying the left-hand side of (13), namely:

$$V_0(\phi(0)) \geq \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right). \tag{54}$$

If $V_0(\phi(0)) \geq \tilde{\gamma}(|v|)$, we readily get from (53) that $\nabla V_0(\phi(0))f(\phi, v) \leq -\tilde{\alpha}(|\phi(0)|)$, meaning that the right-hand side of (13) is fulfilled. So we can now focus on the case when $\tilde{\gamma}(|v|) > V_0(\phi(0))$. In view of (51) and (52), we then have that

$$\nabla V_0(\phi(0))f(\phi, v) \leq \alpha_2(|\phi(0)|)\alpha_1(\|\phi\| + |v|). \tag{55}$$

Since $\tilde{\gamma}(|v|) > V_0(\phi(0))$, we get from (45) that

$$|\phi(0)| \leq \underline{\alpha}^{-1}(V_0(\phi(0))) < \underline{\alpha}^{-1} \circ \tilde{\gamma}(|v|),$$

and, using (54), we get that

$$\|\phi\| \leq \underline{\alpha}^{-1} \circ \rho^{-1}(V_0(\phi(0))) < \underline{\alpha}^{-1} \circ \rho^{-1} \circ \tilde{\gamma}(|v|).$$

Consequently, (55) ensures that

$$\nabla V_0(\phi(0))f(\phi, v) \leq \alpha_2 \circ \underline{\alpha}^{-1} \circ \tilde{\gamma}(|v|)\, \alpha_1\left(\underline{\alpha}^{-1} \circ \rho^{-1} \circ \tilde{\gamma}(|v|) + |v|\right) \leq \mu(|v|),$$

where $\mu \in \mathcal{K}_\infty$ is the function defined as $\mu(s) := \alpha_2 \circ \underline{\alpha}^{-1} \circ \tilde{\gamma}(s)\, \alpha_1\left(\underline{\alpha}^{-1} \circ \rho^{-1} \circ \tilde{\gamma}(s) + s\right)$ for all $s \geq 0$. Using again the fact that $\tilde{\gamma}(|v|) > V_0(\phi(0))$ and (45), we get that

$$\begin{aligned}
\nabla V_0(\phi(0))f(\phi, v) &\leq \mu(|v|) \\
&\leq -\tilde{\alpha}(|\phi(0)|) + \tilde{\alpha}(|\phi(0)|) + \mu(|v|) \\
&\leq -\tilde{\alpha}(|\phi(0)|) + \tilde{\alpha} \circ \underline{\alpha}^{-1} \circ \tilde{\gamma}(|v|) + \mu(|v|).
\end{aligned}$$

Hence, when (54) holds, we have that

$$\nabla V_0(\phi(0))f(\phi, v) \leq -\alpha(|\phi(0)|) + \gamma(|v|),$$

where $\alpha := \tilde{\alpha}$ and $\gamma := \tilde{\alpha} \circ \underline{\alpha}^{-1} \circ \tilde{\gamma} + \mu \in \mathcal{K}_\infty$. This ensures that (13) holds. Therefore the equivalence holds and the proof of Proposition 1 is complete.

5.2 Proof of Theorem 7

Let $c := \frac{1}{\Delta} \ln\left(\frac{1}{\ell}\right)$. Then it holds from (10) that $c > 0$ and $\rho(s) \leq se^{-c\Delta}$ for all $s > 0$. Since this inequality also trivially holds for $s = 0$, we get that

$$\rho(s) \leq se^{-c\Delta}, \quad \forall s \geq 0. \tag{56}$$

With this choice of c, the functional V defined in (11) reads

$$V(\phi) = \max_{\tau\in[-\Delta,0]} \frac{1}{\ell^{\tau/\Delta}} V_0(\phi(\tau)) = \max_{\tau\in[-\Delta,0]} e^{c\tau} V_0(\phi(\tau)).$$

It follows in particular from (45) that, for all $\phi \in \mathcal{X}^n$,

$$e^{-c\Delta}\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|),$$

and we get from Lemma 1 that V is a coercive LKF and the following implications hold:

$$V(\phi) > V_0(\phi(0)) \quad \Rightarrow \quad D^+V(\phi, f(\phi, v)) \leq -cV(\phi). \tag{57}$$

$$V(\phi) = V_0(\phi(0)) \quad \Rightarrow \quad D^+V(\phi, f(\phi, v)) \leq \max\left\{-cV(\phi)\,,\, \nabla V_0(\phi(0))f(\phi, v)\right\}. \tag{58}$$

In view of (56), it holds that

$$\rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right) \leq e^{-c\Delta} \max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)) \leq \max_{\tau\in[-\Delta,0]} e^{c\tau}V_0(\phi(\tau)) = V(\phi).$$

It follows that

$$V(\phi) = V_0(\phi(0)) \quad \Rightarrow \quad \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right) \leq V_0(\phi(0)).$$

