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Global Well-Posedness for the 3D Rotating Boussinesq Equations in Variable Exponent Fourier-Besov Spaces

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Abstract

We study the small initial data Cauchy problem for the three-dimensional Boussinesq equations with the Coriolis force in variable exponent Fourier-Besov spaces. By using the Fourier localization argument and Littlewood-Paley decomposition, we obtain the global well-posedness result for small initial data (u_0, θ_0) belonging to the critical variable exponent Fourier-Besov spaces $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}$.

KEYWORDS:
Boussinesq equations; Coriolis force; global well-posedness; variable exponent Fourier-Besov spaces

1 | INTRODUCTION

In this paper, we consider the three-dimensional Boussinesq equations with the Coriolis force:

$$\begin{cases} \partial_t u - \nu\Delta u + \Omega e_3 \times u + (u\cdot\Delta)\cdot u + \nabla P = g\theta e_3, & \text{in} \quad \mathbb{R}^3\times(0,\infty), \\ \partial_t\theta - \mu\Delta\theta + (u\cdot\Delta)\cdot\theta = 0, & \text{in} \quad \mathbb{R}^3\times(0,\infty), \\ \mathrm{div} u = 0, & \text{in} \quad \mathbb{R}^3\times(0,\infty), \\ u(x,0) = u_0, \quad \theta(x,0) = \theta_0, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ denotes the velocity field of the fluid, θ is the fluctuation, P is the the pressure. The positive constants ν, μ and g are the kinetic viscosity, the thermal diffusivity and the gravity. $\Omega \in \mathbb{R}$ is the Coriolis parameter, which denotes twice the speed of rotation around the vertical unit vector $e_3 = (0, 0, 1)$. The term $g\theta e_3$ represents buoyancy force using the Boussinesq approximation, which consists in neglecting the density dependence in all the terms but the one involving the gravity. The parameters ν and μ do not play any important role and we set $\nu = \mu = 1$ throughout the rest of this paper. For more detailed explanation, we can refer the readers to Babin[1], Charve[2], Cushman-Roisin[3] and Pedlosky[4].

When $\Omega = 0$, (1.1) reduces to the classical Boussinesq equations. Abidi, Hmidi and Keraani[5] proved a global well-posedness result for tridimensional Navier-Stokes-Boussinesq system with axisymmetric initial data. Danchin and Paicu[6] studied the Cauchy problem for the Boussinesq equations with partial viscosity in dimension $N \geq 3$ and obtained a global existence and uniqueness result for small data in Lorentz spaces. They[7] also proved the global existence of finite energy weak solutions in any dimension, and global well-posedness in dimension $N \geq 3$ for small data. In the two-dimensional case, the finite energy global solutions were shown to be unique for any data in $L^2(\mathbb{R}^2)$. Hmidi and Rousset[8] proved the global well-posedness for a three-dimensional Boussinesq system with axisymmetric initial data. Karch and Prioux[9] studied the existence and the asymptotic stability as the time variable escapes to infinity of self-similar solutions to the viscous Boussinesq equations posed in the whole three-dimensional space. Sulaiman[10] obtained to the global existence and uniqueness results for the three-dimensional Boussinesq equations with axisymmetric initial data $v^0 \in \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)$ and $\rho^0 \in \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $p > 6$.

When $\Omega \neq 0$, but $\theta \equiv 0$, (1.1) reduces to the Navier-Stokes equations with the Coriolis force. Babin Mahalov and Nicolaenko [11] proved existence on infinite time intervals of regular solutions to the 3D rotating Navier-Stokes equations in the limit of strong rotation. Iwabuchi and Takada [12] proved the global in time existence and the uniqueness of the mild solution for small initial data in $\mathcal{B}^{-1}_{1,2}$ near $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ and obtained the ill-posedness for the Navier-Stokes equations with the Corilis force. Fang, Han and Hieber [13] proved the uniqueness of the global mild solution to the rotating Navier-Stokes equations with only horizontal dissipation in the Fourier-Besov space $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p}}_{p,r}(\mathbb{R}^3)$ for $p \in [2,\infty], r \in [1,\infty)$. Hieber and Shibata [14] proved that the Navier-Stokes equations with the Coriolis force possess a unique global mild solution for arbitrary speed of rotation provided the initial data u_0 is small enough in the $H^{\frac{1}{2}}_{\sigma}(\mathbb{R}^3)$. We refer the readers to Babin, Mahalov, Nicolaenko [1 15 16], Iwabuchi, Takada [17], Giga, Inui, Mahalov, Saal [18], Koh, Lee, Takada [19], Konieczny, Yoneda [20], Sun, Yang, Cui [21] and Sun, Liu, Zhang [22].

