Generalized homogeneous Rogers-Szegö polynomials and identities

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Abstract: This paper addresses the construction of Cauchy operators and related identities from $\mathcal{R}(p,q)$-deformed quantum algebras. The generating function, Mehler and Rogers formulae, and their extended identities for the homogeneous Rogers-Szegö polynomials are computed and discussed. Relevant particular identities extracted from known quantum algebras are highlighted.

Key words: $\mathcal{R}(p,q)$-deformed quantum algebras, $\mathcal{R}(p,q)$-calculus, Roger's-Szegö polynomials, generating function, Mehler's formula, Rogers formula.

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1. Introduction

The Cauchy augmentation operator for basic hypergeometric series was introduced in [1], and by using the symmetric property of some parameters in the operator identities, the Heine ${}_2\phi_1$ and Sears ${}_3\phi_2$ transformation formulas were obtained. Besides, the extensions of the Askey-Wilson integral, the Askey-Roy integral, Sears two-term summation formula, and the q-analogs of Barnes lemmas were established.

Let us also note that, in [2], Saad and Abdlhusein constructed the homogeneous q-deformed Rogers-Szegö polynomials by using the q-deformed Cauchy operator. They also derived some operator identities for the q-deformed Cauchy operator, and gave a representation of homogeneous Rogers-Szegö polynomials by the Cauchy operator. Furthermore, the Cauchy operator was also used to deduce some basic identities related to the generating function, Mehler's formula and Rogers formula for homogeneous q-deformed Rogers-Szegö polynomials. Their extended counterparts were also discussed.

The importance of these results and the advantages they provide in the characterization of polynomials naturally lead us to consider them in a more global framework of the $\mathcal{R}(p,q)$-deformed quantum algebras generalizing deformed quantum algebras known in the literature. See [3] for more details, but also [4] about their differentiation, integration and particular cases. Furthermore, in [5], the $\mathcal{R}(p,q)$-deformed Rogers-Szegö polynomials were characterized and their three-term recursion relation were discussed; the continuous $\mathcal{R}(p,q)$-deformed Hermite polynomials were deduced and their properties were also investigated. In the same vein, the $\mathcal{R}(p,q)$-deformation of orthogonal polynomials, basic univariate discrete distributions of the probability theory and their related properties were constructed and discussed in [6].

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The aims of this work is therefore to provide a generalization of the Cauchy operator, compute homogeneous Roger's-Szegö polynomials from the $\mathcal{R}(p,q)$-deformed quantum algebras, and derive novel generalized identities including the generating function, Mehler and Rogers formulae, and their extended versions.

This paper is organized as follows. In section 2, we give an overview on definitions, notations and known results used in the sequel from $\mathcal{R}(p,q)$-deformed calculus and differentiation, $\mathcal{R}(p,q)$-deformed quantum algebras, Jagannathan-Srinivasa deformed algebra and identities. Section 3 is devoted to the generalization of the hypergeometric series and combinatorics from the $\mathcal{R}(p,q)$-deformed quantum algebras [3] used in this work. In section 4, we define some $\mathcal{R}(p,q)$-deformed polynomials such as Cauchy, Rogers-Szegö, and Hahn polynomials. Furthermore, their properties are determined. In Section 5, we introduce the generalized Cauchy operator and derive its properties. Section 6, is dedicated to the construction of homogeneous Roger's-Szegö polynomials and identities such as the generating function, Mehler and Rogers formulas. In section 7, we derive extended indentities for the genenralized homogeneous Roger's-Szegö polynomials. Finally, we conclude with some concluding remarks in section 8.

2. Basic definitions and notations

In this section, we briefly recall the definitions, notations and known results used in the sequel.

Let p and q be two positive real numbers such that $0 < q < p < 1$. We consider a meromorphic function $\mathcal{R}$ defined on $\mathbb{C}\times\mathbb{C}$ by[3]:

$$\mathcal{R}(u,v) = \sum_{s,t=-l}^{\infty} r_{st}u^s v^t, \tag{1}$$

with an eventual isolated singularity at the zero, where r_{st} are complex numbers, $l \in \mathbb{N}\cup\{0\}$, $\mathcal{R}(p^n,q^n) > 0, \forall n \in \mathbb{N}$, and $\mathcal{R}(1,1) = 0$ by definition. We denote by $\mathbb{D}_R$ the bidisk

$$\begin{aligned}\mathbb{D}_R &:= \prod_{j=1}^{2}\mathbb{D}_{R_j}\\ &= \left\{w = (w_1,w_2) \in \mathbb{C}^2 : |w_j| < R_j\right\},\end{aligned}$$

where R is the convergence radius of the series (1) defined by Hadamard formula as follows:

$$\lim\sup_{s+t\longrightarrow\infty}\sqrt[s+t]{|r_{st}|R_1^s\,R_2^t} = 1.$$

For the proof and more details see [7]. Let us also consider $\mathcal{O}(\mathbb{D}_R)$, the set of holomorphic functions defined on $\mathbb{D}_R$, and the $\mathcal{R}(p,q)$-deformed numbers [3]:

$$[n]_{\mathcal{R}(p,q)} := \mathcal{R}(p^n,q^n), \quad n \in \mathbb{N} \tag{2}$$

generalizing known numbers from particular deformations as follows:

(i) q-Arick-Coon-Kuryskin deformation [8]

$$\mathcal{R}(p,q) := \mathcal{R}(1,q) = 1 \quad\text{and}\quad [n]_q = \frac{1-q^n}{1-q}.$$

(ii) q-Quesne deformation [9]

$$\mathcal{R}(p,q) := \mathcal{R}(1,q) = \frac{1-q^{-1}}{1-q} \quad \text{and} \quad [n]_q^Q = \frac{1-q^{-n}}{q-1}.$$