So, using assumption (12), we get from (58) that, whenever $V(\phi) = V_0(\phi(0))$,

$$D^+V(\phi, f(\phi, v)) \leq \max\left\{-cV(\phi), -\alpha(|\phi(0)|) + \gamma(|v|)\right\}.$$

It follows from (45) that, whenever $V(\phi) = V_0(\phi(0))$,

$$\begin{aligned} D^+V(\phi, f(\phi, v)) &\leq \max\left\{-cV(\phi)\,,\, -\alpha\circ\overline{\alpha}^{-1}(V_0(\phi(0))) + \gamma(|v|)\right\} \\ &\leq -\min\left\{cV(\phi)\,,\, \alpha\circ\overline{\alpha}^{-1}(V(\phi))\right\} + \gamma(|v|) \\ &\leq -\tilde{\alpha}(V(\phi)) + \gamma(|v|), \end{aligned} \tag{59}$$

with $\tilde{\alpha}(s) := \min\{cs, \alpha\circ\overline{\alpha}^{-1}(s)\}$ for all $s \geq 0$. Finally, combining (57) and (59), we have for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$ that

$$D^+V(\phi, f(\phi, v)) \leq -\tilde{\alpha}(V(\phi)) + \gamma(|v|), \tag{60}$$

meaning that V is a coercive ISS LKF for (1). ISS follows from Theorem 1, which completes the proof.

5.3 Proof of Corollary 2

First observe that, as ensured by Proposition 1, condition (17) implies condition (16) (up to to the division of a by 2). So we just need to establish the result under (16), namely:

$$V_0(\phi(0)) \geq \max\left\{\rho_0 \max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)), \gamma(|v|)\right\} \quad \Rightarrow \quad \nabla V_0(\phi(0))f(\phi, v) \leq -a|\phi(0)|^p.$$

This corresponds to the main assumption of Theorem 7, in which $\rho(s) = \rho_0 s$ and $\alpha(s) = as^p$ for all $s \geq 0$. With the notations of Theorem 7, we also have $\ell = \rho_0$, which is assumed to be smaller than 1. The sandwich condition (45) also holds with $\underline{\alpha}(s) = \underline{a}s^p$ and $\overline{\alpha}(s) = \overline{a}s^p$. Following the proof steps of Theorem 7 (see (60)), we get that the functional defined in (18) is Lipschitz on bounded sets and satisfies, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$, $D^+V(\phi, f(\phi, v)) \leq -\tilde{\alpha}(V(\phi)) + \gamma(|v|)$, where $\tilde{\alpha}(s) = \min\{cs, \alpha\circ\overline{\alpha}^{-1}(s)\} = \min\{c, a/\overline{a}\}s$ with the constant $c := \frac{1}{\Delta}\ln\left(\frac{1}{\rho_0}\right) > 0$. In other words:

$$D^+V(\phi, f(\phi, v)) \leq -\min\left\{c, a/\overline{a}\right\}V(\phi) + \gamma(|v|).$$

It also holds from (15) and (18) that, for all $\phi \in \mathcal{X}^n$,

$$\rho_0\underline{a}\|\phi\|^p \leq V(\phi) \leq \overline{a}\|\phi\|^p.$$

Thus, V is indeed a coercive exp-ISS LKF for (1).

5.4 Proof of Theorem 8

Consider the functional V proposed in the statement, namely $V(\phi) := \max_{\tau\in[-\Delta,0]} q^{\tau/\Delta}V_0(\phi(\tau))$, where $q > 1$ is such that $s \mapsto \alpha(s) - \rho(qs)$ is of class $\mathcal{K}_\infty$. Then it holds from (45) that

$$\frac{1}{q}\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}^n.$$

Note that V can equivalently be written as $V(\phi) = \max_{\tau\in[-\Delta,0]} e^{c\tau/\Delta} V_0(\phi(\tau))$, where $c := \ln(q) > 0$. Lemma 1 then ensures that V is a coercive LKF and satisfies, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$V(\phi) > V_0(\phi(0)) \quad \Rightarrow \quad D^+V(\phi, f(\phi, v)) \leq -\frac{c}{\Delta} V(\phi) \tag{61}$$

$$V(\phi) = V_0(\phi(0)) \quad \Rightarrow \quad D^+V(\phi, f(\phi, v)) \leq \max \left\{ -\frac{c}{\Delta} V(\phi)\,,\, \nabla V_0(\phi(0)) f(\phi, v) \right\}. \tag{62}$$

Consider first all the $\phi \in \mathcal{X}^n$ satisfying

$$V(\phi) = V_0(\phi(0)). \tag{63}$$

Then we have by (62) and (19) that

$$D^+V(\phi, f(\phi, v)) \leq \max \left\{ -\frac{c}{\Delta} V(\phi)\,,\, -\alpha(V_0(\phi(0))) + \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)) \right) + \gamma(|v|) \right\}.$$