When $\Omega = 0$, and $\theta \equiv 0$, (1.1) reduces to the classical Navier-Stokes equations. Abidi, Gui and Zhang [23] proved the local well-posedness of three-dimensional incompressible inhomogeneous Navier-Stokes equations with initial data (a_0, u_0) in the critical Besov spaces and proved this system is globally well-posed provided that $\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}$ is sufficiently small. Sun and Liu [24] demonstrated uniqueness of the weak solution to the fractional anisotropic Navier-Stokes system with only horizontal dissipation. Kozono, Ogawa and Taniuchi [25] proved a local existence for the Navier-Stokes equations with the initial in $\mathcal{B}^0_{\infty,\infty}(\mathbb{R}^n)$ containing functions which do not decay at infinity and established an extension criterion on our local solutions in terms of vorticity in the homogeneous Besov space $\mathcal{B}^0_{\infty,\infty}(\mathbb{R}^n)$. Bourgain and Pavlovic [26] proved the Cauchy problem for the three-dimensional Navier–Stokes equations is ill-posed in $\dot{B}^{-1,\infty}_{\infty}(\mathbb{R}^3)$. Ru and Abidin [27] studied the Cauchy problem of the fractional Navier-Stokes equations in critical variable exponent Fourier-Besov spaces $\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}(\mathbb{R}^3)$. Yu and Zhai [28] studied the well-posedness of the fractional Navier-Stokes equations in some supercritical as well as in the largest critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$ for $\beta \in (\frac{1}{2}, 1)$ and the well-posedness for fractional magnetohydrodynamics equations in these Besov spaces.

There are many differences between variable exponent Fourier-Besov spaces and Fourier-Besov Spaces. Some classical theories such as Young's inequality and the multiplier theorem do not hold in variable exponent Fourier-Besov spaces. Because of this, it is difficult to consider the well-posedness of equations on such spaces. In this paper, we mainly use the properties introduced in Section 2, 3 and combine with the Banach's contraction mapping principle to consider the global well-posedness of the Boussinesq equations with the Coriolis force in variable exponent frequency spaces $\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^3)$. The main results are as follows.

Theorem 1.1. Let $p(\cdot) \in C_{\log}(\mathbb{R}^3) \cap \mathcal{P}_0(\mathbb{R}^3)$, $2 \leq p(\cdot) \leq 6$, $1 \leq q, \rho \leq \infty$, and there exist a sufficiently small $\epsilon > 0$, such that

$$\|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} < \epsilon$$

for $\Omega \in \mathbb{R}$. Then problem (1.1) has a unique global solution

$$(u,\theta) \in \widetilde{L}^{\infty}(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}) \cap \widetilde{L}^{\rho}(0,\infty;\dot{\mathcal{B}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q}) \cap \widetilde{L}^{\infty}(0,\infty;\dot{\mathcal{B}}^{\frac{1}{2}}_{2,q}).$$

Moreover, let $p(\cdot) \in C_{\log}(\mathbb{R}^3) \cap \mathcal{P}_0(\mathbb{R}^3)$, $s_1(\cdot) \in C_{\log}(\mathbb{R}^3)$, and $s_1(\cdot) = \frac{2}{\rho} + 2 - \frac{3}{p_1(\cdot)}$, if there exist a constant $c > 0$ such that $2 \leq p_1(\cdot) \leq c \leq p(\cdot)$, then the above solution is still satisfied

$$(u,\theta) \in \mathcal{C}(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}) \cap \widetilde{L}^{\rho}(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q}).$$

Remark 1. The Fourier-Besov space $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}$ is critical for (1.1). In fact, if $u(t,x)$ is the solution of Eq.(1.1), then

$$u_{\lambda}(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

is also a solution of the same equation and

$$\|u_0(0,x)\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} \sim \|u_{\lambda}(0,x)\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.$$

Remark 2. From the structure of variable exponent Fourier-Besov space, we can find that this kind of space is quite different from variable exponent Besov space. Compared with variable exponent Besov space, this kind of space is more favorable for us to consider the boundedness of semigroup operators and the estimation of nonlinear terms.

In section1, we mainly introduce some background and main results; section2, we recall some basic facts about Littlewood-Paley theory and function spaces; section3, we establish the linear estimates of the semigroup $\{T_\Omega(t)\}_{t>0}$; section4, we devoted to the proof of Theorem1.1.

2 | FUNCTION SPACES

$\mathcal{S}(\mathbb{R}^n)$ denotes the space of smooth rapidly decreasing functions on $\mathbb{R}^n$. $\mathcal{S}'(\mathbb{R}^n)$ denotes the topological dual space of the $\mathcal{S}(\mathbb{R}^n)$, also be called temperate distribution. For any $f \in X$, there exists a constant $c > 0$ such that $\|f\|_a \le c\|f\|_b$, then it is written as $\|\cdot\|_a \lesssim \|\cdot\|_b$. We first recall the homogeneous Littlewood-Paley decomposition[29].

