Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ARTICLE TYPE

The Dawson Transform and its Applications

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Summary

Some properties of the Dawson Integral are presented first in the current work, followed by the introduction of the Dawson Integral Transform. Iteration identities and relationships, similar to the Parseval Goldstein type, are established involving various well-known integral transforms, such as the Laplace Transform, the $\mathscr{L}_2$-Transform, and the Dawson Integral for the new integral transform. Furthermore, improper integrals of well-known functions, including the Dawson Integral, Exponential Integral, and the Macdonald Function, are evaluated using the results obtained.

KEYWORDS:
the Dawson transform, the Glasser transform, the Widder transform, the $\mathscr{L}_2$-transform, the Laplace transform, Parseval–Goldstein type theorems

1 | INTRODUCTION AND DEFINITIONS

Dawson's integral, sometimes called Dawson's function, is defined by the integral

$$\operatorname{daw}(t) = \int_0^t \exp(s^2 - t^2)\, ds, \tag{1}$$

[1] p. 427, Eq. 42:3:1. It has various applications such as heat conduction, spectroscopy, electrical oscillations in certain special vacuum tubes. Complex error function is closely related to Dawson's integral:

$$\operatorname{erfi}(t) = \frac{\operatorname{erf}(it)}{i} = \frac{2}{\sqrt{\pi}} \exp(t^2)\operatorname{daw}(t). \tag{2}$$

The important mathematical properties of Dawson's integral are given in Chapter 7 of [2]. Some applications and computational methods are discussed in [3]. Rational Chebyshev approximations to Dawson's integral are shown in [4].

In this paper, we introduce a new integral transform

$$\mathscr{D}\left[f(t)\right](s) = \int_0^\infty \operatorname{daw}(st) f(t)\, dt \tag{3}$$

with the kernel Dawson's integral. We will refer the transform as the Dawson transform.

We obtain iteration identities for the Dawson transform (3), the Fourier sine transform, the Fourier cosine transform [8], the Glasser transform [11], the Widder transform [10] and the $\mathscr{L}_2$-transform

$$\mathscr{L}_2\left[f(t)\right](s) = \int_0^\infty t \exp\left(-t^2 s^2\right) f(t)\, dt. \tag{4}$$

A systematic account the Glasser transform[11] is discussed in Yürekli et al.[5]. The $\mathscr{L}_2$-transform is introduced by the fourth author[6]. An expository article about recent results related various integral transforms including the $\mathscr{L}_2$-transform can be found in Chapter 4 of the recent book[7].

The $\mathscr{L}_2$-transform is related to the well known Laplace transform[8] with the following identity with the following identity

$$\mathscr{L}_2\big[f(t)\big](s) = \frac{1}{2}\mathscr{L}\Big[f\big(\sqrt{t}\big)\Big]\big(s^2\big). \tag{5}$$

2 | ITERATION IDENTITIES

We start with the iteration identities for Dawson integral, Fourier sine, Fourier cosine and $\mathscr{L}_2$-transforms.

Lemma 1. The iteration identities

$$\mathscr{D}\Big[\mathscr{F}_s[f(t)](u)\Big](s) = \frac{\pi}{4s}\mathscr{L}_2\left[\frac{f(t)}{t}\right]\left(\frac{1}{2s}\right) \tag{6}$$

and

$$\mathscr{F}_s\Big[\mathscr{D}[f(t)](u)\Big](s) = \frac{\pi}{4}\mathscr{L}_2\left[\frac{1}{t^2}f\left(\frac{1}{t}\right)\right]\left(\frac{s}{2}\right). \tag{7}$$

hold true, provided that the integrals involved converge absolutely.

Proof. Using the definition (3) of the Dawson transform and the Fourier sine transform, we have

$$\mathscr{D}\Big[\mathscr{F}_s[f(t)](u)\Big](s) = \int_0^\infty \operatorname{daw}(us)\left(\int_0^\infty \sin(ut)f(t)\,dt\right)du \tag{8}$$

where the Fourier sine transform is defined as

$$\mathscr{F}_s\big[f(t)\big](s) = \int_0^\infty \sin(st)f(t)\,dt. \tag{9}$$

Changing the order of the integration in Equation (8), we have

$$\mathscr{D}\Big[\mathscr{F}_s[f(t)](u)\Big](s) = \int_0^\infty f(t)\left(\int_0^\infty \sin(ut)\operatorname{daw}(us)\,du\right)dt. \tag{10}$$

Using the formula[1] p. 431, Eq. 42:10:5, the inner integral on the right-hand side of Equation (10) is

$$\int_0^\infty \sin(ut)\operatorname{daw}(us)\,du = \frac{\pi}{4s}\exp\left(-\frac{t^2}{4s^2}\right), \qquad t > 0. \tag{11}$$

Substituting Equation (11) into Equation (10) and using the definition (4) of the $\mathscr{L}_2$-transform yield the claim (6) of Lemma 1.