Observe that

$$\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)) \leq e^c \max_{\tau\in[-\Delta,0]} e^{c\tau/\Delta} V_0(\phi(\tau)) = e^c V(\phi).$$

From (63) and the fact that $\rho \in \mathcal{P}$, it follows that

$$\begin{aligned} D^+V(\phi, f(\phi, v)) &\leq \max \left\{ -\frac{c}{\Delta} V(\phi)\,,\, -\alpha(V(\phi)) + \rho\left(e^c V(\phi)\right) \right\} + \gamma(|v|) \\ &\leq -\min \left\{ \frac{c}{\Delta} V(\phi)\,,\, \alpha(V(\phi)) - \rho(qV(\phi)) \right\} + \gamma(|v|). \end{aligned} \tag{64}$$

Let $\tilde{\alpha} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as $\tilde{\alpha}(s) := \min\{cs/\Delta\,,\, \alpha(s) - \rho(qs)\}$ for all $s \geq 0$. Recalling that $s \mapsto \alpha(s) - \rho(qs)$ is of class $\mathcal{K}_\infty$ by assumption, it holds that $\tilde{\alpha} \in \mathcal{K}_\infty$ and, combining (61) and (64), one finally has that, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$,

$$D^+V(\phi, f(\phi, v)) \leq -\tilde{\alpha}(V(\phi)) + \gamma(|v|), \tag{65}$$

meaning that V is indeed a coercive ISS LKF for (1).

5.5 *Proof of Corollary 3*

The assumptions of Corollary 3 are a special case of those of Theorem 8, in which $\alpha(s) = as$ and $\rho(s) = \rho_0 s$ for all $s \geq 0$. Moreover, the sandwich condition (20) is a special case of (45) in which $\underline{\alpha}(s) = \underline{a}s^p$ and $\overline{\alpha}(s) = \overline{a}s^p$. Theorem 8 imposes that the function $s \mapsto \alpha(s) - \rho(qs)$ is of class $\mathcal{K}_\infty$. This boils down here to $a - \rho_0 q > 0$, which is indeed fulfilled for $q \in (1, a/\rho_0)$. Following the proof steps of Theorem 8 (see (65)) and letting $c := \ln(q)$, the functional defined in (21) is Lipschitz on bounded sets and satisfies, for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$, $D^+V(\phi, f(\phi, v)) \leq -\tilde{\alpha}(V(\phi)) + \gamma(|v|)$ with $\tilde{\alpha}(s) := \min\{cs/\Delta, \alpha(s) - \rho(qs)\} = \min\{c/\Delta, a - \rho_0 q\}s$ for all $s \geq 0$. The quantity $\tilde{a} := \min\{c/\Delta, a - \rho_0 q\}$ is positive by assumption and we get

$$D^+V(\phi, f(\phi, v)) \leq -\tilde{a}V(\phi) + \gamma(|v|).$$

Finally, it holds from (20) and (21) that, for all $\phi \in \mathcal{X}^n$,

$$\frac{\underline{a}}{q}\|\phi\|^p \leq V(\phi) \leq \overline{a}\|\phi\|^p.$$

Thus, V is indeed a coercive exp-ISS LKF for (1).

6 Conclusion

In this paper, we have shown that, in most cases, it is possible to explicitly derive a coercive Lyapunov-Krasovkii functional for systems that satisfy Razumikhin or Halanay conditions. The LKF construction we propose can be applied to both ISS and exp-ISS, and therefore to both global asymptotic stability and global exponential stability by simply considering a zero-input. We have shown through the example of a chemical reactor whose temperature is regulated by a cooling fluid, that this LKF construction can lead to enhanced stability results not only by allowing to involve tighter norms in the ISS estimate, but also by opening the door to more systematic robustness analysis with respect to parameter uncertainties or model imprecisions.

The results presented here suggest two future lines of investigation. The first one is about the slight additional conservatism introduced to be able to construct the LKF. In the Razumikhin condition, the nonlinear gain ρ is not allowed here to get arbitrarily close to identity although it is in the general Razumikhin approach. Similarly, a slightly more demanding constraint is imposed in the Halanay context. We believe that the systematic construction of an LKF in the exact same conditions as the original Razumikhin and Halanay results would be worth investigating. A second possible line of research is the use of vector versions of Halanay conditions. It was shown in [Mazenc et al., 2022, Mazenc and Malisoff, 2021, Mazenc and Malisoff, 2022] that matrix-based conditions can be derived to conclude stability of systems involving several Lyapunov functions, which turns out particularly useful for networks of interconnected dynamical systems. While the techniques presented here allow for the construction of an LKF in that setup, the conditions we obtained so far appear too demanding as compared to the flexibility offered the original vector extension of Halanay's inequality.

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