Let (χ, φ) be a couple of smooth functions with values in $[0,1]$, χ is supported in the ball $B(0,\frac{3}{4}) = \{\xi \in \mathbb{R}^3 | |\xi| \le \frac{3}{4}\}$, φ is supported in the shell $C(0,\frac{3}{4},\frac{8}{3}) = \{\xi \in \mathbb{R}^3 | \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. We use $\varphi_j(\xi)$ to denote $\varphi(2^{-j}\xi)$ and

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \qquad \forall \xi \in \mathbb{R}^3 \backslash \{0\}.$$

The localization operators are defined by

$$\dot{\Delta}_j u = \varphi_j(D)u = 2^{3j} \int\limits_{\mathbb{R}^3} \psi(2^j y)u(x-y)\mathrm{d}y, \qquad \forall j \in \mathbb{Z},$$

$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{3j} \int\limits_{\mathbb{R}^3} h(2^j y)u(x-y)\mathrm{d}y, \qquad \forall j \in \mathbb{Z},$$

where $\psi = \mathcal{F}^{-1}\varphi$ and $h = \mathcal{F}^{-1}\chi$.

From the definition above there hold that

$$\dot{\Delta}_k \dot{\Delta}_j u = 0, \qquad \text{if} \quad |j-k| \ge 2,$$
$$\dot{\Delta}_k(\dot{S}_{j-1}u\dot{\Delta}_j u) = 0, \quad \text{if} \quad |j-k| \ge 5.$$

If $u \in \mathcal{S}'_h$, there holds that

$$\dot{S}_j u = \sum_{i\le j-1} \dot{\Delta}_i u.$$

Let $\mathcal{P}_0(\mathbb{R}^n)$ be the set of all measure functions $p(\cdot) : \mathbb{R}^n \to (0,\infty]$ such that $p_- = \operatorname{essinf}_{x\in\mathbb{R}^n} p(x)$, $p_+ = \operatorname{esssup}_{x\in\mathbb{R}^n} p(x)$. For $p \in \mathcal{P}_0(\mathbb{R}^n)$, let $L^{p(\cdot)}(\mathbb{R}^n)$ be the set of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\begin{aligned}\|f\|_{L^{p(\cdot)}} &:= \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \le 1\} \\ &= \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1\right\}.\end{aligned}$$

We postulate the following standard conditions to ensure that the Hardy-Maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$:

1. p is said to satisfy the Locally log-Hölder's continuous condition if there exists a positive constant $C_{\log}(p)$ such that $|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e+|x-y|^{-1})}$, (for all $x, y \in \mathbb{R}^n, x \ne y$).
2. p is said to satisfy the Globally log-Hölder's continuous condition if there exists a positive constant $C_{\log}(p)$ and p_∞, such that $|p(x) - p_\infty| \le \frac{C_{\log}(p)}{\log(e+|x|)}$, (for all $x \in \mathbb{R}^n$).

We use $C_{\log}(\mathbb{R}^n)$ as the set of all real valued functions $p : \mathbb{R}^n \to \mathbb{R}$ satisfying 1 and 2.

Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, we use $l^{q(\cdot)}(L^{p(\cdot)})$ to denote the space consisting of all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\mu > 0, \varrho_{l^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\mu}\right\}_{j\in\mathbb{Z}}\right) \le 1\right\} \le \infty,$$

where

$$\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^n} \left(\frac{|f_j(x)|}{\lambda^{\frac{1}{q(x)}}}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

Since we assume that $q_+ < \infty$, $\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} \left\| |f_j|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}$ holds.

Definition 2.1. [30] Let $p(\cdot), q(\cdot) \in C_{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in C_{\log}(\mathbb{R}^n)$. The homogeneous Besov space with variable exponents $\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)} = \{ f \in \mathcal{S}' : \|f\|_{\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} < \infty \},$$

$$\|f\|_{\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\Delta_j f\}_{j\in\mathbb{Z}}\|_{\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}} < \infty,$$

where $\mathcal{S}'$ denote the dual of $\mathcal{S}(\mathbb{R}^n) = \{ f \in \mathcal{S}(\mathbb{R}^n) : (\mathrm{D}^\alpha \hat{f})(0) = 0, \forall \alpha \}$.

For $T > 0$ and $\rho \in [1, \infty]$, we denote by $L^\rho(0, T; \dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$ the set of all tempered distribution u satisfying

$$\|u\|_{L^\rho(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} := \left\| \left(\sum_{j=0}^{\infty} \|2^{js(\cdot)}\Delta_j u\|^r_{L^{p(\cdot)}} \right)^{\frac{1}{r}} \right\|_{L^\rho_T} < \infty.$$

The mixed $\widetilde{L}^\rho(0, T; \dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$ is the set of all tempered distribution u satisfying

$$\|u\|_{\widetilde{L}^\rho(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} := \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\Delta_j u\|^r_{L^\rho_T L^{p(\cdot)}} \right)^{\frac{1}{r}} < \infty.$$

For simplicity, we denote

$$L^\rho_T \dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r} := L^\rho(0, T; \dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r}) \quad \text{and} \quad \widetilde{L}^\rho_T \dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r} := \tilde{L}^\rho(0, T; \dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$$

By virtue of the Minkowski's inequality, we have

$$\|u\|_{\widetilde{L}^\rho(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} \le \|u\|_{L^\rho(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} \quad \text{if} \quad \rho \le r,$$

$$\|u\|_{L^\rho(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} \le \|u\|_{\widetilde{L}^\rho(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} \quad \text{if} \quad r \le \rho.$$

To obtain the global well-posedness of the small initial data Cauchy problem for the three-dimensional Boussinesq equations with the Coriolis force in Variable Exponent Fourier-Besov Spaces, we need to introduce the following spaces.