Assertion (7) of Lemma 1's proof is similar. □

An immediate corollary of Equation (6) of Lemma 1 is the following Corollary.

Corollary 1. We have

$$\mathscr{D}\big[\operatorname{daw}(at)\big](s) = \frac{\pi^{3/2}}{8}\frac{1}{\sqrt{s^2+a^2}}. \tag{12}$$

Proof. We put

$$f(t) = \exp\left(\frac{-t^2}{4a^2}\right) \tag{13}$$

in (6) of Lemma 1. Using the formula[8] Entry (18), p. 73, we have

$$\mathscr{F}_s\big[f(t)\big](u) = \mathscr{F}_s\left[\exp\left(\frac{-t^2}{4a^2}\right)\right](u) = -i\sqrt{\pi}\exp\big(-a^2u^2\big)\operatorname{erf}(iau) \tag{14}$$

Using Equation (2), we have

$$\mathscr{F}_s\left[\exp\left(\frac{-t^2}{4a^2}\right)\right](u) = 2a\,\mathrm{daw}(au) \tag{15}$$

Using Equation (5) and the formula[8] Entry (3), p. 144, we have

$$\mathscr{L}_2\left[\frac{1}{t}\exp\left(\frac{-t^2}{4a^2}\right)\right]\left(\frac{1}{2s}\right) = \frac{1}{2}\mathscr{L}\left[\frac{1}{t^{1/2}}\exp\left(\frac{-t}{4a^2}\right)\right]\left(\frac{1}{2s^2}\right) = \frac{\pi^{1/2}\,as}{\sqrt{s^2+a^2}} \tag{16}$$

Now the assertion (12) of Corollary 1 follows upon substituting Equations (13), (15), and (16) into (6) of Lemma 1. □

The next iteration identity shows the second iteration of the Dawson transform (3) is the Glasser transform defined by

$$\mathscr{G}\left[f(t)\right](s) = \int_0^\infty \frac{f(t)}{\sqrt{t^2+s^2}}\,dt. \tag{17}$$

Lemma 2. The iteration identity

$$\mathscr{D}\Big[\mathscr{D}[f(t)](u)\Big](s) = \frac{\pi^{3/2}}{8}\mathscr{G}[f(t)](s) \tag{18}$$

holds true, if the integrals involved converge absolutely.

Proof. Using the definition (3) of Dawson transform, we get

$$\mathscr{D}\Big[\mathscr{D}[f(t)](u)\Big](s) = \int_0^\infty \mathrm{daw}(us)\left(\int_0^\infty \mathrm{daw}(ut)f(t)\,dt\right)du. \tag{19}$$

Changing the order of the integration in Equation (19), we have

$$\mathscr{D}\Big[\mathscr{D}[f(t)](u)\Big](s) = \int_0^\infty f(t)\left(\int_0^\infty \mathrm{daw}(us)\mathrm{daw}(ut)\,du\right)dt. \tag{20}$$

and again using the definition (3) of Dawson transform, we express Equation (20) as

$$\mathscr{D}\Big[\mathscr{D}[f(t)](u)\Big](s) = \int_0^\infty f(t)\mathscr{D}\big[\mathrm{daw}(tu)\big](s)\,dt. \tag{21}$$

Using Equation (12) of Corollary (1), we obtain

$$\mathscr{D}\Big[\mathscr{D}[f(t)](u)\Big](s) = \frac{\pi^{3/2}}{8}\int_0^\infty \frac{f(t)}{\sqrt{t^2+s^2}}\,dt. \tag{22}$$

Now the assertion (18) of Lemma 2 follows from Equation (22) and the definition (17) of the Glasser transform. □

The next iteration identities are for the Dawson transform (3), the $\mathscr{L}_2$-transform (4), and the Widder transform defined by

$$\mathscr{W}\left[f(t)\right](s) = \int_0^\infty \frac{tf(t)}{t^2+s^2}\,dt. \tag{23}$$