Definition 2.2. [27][Homogeneous Fourier-Besov spaces with variable exponents] Let $p(\cdot), q(\cdot) \in C_{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in C_{\log}(\mathbb{R}^n)$. The homogeneous Fourier-Besov space with variable exponents $\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)} = \{ f \in \mathcal{S}' : \|f\|_{\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} < \infty \},$$

$$\|f\|_{\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j \hat{f}\}^{\infty}_{-\infty}\|_{l^{q(\cdot)}L^{p(\cdot)}} < \infty.$$

Similarly, we denote by $L^\rho(0, T; \mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$ the set of all tempered distribution u satisfying

$$\|u\|_{L^\rho(0,T;\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} := \left\| \left(\sum_{j=0}^{\infty} \|2^{js(\cdot)}\varphi_j \hat{u}\|^r_{L^{p(\cdot)}} \right)^{\frac{1}{r}} \right\|_{L^\rho_T} < \infty.$$

The mixed $\widetilde{L}^\rho(0, T; \mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$ is the set of all tempered distribution u satisfying

$$\|u\|_{\widetilde{L}^\rho(0,T;\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} := \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\varphi_j \hat{u}\|^r_{L^\rho_T L^{p(\cdot)}} \right)^{\frac{1}{r}} < \infty.$$

Definition 2.3. [29] Let $u, v \in \mathcal{S}'_h$, the product uv has the homogeneous Bony decomposition as follows

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),$$

where

$$\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \qquad \dot{T}_v u = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} v \dot{\Delta}_j u,$$

$$\dot{R}(u, v) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \tilde{\dot{\Delta}}_j v, \quad \tilde{\dot{\Delta}}_j v = \sum_{|j-k|\le 1} \dot{\Delta}_k v.$$

Lemma 2.1. The following inclusions holds for the variable exponent function spaces.

(I) (Hölder inequality[31]) Given a measurable set A and exponent functions $r(\cdot), q(\cdot) \in \mathcal{P}_0(A)$ define $p(\cdot) \in \mathcal{P}_0(A)$ by

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Then there exists a constant C such that for all $f \in L^{q(\cdot)}(A)$ and $g \in L^{r(\cdot)}(A)$, $fg \in L^{p(\cdot)}(A)$ and

$$\|fg\|_{p(\cdot)} \leq C\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

In particular, given A and $p(\cdot) \in \mathcal{P}_0(A)$, for all $f \in L^{p(\cdot)}(A)$ and $g \in L^{p'(\cdot)}(A)$, $fg \in L^1(A)$ and

$$\int_A |f(x)g(x)|\mathrm{d}x \leq C_{p(\cdot)}\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

where the function p' is called the dual variable exponent of p and A_*, A_1, A_∞ are disjoint sets, i.e.,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad C_{p(\cdot)} = \left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\|\chi_{A_*}\|_\infty + \|\chi_{A_\infty}\|_\infty + \|\chi_{A_1}\|_\infty.$$

(II) (Sobolev inequality[30]) Let $p_0, p_1, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $s_0, s_1 \in L^\infty(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ with $s_0 > s_1$. If $\frac{1}{q}$ and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$$

are locally log-Hölder continuous, then

$$\dot{\mathcal{B}}^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

(III) ([30]) Let $p_0, p_1, q_0, q_1 \in \mathcal{P}_0(\mathbb{R}^n)$ and $s_0, s_1 \in L^\infty(\mathbb{R}^n) \cap C_{\log}(\mathbb{R}^n)$ with $s_0 > s_1$. If $\frac{1}{q_0}, \frac{1}{q_1}$ and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} + \varepsilon(x)$$

are locally log-Hölder continuous and $\operatorname{essinf}_{x\in\mathbb{R}^n} \varepsilon(x) > 0$, then

$$\dot{\mathcal{B}}^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow \dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}.$$

(IV) (Molification inequality[32]) For $p(\cdot) \in C_{\log}(\mathbb{R}^n)$ and $\psi \in L^1(\mathbb{R}^n)$, assume that $\Psi(x) = \sup_{y\notin B(0,|x|)} |\psi(y)|$ is integrable. Then

$$\|f * \psi_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\Psi\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,where $\psi_\varepsilon = \frac{1}{\varepsilon^n}\psi(\frac{\cdot}{\varepsilon})$ and C depends only on n.

Lemma 2.2. [29][Hausdorff-Young's inequality] Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. Then $\hat{f} \in L^{p'}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and

$$\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}.$$

Lemma 2.3. [29] A constant C exists such that for all $s \in \mathbb{R}$,

$$\begin{aligned} r_1 \leq r_2 \quad &\Rightarrow \quad \|u\|_{\dot{B}^s_{p,r_2}} \leq C\|u\|_{\dot{B}^s_{p,r_1}}, \\ p_1 \leq p_2 \quad &\Rightarrow \quad \|u\|_{\dot{B}^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r}} \leq C\|u\|_{\dot{B}^s_{p_1,r}}. \end{aligned}$$

Lemma 2.4. Let $s > 0$, $1 \leq p, r \leq \infty$, $p_1(\cdot), p_2(\cdot) \in C_{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, and $\frac{1}{p} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Then

$$\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{\dot{B}^0_{p_1(\cdot),r}}\|v\|_{\dot{B}^s_{p_2(\cdot),r}} + \|v\|_{\dot{B}^0_{p_1(\cdot),r}}\|u\|_{\dot{B}^s_{p_2(\cdot),r}}.$$

Proof. According to definition 2.3, for fixed $j \geq 0$, we have

$$\begin{aligned} \Delta_j(uv) &= \sum_{|k-j|\leq 4} \Delta_j(S_{k-1}u\Delta_k v) + \sum_{|k-j|\leq 4} \Delta_j(S_{k-1}v\Delta_k u) + \sum_{k\geq j-2} \Delta_j(\Delta_k u\widetilde{\Delta}_k v) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We will estimate each of the three above. Using Young's inequality and Hölder's inequality from the Lemma 2.1, we have

$$\|2^{js}\Delta_j(S_{k-1}u\Delta_k v)\|_{L^p} \lesssim \|S_{k-1}u\|_{L^{p_1(\cdot)}}\|2^{js}\Delta_k v\|_{L^{p_2(\cdot)}},$$

then

$$\|2^{js}I_1\|_{L^p} \lesssim \sum_{|k-j|\le 4} \|S_{k-1}u\|_{L^{p_1(\cdot)}} \|2^{js}\Delta_k v\|_{L^{p_2(\cdot)}}.$$

Similarly, for I_2 we have

$$\|2^{js}I_2\|_{L^p} \lesssim \sum_{|k-j|\le 4} \|S_{k-1}v\|_{L^{p_1(\cdot)}} \|2^{js}\Delta_k u\|_{L^{p_2(\cdot)}}.$$

Now, it remains to estimates I_3. Using Young's inequality, we have

$$\|\Delta_j(\Delta_k u\widetilde{\Delta}_k v)\|_{L^p} \lesssim \|\Delta_k u\|_{L^{p_1(\cdot)}} \|\widetilde{\Delta}_k v\|_{L^{p_2(\cdot)}}.$$

Hence,

$$\begin{aligned}\|2^{js}I_3\|_{L^p} &\lesssim \sum_{k\ge j-2} \|2^{js}\Delta_k u\|_{L^{p_1(\cdot)}} \|\widetilde{\Delta}_k v\|_{L^{p_2(\cdot)}}\\ &= \sum_{k\ge j-2} 2^{(j-k)s}\|2^{ks}\Delta_k u\|_{L^{p_1(\cdot)}} \|\widetilde{\Delta}_k v\|_{L^{p_2(\cdot)}}.\end{aligned}$$

Taking the norm $\|\cdot\|_{l^r}$ on both side of above inequality, there holds that

$$\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{\dot{B}^0_{p_1(\cdot),r}} \|v\|_{\dot{B}^s_{p_2(\cdot),r}} + \|v\|_{\dot{B}^0_{p_1(\cdot),r}} \|u\|_{\dot{B}^s_{p_2(\cdot),r}}.$$

□

Lemma 2.5. Let $s>0$, $1\le p,r,\rho\le\infty$, $p_1(\cdot),p_2(\cdot)\in C_{\log}(\mathbb{R}^n)\cap\mathcal{P}_0(\mathbb{R}^n)$, and $\frac{1}{p}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$, $\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$. Then

$$\|uv\|_{\widetilde{L}^\rho_T\dot{B}^s_{p,r}} \lesssim \|u\|_{\widetilde{L}^{\rho_1}_T\dot{B}^0_{p_1(\cdot),r}} \|v\|_{\widetilde{L}^{\rho_2}_T\dot{B}^s_{p_2(\cdot),r}} + \|v\|_{\widetilde{L}^{\rho_1}_T\dot{B}^0_{p_1(\cdot),r}} \|u\|_{\widetilde{L}^{\rho_2}_T\dot{B}^s_{p_2(\cdot),r}}.$$

Proof. In the proof of the lemma 2.4, replacing $L^{p(\cdot)}$ with $L^\rho_T L^{p(\cdot)}$, we can get that the conclusion holds. □

3 | LINEAR ESTIMATES

We establish the linear estimates of the semigroups $\{T_\Omega(t)\}_{t>0}$ in this section, and see the specific introduction of the semigroups $\{T_\Omega(t)\}_{t>0}$ in section 4.