Lemma 3. The iteration identities

$$\mathscr{L}_2\Big[\mathscr{D}\big[f(t)\big](u)\Big](s) = \frac{\pi^{1/2}}{4y}\mathscr{W}\big[f(t)\big](s) \tag{24}$$

and

$$\mathscr{D}\Big[u\mathscr{L}_2\big[f(t)\big](u)\Big](s) = \frac{\pi^{1/2}s}{4}\mathscr{W}\left[\frac{f(t)}{t}\right](s) \tag{25}$$

hold true, if the integrals involved converge absolutely.

Proof. Using the definition (4) and the definition (3) of Dawson transform, we have

$$\mathcal{L}_2\Big[\mathcal{D}\big[f(t)\big](u)\Big](s)=\int_0^\infty u\exp\big(-u^2s^2\big)\left(\int_0^\infty \operatorname{daw}(ut)f(t)\,dt\right)du. \tag{26}$$

Changing the order of the integration in Equation (26) and using the definition (4), we get

$$\mathcal{L}_2\Big[\mathcal{D}\big[f(t)\big](u)\Big](s)=\int_0^\infty f(t)\mathcal{L}_2\big[\operatorname{daw}(ut)\big](s)\,dt. \tag{27}$$

Using the identity (5), we express Equation (27) as

$$\mathcal{L}_2\Big[\mathcal{D}\big[f(t)\big](u)\Big](s)=\frac{1}{2}\int_0^\infty f(t)\mathcal{L}\big[\operatorname{daw}(t\sqrt{u})\big](s^2)\,dt. \tag{28}$$

Using the formula[1] 42:10:8, p. 432, we obtain

$$\mathcal{L}_2\Big[\mathcal{D}\big[f(t)\big](u)\Big](s)=\frac{\pi^{1/2}}{4s}\int_0^\infty \frac{tf(t)}{t^2+s^2}\,dt. \tag{29}$$

Now the assertion Equation (24) follows from the definition (23) of the Widder transform.

The proof of the assertion Equation (25) is similar. □

The next iteration identitity is for the Dawson transform (3), the $\mathcal{L}_2$-transform (4), and the Glasser transform (17).

Lemma 4. The iteration identitity

$$\mathcal{D}\Big[\mathcal{L}_2\big[f(t)\big](u)\Big](s)=\frac{1}{2}\mathcal{G}\Big[tf(t)\operatorname{arcsinh}(t/s)\Big](s), \tag{30}$$

holds true, if the integrals involved converge absolutely.

Proof. Using the definition (3) of the Dawson transform, the definition (4) of the $\mathcal{L}_2$-transform, and changing the order of integration, we have

$$\mathcal{D}\Big[\mathcal{L}_2\big[f(t)\big](u)\Big](s)=\int_0^\infty t\,f(t)\mathcal{L}_2\left[\frac{\operatorname{daw}(su)}{u}\right](t)\,dt. \tag{31}$$

Using the formula[1] 42:10:4, p. 431, we obtain

$$\mathcal{L}_2\left[\frac{\operatorname{daw}(su)}{u}\right](t)=\frac{\operatorname{arcsinh}(t/s)}{2(t^2+s^2)^{1/2}}. \tag{32}$$

The proof of the assertion Equation (30) follows upon substituting Equation (32) into Equation (31) and using the definition (17) of the Glasser transform. □

The next iteration identitity is for the Dawson transform (3), and the $\mathcal{L}_2$-transform (4).

Lemma 5. The iteration identitity

$$\mathcal{D}\left[\frac{1}{u}\mathcal{L}_2\big[f(t)\big](u)\right](s)=\frac{\pi^{1/2}}{2}\int_0^\infty t\arctan\left(\frac{s}{t}\right)f(t)\,dt, \tag{33}$$

holds true, if the integrals involved converge absolutely.