Lemma 3.1. Let $p(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathcal{P}_0(\mathbb{R}^3)$, $2\le p(\cdot)\le 6$, $2\le p_1(\cdot)\le c\le p(\cdot)$, $s_1(\cdot)=\frac{2}{\rho}+2-\frac{3}{p_1(\cdot)}$ and $1\le q,\rho\le\infty$. Then

$$\|T_\Omega(t)f\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{B}^{s_1(\cdot)}_{p_1(\cdot),q}\right)} \lesssim \|f\|_{\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}$$

for $\Omega\in\mathbb{R}$ and $f\in\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}$.

Proof. By definition 2.2, we have

$$\|T_\Omega(t)f\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{B}^{s_1(\cdot)}_{p_1(\cdot),q}\right)} = \left\|\left\{\|2^{js_1(\cdot)}\varphi_j\mathcal{F}[T_\Omega(t)f]\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}.$$

Since $T_\Omega(t)f$ is bounded Fourier multiplier, we estimate by a positive constant. Using Lemma 2.1, we have

$$\begin{aligned}\|T_\Omega(t)f\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{B}^{s_1(\cdot)}_{p_1(\cdot),q}\right)} &= \left\|\left\{\|2^{js_1(\cdot)}\varphi_j\mathcal{F}[T_\Omega(t)f]\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\ &\lesssim \|\{\|2^{js_1(\cdot)}\varphi_j e^{-t|\cdot|^2}\hat{f}\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\}_{j\in\mathbb{Z}}\|_{l^q(\mathbb{Z})}\\ &\lesssim \left\|\left\{\sum_{l=0,\pm1}\left\|2^{j(2-\frac{3}{c})}\right\|_{L^c}\left\|2^{j(\frac{2}{\rho}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}e^{-t2^{2(j+l)}}\right\|_{L^\rho(0,\infty;L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\ &\lesssim \|f\|_{\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},\end{aligned}$$

where the second norm in the second line above is estimated as follows

$$
\begin{aligned}
&\left\|2^{j(\frac{2}{\rho}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}e^{-t2^{2(j+l)}}\right\|_{L^\rho(0,\infty;L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}})}\\
&=\|2^{j\frac{2}{\rho}}e^{-t2^{2(j+l)}}\|_{L^\rho(0,\infty)}\|2^{j(\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}\|_{L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}}}\\
&=\|2^{j\frac{2}{\rho}}e^{-t2^{2(j+l)}}\|_{L^\rho(0,\infty)}\inf\{\lambda>0:\int_{\mathbb{R}^3}|2^{j(\frac{3}{c}-\frac{3}{p_1(x)})}\varphi_{j+l}|^{\frac{cp_1(x)}{c-p_1(x)}}\mathrm{d}x\le 1\}\\
&\lesssim \inf\{\lambda>0:\int_{\mathbb{R}^3}|2^{j(\frac{3}{c}-\frac{3}{p_1(x)})}\varphi_{j+l}|^{\frac{cp_1(x)}{c-p_1(x)}}\mathrm{d}x\le 1\}\\
&\lesssim \inf\{\lambda>0:\int_{\mathbb{R}^3}|\varphi_{j+l}|^{\frac{cp_1(x)}{c-p_1(x)}}2^{-3j}\mathrm{d}x\le 1\}\\
&\lesssim \inf\{\lambda>0:\int_{\mathbb{R}^3}|\varphi_{l}|^{\frac{cp_1(2^jx)}{c-p_1(2^jx)}}\mathrm{d}x\le 1\}\\
&\lesssim C.
\end{aligned}
$$

□

Lemma 3.2. Let $p(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathcal{P}_0(\mathbb{R}^3)$, $2\le p(\cdot)\le 6$, $2\le p_1(\cdot)\le c\le p(\cdot)$, $s_1(\cdot)=\frac{2}{\rho}+2-\frac{3}{p_1(\cdot)}$ and $1\le q,\rho\le\infty$. Then

$$
\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}f\,\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\lesssim\|f\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}
$$

for $\Omega\in\mathbb{R}$ and $f\in\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}$.