Proof. Using the definition (3) of the Dawson transform, the definition (4) of the $\mathcal{L}_2$-transform, and changing the order of integration, we have

$$\mathcal{D}\left[\frac{1}{u}\mathcal{L}_2\big[f(t)\big](u)\right](s)=\int_0^\infty t\,f(t)\mathcal{L}_2\left[\frac{\operatorname{daw}(su)}{u^2}\right](t)\,dt. \tag{34}$$

Using the identity (5) and the property of the Laplace transform obtain

$$\mathscr{L}\left[\frac{f(\sqrt{t})}{t}\right](u^2) = \int_u^\infty 2u\mathscr{L}\left[f(\sqrt{t})\right](u^2)\,du, \tag{35}$$

in Equation (34), we obtain

$$\begin{aligned}\mathscr{D}\left[\frac{1}{u}\mathscr{L}_2\left[f(t)\right](u)\right](s) &= \frac{1}{2}\int_0^\infty t\,f(t)\mathscr{L}\left[\frac{\operatorname{daw}\left(s\,u^{1/2}\right)}{u}\right](t^2)\,dt \\ &= \frac{1}{2}\int_0^\infty t\,f(t)\left(\int_t^\infty 2v\mathscr{L}\left[\operatorname{daw}\left(s\,u^{1/2}\right)\right](v^2)\,dv\right)dt \end{aligned}\tag{36}$$

Now the assertion Equation (33) follows from the definition[8] of the Laplace transform and the formula[1] 42:10:8, p. 431. □

3 | PARSEVAL-GOLDSTEIN RELATIONSHIPS AND EXCHANGE IDENTITIES

We start with the Parseval-Goldstein relationships for the Dawson integral (3), the Fourier sine transform (9), and the $\mathscr{L}_2$-transform (4).

Theorem 1. The Parseval-Goldstein relationships

$$\int_0^\infty \mathscr{D}\left[f(t)\right](s)\mathscr{F}_s\left[g(u)\right](s)\,ds = \frac{\pi}{4}\int_0^\infty \frac{f(t)}{t}\mathscr{L}_2\left[\frac{g(u)}{u}\right]\left(\frac{1}{2t}\right)dt \tag{37}$$

and

$$\int_0^\infty \mathscr{D}\left[f(t)\right](s)\mathscr{F}_s\left[g(u)\right](s)\,ds = \frac{\pi}{4}\int_0^\infty g(u)\mathscr{L}_2\left[\frac{1}{t^2}f\left(\frac{1}{t}\right)\right]\left(\frac{u}{2}\right)du \tag{38}$$

hold true, if the integrals involved converge absolutely.

Proof. Using the definition (3) of the Dawson transform, we get

$$\int_0^\infty \mathscr{D}\left[f(t)\right](s)\mathscr{F}_s\left[g(u)\right](s)\,ds = \int_0^\infty \mathscr{F}_s\left[g(u)\right](s)\left(\int_0^\infty \operatorname{daw}(st)\,f(t)\,dt\right)ds \tag{39}$$

Changing the order of the integration in Equation (39) and using the definition (3) of the Dawson transform, we have

$$\int_0^\infty \mathscr{D}\left[f(t)\right](s)\mathscr{F}_s\left[g(u)\right](s)\,ds = \int_0^\infty f(t)\mathscr{D}\left[\mathscr{F}_s\left[g(u)\right](s)\right](t)\,dt \tag{40}$$

Using the iteration identity (6), we obtain the assertion (37) of Theorem 1.

The proof of the assertion Equation (38) of Theorem 1 similarly follows first using the definition (9) of the Fourier sine transform, changing the order of integration, and using the iteration identity (7). □

The following exchange identity for $\mathscr{L}_2$-transform (4) is an immediate result of the iteration identities (37) and (38) of Theorem 1.

Corollary 2. The exchange identity

$$\int_0^\infty \frac{f(t)}{t}\mathscr{L}_2\left[\frac{g(u)}{u}\right]\left(\frac{1}{2t}\right)dt = \int_0^\infty g(u)\mathscr{L}_2\left[\frac{1}{t^2}f\left(\frac{1}{t}\right)\right]\left(\frac{u}{2}\right)du, \tag{41}$$

hold true, provided that the integrals involved converge absolutely.

The next Parseval-Goldstein relationships are for the Dawson integral (3), the Widder transform (23), and the $\mathscr{L}_2$-transform (4).

Theorem 2. The Parseval-Goldstein relationships

$$\int_0^\infty s\mathscr{L}_2\big[f(t)\big](s)\mathscr{D}\big[g(u)\big](s)\,ds = \frac{\pi^{1/2}}{4}\int_0^\infty f(t)\mathscr{W}\big[g(u)\big](t)\,dt \tag{42}$$

and

$$\int_0^\infty s\mathscr{L}_2\big[f(t)\big](s)\mathscr{D}\big[g(u)\big](s)\,ds = \frac{\pi^{1/2}}{4}\int_0^\infty ug(u)\mathscr{W}\left[\frac{f(t)}{t}\right](u)\,du \tag{43}$$

hold true, if the integrals involved converge absolutely.

Proof. Using the definition (4) of the $\mathscr{L}_2$-transform, we get

$$\int_0^\infty s\mathscr{L}_2\big[f(t)\big](s)\mathscr{D}\big[g(u)\big](s)\,ds = \int_0^\infty s\mathscr{L}_2\big[f(t)\big](s)\left(\int_0^\infty \mathrm{daw}(us)\,g(u)\,du\right)ds \tag{44}$$

Changing the order of the integration in Equation (44) and using the definition (4) of the $\mathscr{L}_2$-transform, we have

$$\int_0^\infty s\mathscr{L}_2\big[f(t)\big](s)\mathscr{D}\big[g(u)\big](s)\,ds = \int_0^\infty g(u)\mathscr{D}\Big[s\mathscr{L}_2\big[f(t)\big](s)\Big](u)\,du \tag{45}$$

Using the iteration identity (25) of Lemma 3, we obtain the assertion (43) of Theorem 2.

The proof of the assertion Equation (42) of Theorem 2 similarly follows first using the definition (3) of the Dawson transform, changing the order of integration, and using the iteration identity (24).

□

The following exchange identity for the Widder transform (23) is an immediate result of the iteration identities (42) and (43) of Theorem 2.

Corollary 3. The exchange identity

$$\int_0^\infty f(s)\mathscr{W}\big[g(t)\big](s)\,ds = \int_0^\infty tg(t)\mathscr{W}\left[\frac{f(s)}{s}\right](t)\,dt \tag{46}$$

hold true, if the integrals involved converge absolutely.

4 | SOME ILLUSTRATIVE EXAMPLES

The next three examples are illustration for Equation (6) of Lemma 1.

Example 1. If $0 < \mu < 1$, then

$$\mathscr{D}\big[t^{\mu-1}\big](s) = \frac{\pi^{1/2}}{4s^\mu}\tan\left(\frac{\pi\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right). \tag{47}$$

Proof. We set

$$f(t) = t^{-\mu}. \tag{48}$$

in Equation (6) of Lemma 1. Using the formula[8] Entry (1), p. 68, and the identity (5), respectively, we have

$$\mathscr{F}_s\big[f(t)\big](s) = s^{\mu-1}\Gamma(1-\mu)\cos\left(\frac{\mu\pi}{2}\right). \tag{49}$$

and

$$\mathscr{L}_2\left[\frac{1}{t^{1+\mu}}\right]\left(\frac{1}{2s}\right) = \frac{1}{2}\mathscr{L}\left[\frac{1}{t^{(1+\mu)/2}}\right]\left(\frac{1}{4s^2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}-\frac{\mu}{2}\right)(2s)^{1-\mu}. \tag{50}$$

Now the assertion (47) follows upon substituting Equations (48), (49) and (50) to Equation (6) of Lemma 1. □

Example 2. If $\mathfrak{R}(a) > 0$ and $|\arg a| < \frac{\pi}{2}$, then

$$\mathscr{D}\big[\exp(-at)\big](s) = -\frac{1}{4s}\exp\left(\frac{a^2}{4s^2}\right)\mathrm{Ei}\left(-\frac{a^2}{4s^2}\right). \tag{51}$$

Proof. We set

$$f(t) = \frac{t}{t^2+a^2}. \tag{52}$$

in Equation (6) of Lemma 1. Using the formula[8] Entry (15), p. 65, we have

$$\mathscr{F}_s\left[\frac{t}{t^2+a^2}\right](s) = \frac{\pi}{2}\exp(-as). \tag{53}$$

Using the identity (5) and the formula[8] Entry (7), p. 137

$$\mathscr{L}_2\left[\frac{1}{t^2+a^2}\right]\left(\frac{1}{2s}\right) = \frac{1}{2}\mathscr{L}\left[\frac{1}{t+a^2}\right]\left(\frac{1}{4s^2}\right) = -\frac{1}{2}\exp\left(\frac{a^2}{4s^2}\right)\mathrm{Ei}\left(-\frac{a^2}{4s^2}\right). \tag{54}$$