Proof. Using Lemma 2.1,2.2 and Young' inequality, we obtain

$$
\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}f\,\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\\
&=\left\|\left\{\left\|2^{js_1(\cdot)}\varphi_j\mathcal{F}[\int_0^t T_\Omega(t-\tau)\mathbb{P}f\,\mathrm{d}\tau]\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\
&\lesssim\left\|\left\{\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}\widehat{f}\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\
&\lesssim\|f\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})},
\end{aligned}
$$

where the inner norm of the second line above is estimated as follows

$$
\begin{aligned}
&\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}\widehat{f}\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\\
&\lesssim\left\|\int_0^t\|2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}\|_{L^{\frac{2p_1(\cdot)}{2-p_1(\cdot)}}}\|\varphi_j\widehat{f}\|_{L^2}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
&\lesssim\left\|\int_0^t\|2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}\|_{L^{\frac{2p_1(\cdot)}{2-p_1(\cdot)}}}\|\Delta_j f\|_{L^2}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
&\lesssim\left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac{1}{2})}e^{-(t-\tau)2^{2j}}\|2^{-3j\frac{2-p_1(\cdot)}{2p_1(\cdot)}}\varphi_j\|_{L^{\frac{2p(\cdot)}{2-p(\cdot)}}}\|\Delta_j f\|_{L^2}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
&\lesssim\left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac{1}{2})}e^{-(t-\tau)2^{2j}}\|\Delta_j f\|_{L^2}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
&\lesssim\left\|2^{j(\frac{2}{\rho}+\frac{1}{2})}\|\Delta_j f\|_{L^2}\right\|_{L^\rho(0,\infty)}\|e^{-t2^{2j}}\|_{L^1(0,\infty)}\\
&\lesssim\left\|2^{j(\frac{2}{\rho}+\frac{1}{2})}\|\Delta_j f\|_{L^2}\right\|_{L^\rho(0,\infty)}.
\end{aligned}
$$

□

4 PROOF OF THEOREM 1.1

In order to solve the Boussinesq equations with Coriolis force, we consider the following linear generalized problem

$$\begin{cases} \partial_t u - \Delta u + \Omega e_3 \times u + \nabla P = 0, & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ \mathrm{div} u = 0, & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u|_{t=0} = u_0, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{4.1}$$

The solution of equation (4.1) can be given by the generalized Stokes-Coriolis semi group $T_\Omega(t)$, which has the following explicit expression

$$\begin{aligned} T_\Omega(t)u &= \mathcal{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^2}I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^2}R(\xi)\right] * u \\ &= \mathcal{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)R(\xi)\right] * (e^{t\Delta}u), \end{aligned}$$

where divergence free vector field $u \in \mathcal{S}(\mathbb{R}^3)$, I is the unit matrix in $M_{3\times 3}(\mathbb{R})$ and $R(\xi)$ is skew-symmetric matrix defined by

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ \xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}, \quad \xi \in \mathbb{R}^3\backslash\{0\}.$$

Hence, the solution of the equation (1.1) can be rewritten as

$$\begin{cases} u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau + \int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau, \\ \theta(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\theta]\mathrm{d}\tau. \end{cases}$$

For the derivation of explicit form of $T_\Omega(\cdot)$, we refer to Babin, Mahalov and Nicolaenko [11], Giga, Inui, Mahalov, Matsui [18] and Hieber, Shibata [14].

Proof of Theorem 1.1. Let $M > 0$, $\delta > 0$ to be determined. Set

$$\begin{aligned} X = \Bigg\{ (u,\theta) : \|u\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})} + \|\theta\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})} \le M, \\ \|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} + \|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} \le \delta \Bigg\}, \end{aligned}$$

which is equipped with the metric

$$\begin{aligned} d((u,\theta),(w,v)) = \|u-w\|&_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})\cap\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} \\ &+ \|\theta - v\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})\cap\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}. \end{aligned}$$

It is easy to see that (X,d) is a complete metric space. Next we consider the following mapping

$$\begin{aligned} \Phi : (u,\theta) \to (T_\Omega(t)u_0, e^{t\Delta}\theta_0) - \left(\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau, \int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\theta]\mathrm{d}\tau\right) \\ + \left(\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau, 0\right), \end{aligned}$$

where $\mathbb{P} := I - \nabla(-\Delta)^{-1}$ denotes the Helmholtz projection onto the divergence free vector fields.

We shall prove there exist $M, \delta > 0$ such that $\Phi : (X,d) \to (X,d)$ is a strict contraction mapping.

First, we establish that the estimate of $(T_\Omega(t)u_0, e^{t\Delta}\theta_0)$. According to Lemma 3.1, it follows that

$$\|T_\Omega(t)u_0\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{B}^{s_1(\cdot)}_{p_1(\cdot),q}\right)} \lesssim \|u_0\|_{\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},$$

and we have

$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{B}^{s_1(\cdot)}_{p_1(\cdot),q}\right)} \lesssim \|\theta_0\|_{\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}$$

when $\Omega = 0$.