Now the assertion (51) follows upon substituting Equations (52), (53) and (54) to Equation (6) of Lemma 1. □

Example 3. If $\mathfrak{R}(a) > 0$, then

$$\mathscr{D}\left[\frac{t}{t^2+a^2}\right](s) = \frac{\pi^{3/2}}{4}\exp\left(a^2s^2\right)\mathrm{erfc}\,(as). \tag{55}$$

Proof. We set

$$f(t) = \exp(-at). \tag{56}$$

in Equation (6) of Lemma 1. Using the formula[8] Entry (1), p. 72, we have

$$\mathscr{F}_s\Big[\exp(-at)\Big](s) = \frac{s}{s^2+a^2}. \tag{57}$$

Using the identity (5) and the formula[8] Entry (33), p. 147, we have

$$\begin{aligned}\mathscr{L}_2\left[\frac{1}{t}\exp(-at)\right]\left(\frac{1}{2s}\right) &= \frac{1}{2}\mathscr{L}\left[\frac{1}{t^{1/2}}\exp\left(-at^{1/2}\right)\right]\left(\frac{1}{4s^2}\right)\\ &= \pi^{1/2}s\exp\left(a^2s^2\right)\mathrm{erfc}\,(as).\end{aligned} \tag{58}$$

Now the assertion (55) follows upon substituting Equations (56), (57), and (58) to Equation (6) of Lemma 1. □

The next two examples are illustrations for Equation (18) of Lemma 2.

Example 4. We have

$$\mathscr{D}\left[\frac{\mathrm{arcsinh}(a/t)}{(t^2+a^2)^{1/2}}\right](s) = \frac{\pi^{3/2}}{8}\exp\left(\frac{a^2s^2}{2}\right)\mathrm{K}_0\left(\frac{a^2s^2}{2}\right), \tag{59}$$

where K_0 is the Macdonald function of order zero.

Proof. We set

$$f(t) = \exp(-a^2t^2) \tag{60}$$

in Equation (18) of Lemma 2. Using the formula[1] 42:10:4, p. 431, we have

$$\mathscr{D}\left[\exp\left(-a^2t^2\right)\right](s) = \frac{\mathrm{arcsinh}(a/s)}{2(s^2+a^2)^{1/2}}. \tag{61}$$

Using the formula[8] Entry (13), p. 138 and the identity (5), we have

$$\begin{aligned}\mathscr{G}\left[\exp\left(-a^2t^2\right)\right](s) &= \mathscr{L}_2\left[\frac{1}{t(t^2+s^2)^{1/2}}\right](a)\\ &= \frac{1}{2}\mathscr{L}\left[\frac{1}{t^{1/2}(t+s^2)^{1/2}}\right]\left(a^2\right)\\ &= \frac{1}{2}\exp\left(\frac{a^2s^2}{2}\right)\mathrm{K}_0\left(\frac{a^2s^2}{2}\right).\end{aligned} \tag{62}$$

Now the assertion (59) follows upon substituting Equations (60), (61) and (62) to Equation (18) of Lemma 2. □

Example 5. We have

$$\mathscr{D}\left[\frac{1}{t}\exp\left(\frac{-\beta^2}{4t^2}\right)\right](s)=\frac{\pi^{3/2}}{4}\left[\mathrm{I}_0(\beta s)-\mathrm{L}_0(\beta s)\right], \tag{63}$$

where I_0 is the modified Bessel function of the first kind of order zero and L_0 is the modified Struve function order zero.

Proof. We set

$$f(t)=\sin(\beta t) \tag{64}$$

in Equation (18) of Lemma 2. Using the definition (17), the definition (9) and the formula[1] Entry (6), p. 68, we have

$$\mathscr{G}\left[\sin(\beta t\right](s)=\mathscr{F}_s\left[(t^2+s^2)^{-1/2}\right](\beta)=\frac{\pi}{2}\left[\mathrm{I}_0(\beta s)-\mathrm{L}_0(\beta s)\right]. \tag{65}$$

Now the assertion (63) follows upon substituting Equations (64) and (65) to the iteration identity (18) of Lemma 2, and finally using Equation (11). □

Example 6. We have

$$\mathscr{D}\left[\frac{1}{t^2}\mathrm{daw}\left(\frac{a}{2t}\right)\right](s)=\frac{\pi^{3/2}}{4a}\left[1-\exp(-as)\right]. \tag{66}$$

Proof. If we set

$$f(t)=\frac{\sin(at)}{t} \tag{67}$$

in Equation (33) of Lemma 5, we have

$$\mathscr{D}\left[\frac{1}{u}\mathscr{L}_2\left[\frac{\sin(at)}{t}\right](u)\right](s)=\frac{\pi^{1/2}}{2}\int_0^\infty \arctan\left(\frac{s}{t}\right)\sin(at)\,dt. \tag{68}$$

Using the formula[12] 3.23, p. 126 and using Equation (2), we have

$$\mathscr{L}_2\left[\frac{\sin(at)}{t}\right](u)=\mathscr{F}_s\left[\exp\left(-u^2t^2\right)\right](a)=\frac{1}{u}\mathrm{daw}\left(\frac{a}{2u}\right)\cdot \tag{69}$$

Using the integration by parts on the right-hand side of Equation (68) and the definition of the Fourier cosine transform, we have

$$\int_0^\infty \arctan\left(\frac{s}{t}\right)\sin(at)\,dt=\frac{\pi}{2a}-\frac{s}{a}\int_0^\infty\frac{\cos(at)}{t^2+s^2}\,dt=\frac{\pi}{2a}-\frac{s}{a}\mathscr{F}_c\left[\frac{1}{t^2+s^2}\right](a) \tag{70}$$

where the Fourier cosine transform is defined by

$$\mathscr{F}_c\left[f(t)\right](s)=\int_0^\infty\cos(ts)f(t)\,dt, \tag{71}$$

Using the formula[8] Entry (11), p. 8

$$\int_0^\infty \arctan\left(\frac{s}{t}\right)\sin(at)\,dt=\frac{\pi}{2a}\left[1-\exp(-as)\right] \tag{72}$$

Now the assertion (66) follows upon substituting Equation (70) into (69), substituting Equation (72) to (68) and using Lemma 5. □

Example 7. We have

$$\mathscr{D}\left[\frac{1}{t^2}\exp\left(\frac{-a^2}{4t^2}\right)\right](s)=\frac{1}{2a}\left[\exp(-as)\overline{\mathrm{Ei}}(as)-\exp(as)\mathrm{Ei}(-as)\right], \tag{73}$$

where $\mathrm{Ei}(t)$ is the exponential integral function.

Proof. If we set

$$f(t)=\frac{\cos(at)}{t} \tag{74}$$

in Equation (33) of Lemma 5, we have

$$\mathscr{D}\left[\frac{1}{u}\mathscr{L}_2\left[\frac{\cos(at)}{t}\right](u)\right](s) = \frac{\pi^{1/2}}{2}\int_0^\infty \arctan\left(\frac{s}{t}\right)\cos(at)\,dt. \tag{75}$$

Using the definition (4), the definition (71), and the formula[12] 3.17, p. 11, we have

$$\mathscr{L}_2\left[\frac{\cos(at)}{t}\right](s) = \mathrm{F}_c\left[\exp\left(-t^2s^2\right)\right](a) = \frac{\pi^{1/2}}{2s}\exp\left(-\frac{a^2}{4s^2}\right). \tag{76}$$

Using the integration by parts on the right hand side of Equation (68) and the definition (23) of the Widder transform, we have

$$\int_0^\infty \arctan\left(\frac{s}{t}\right)\cos(at)\,dt = -\frac{s}{a}\int_0^\infty \frac{\sin(at)}{t^2+y^2}\,dt = \frac{s}{a}\mathscr{W}\left[\frac{\sin(at)}{t}\right](s) \tag{77}$$

Using the formula[13] Entry (A5), p. 248

$$\mathscr{W}\left[\frac{\sin(at)}{t}\right](s) = \frac{1}{2s}\left[\exp(-as)\overline{\mathrm{Ei}}(as) - \exp(as)\mathrm{Ei}(-as)\right] \tag{78}$$

Now the assertion (73) follows upon substituting Equation (78) into (76), substituting Equation (76) to (68) and using Lemma 5. □

5 | CONCLUSION

In this work, after introducing the Dawson Transform, we establish various iteration identities and Parseval-Goldstein relationships involving the Dawson Transform, the Laplace Transform, and the $\mathscr{L}_2$-Transform. Our results show that evaluating integral transforms or improper integrals of well-known special functions can be done in an elementary manner using the iteration identities and Parseval-Goldstein type relations presented in this work.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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