Similarly we can obtain

$$
\begin{aligned}
&\|T_\Omega(t)u_0\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},\\
&\|e^{t\Delta}\theta_0\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})} \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},
\end{aligned}
$$

It is easy to show that the estimate for $T_\Omega(t)u_0$ and $e^{t\Delta}\theta_0$ also hold for $\rho = \infty$ and $p_1(\cdot) = p(\cdot)$, i.e.,

$$
\begin{aligned}
&\|T_\Omega(t)u_0\|_{\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},\\
&\|T_\Omega(t)u_0\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},\\
&\|e^{t\Delta}\theta_0\|_{\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})} \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},\\
&\|e^{t\Delta}\theta_0\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.
\end{aligned}
$$

Next we show that the estimate of the remaining terms. Using Lemma 2.12.22.32.5, we can show that

$$
\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\\
=&\left\|\left\{\left\|2^{js_1(\cdot)}\varphi_j\mathcal{F}[\textstyle\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau]\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\
\lesssim&\left\|\left\{\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}[\widehat{(u\cdot\nabla)u}]\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^0_{3,q})}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})},
\end{aligned}
$$

where the inner norm of the third line is estimated as follows

$$
\begin{aligned}
&\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}[\widehat{(u\cdot\nabla)u}]\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\\
\lesssim&\left\|\int_0^t \|2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}\|_{L^{\frac{6p_1(\cdot)}{6-p_1(\cdot)}}}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
\lesssim&\left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac{5}{2})}e^{-(t-\tau)2^{2j}}\|2^{-3j\frac{6-p_1(\cdot)}{6p_1(\cdot)}}\varphi_j\|_{L^{\frac{6p_1(\cdot)}{6-p_1(\cdot)}}}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
\lesssim&\left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac{5}{2})}e^{-(t-\tau)2^{2j}}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
\lesssim&\left\|2^{j(\frac{2}{\rho}+\frac{5}{2})}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\right\|_{L^\rho(0,\infty)}\|e^{-t2^{2j}}\|_{L^1(0,\infty)}\\
\lesssim&\left\|2^{j(\frac{2}{\rho}+\frac{1}{2})}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac{6}{5}}}\right\|_{L^\rho(0,\infty)}.
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})} \lesssim \|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})},\\
&\left\|\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\theta]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})} \lesssim \|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}.
\end{aligned}
$$

In addition, we can also get

$$\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\\
=&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}_{2,q}^{\frac{1}{2}})}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})},\\
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\\
\lesssim&\|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})},\\
&\left\|\int_0^t e^{(t-\tau)\Delta}[(u\cdot\Delta)\theta]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}.
\end{aligned}$$

We finally prove that the existence and uniqueness.

Let $Y=\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}})\bigcap\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\bigcap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})$, then

$$\begin{aligned}
&\|\Phi(u,\theta)\|_Y\\
=&\|\Phi(u)\|_Y+\|\Phi(\theta)\|_Y\\
\lesssim&\|u_0\|_{\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}}+\|\theta_0\|_{\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}}+\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\\
&+\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}+\|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}.
\end{aligned}$$

Denote $\delta=M=2\left(\|u_0\|_{\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}}+\|\theta_0\|_{\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}}\right)<2C_\epsilon$, if ϵ is small enough, then we have

$$\|\Phi(u,\theta)\|_Y\le\frac{\delta}{2}+\frac{\delta}{2}=\delta,$$

and

$$d(\Phi(u,\theta),\Phi(w,v))\le\frac{1}{2}d((u,\theta),(w,v)).$$

It follows from the Banach's contraction mapping principle that the rotating Boussinesq equation has a unique global solution and satisfies

$$(u,\theta)\in\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}})\cap\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})$$

when ϵ is small enough.

On the other hand, let

$$\begin{aligned}
Z=&\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}})\cap\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\\
&\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}),
\end{aligned}$$

then we have

$$\begin{aligned}
&\|\Phi(u,\theta)\|_Z\\
=&\|\Phi(u)\|_Z+\|\Phi(\theta)\|_Z\\
\lesssim&\|u_0\|_{\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}}+\|\theta_0\|_{\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}}+\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\\
&+\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}+\|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}_{2,q}^{\frac{2}{\rho}+\frac{1}{2}})\cap\widetilde{L}^\infty(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}.
\end{aligned}$$

Set $\delta=M=2\left(\|u_0\|_{\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}\bigcap\mathcal{F}\dot{B}_{p_1(\cdot),q}^{2-\frac{3}{p_1(\cdot)}}}+\|\theta_0\|_{\mathcal{F}\dot{B}_{p(\cdot),q}^{2-\frac{3}{p(\cdot)}}\bigcap\mathcal{F}\dot{B}_{p_1(\cdot),q}^{2-\frac{3}{p_1(\cdot)}}}\right)<2C_\epsilon$, if ϵ is small enough, then we have

$$\|\Phi(u,\theta)\|_Z\le\frac{\delta}{2}+\frac{\delta}{2}=\delta,$$

and

$$d(\Phi(u,\theta),\Phi(w,v))\le\frac{1}{2}d((u,\theta),(w,v)).$$

According to the Banach's contraction mapping principle, it follows that the rotating Boussinesq equations has a unique global solution and satisfies

$$(u,\theta) \in \widetilde{L}^{\rho}(0,\infty;\mathcal{F}\dot{B}^{s_1(\cdot)}_{p_1(\cdot),q}) \cap \widetilde{L}^{\infty}(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})$$

when ϵ is small enough. □

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CONFLICT OF INTEREST

All authors declare no conflicts of interest in this paper.

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