(iii) (p,q)-Jagannathan-Srinivasa deformation [10]

$$\mathcal{R}(p,q) = 1 \quad \text{and} \quad [n]_{p,q} = \frac{p^n-q^n}{p-q}. \tag{3}$$

(iv) (p^{-1},q)-Chakrabarty-Jagannathan deformation [11]

$$\mathcal{R}(p,q) = \frac{1-p\,q}{(p^{-1}-q)p} \quad \text{and} \quad [n]_{p,q} = \frac{p^{-n}-q^n}{p^{-1}-q}.$$

(v) Hounkonnou-Ngompe generalization of q-Quesne deformation [12]

$$\mathcal{R}(p,q) = \frac{p\,q-1}{(q-p^{-1})q} \quad \text{and} \quad [n]_{p,q}^Q = \frac{p^n-q^{-n}}{q-p^{-1}}.$$

Are also defined, (see [3] for more details), the $\mathcal{R}(p,q)$-deformed factorials

$$[n]!_{\mathcal{R}(p,q)} := \begin{cases} 1 \quad \text{for} \quad n=0 \\ \\ \mathcal{R}(p,q)\cdots\mathcal{R}(p^n,q^n) \quad \text{for} \quad n\geq 1, \end{cases}$$

the $\mathcal{R}(p,q)$-deformed binomial coefficients

$$\left[\begin{array}{c} m \\ n \end{array}\right]_{\mathcal{R}(p,q)} := \frac{[m]!_{\mathcal{R}(p,q)}}{[n]!_{\mathcal{R}(p,q)}[m-n]!_{\mathcal{R}(p,q)}}, \quad (m,n)\in\mathbb{N}\times\mathbb{N}, \quad m\geq n, \tag{4}$$

the linear operators on $\mathcal{O}(\mathbb{D}_R)$:

$$\begin{aligned} Q:\Psi \longmapsto Q\Psi(z): &= \Psi(qz), \\ P:\Psi \longmapsto P\Psi(z): &= \Psi(pz), \end{aligned}$$

and the $\mathcal{R}(p,q)$-derivative:

$$\partial_{\mathcal{R}(p,q)} := \partial_{p,q}\frac{p-q}{P-Q}\mathcal{R}(P,Q) = \frac{p-q}{p^P-q^Q}\mathcal{R}(p^P,q^Q)\partial_{p,q},$$

where $\partial_{p,q}$ is the (p,q)-derivative defined as follows:

$$\partial_{p,q}\phi(z) = \frac{\phi(pz)-\phi(qz)}{(p-q)z}. \tag{5}$$

The algebra associated with the $\mathcal{R}(p,q)$-deformation is a quantum algebra, denoted $\mathcal{A}_{\mathcal{R}(p,q)}$, generated by the set of operators $\{1, A, A^\dagger, N\}$ satisfying the following commutation relations:

$$\begin{aligned} AA^\dagger = [N+1]_{\mathcal{R}(p,q)}, &\qquad A^\dagger A = [N]_{\mathcal{R}(p,q)}. \\ [N,\ A] = -A, &\qquad \left[N,\ A^\dagger\right] = A^\dagger \end{aligned}$$

with its realization on $\mathcal{O}(\mathbb{D}_R)$ given by:

$$A^\dagger := z, \qquad A := \partial_{\mathcal{R}(p,q)}, \qquad N := z\partial_z,$$

where $\partial_z := \frac{\partial}{\partial z}$ is the usual derivative on $\mathbb{C}$.

The particular case of the Jagannathan-Srinivasa quantum algebra [10] gives the (p,q)-numbers (3) and (p,q)-factorials:

$$[n]!_{p,q} = \begin{cases} 1 \quad \text{for} \quad n=0 \\ \frac{((p,q);(p,q))_n}{(p-q)^n} \quad \text{for} \quad n \geq 1, \end{cases}$$

with the following relevant properties

$$\begin{aligned}
[n]_{p,q} &= \sum_{k=0}^{n-1} p^{n-1-k} q^k, \\
[n+m]_{p,q} &= q^m [n]_{p,q} + p^n [m]_{p,q} = p^m [n]_{p,q} + q^n [m]_{p,q}, \\
[-m]_{p,q} &= -q^{-m} p^{-m} [m]_{p,q}, \\
[n-m]_{p,q} &= q^{-m} [n]_{p,q} - q^{-m} p^{n-m} [m]_{p,q} = p^{-m} [n]_{p,q} - q^{n-m} p^{-m} [m]_{p,q}, \\
[n]_{p,q} &= [2]_{p,q} [n-1]_{p,q} - pq [n-2]_{p,q},
\end{aligned}$$

where n and m are nonnegative integers. Furthermore, the related (p,q)-binomial coefficients

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} = \frac{((p,q);(p,q))_n}{((p,q);(p,q))_k ((p,q);(p,q))_{n-k}}, \quad 0 \leq k \leq n; \ \ n \in \mathbb{N},$$

where $((p,q);(p,q))_m = (p-q)(p^2-q^2)\cdots(p^m-q^m)$, $m \in \mathbb{N}$, satisfy the following identities:

$$\begin{aligned}
\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} &= \left[\begin{array}{c} n \\ n-k \end{array}\right]_{p,q} = p^{k(n-k)} \left[\begin{array}{c} n \\ k \end{array}\right]_{q/p} = p^{k(n-k)} \left[\begin{array}{c} n \\ n-k \end{array}\right]_{q/p}, \\
\left[\begin{array}{c} n+1 \\ k \end{array}\right]_{p,q} &= p^k \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} + q^{n+1-k} \left[\begin{array}{c} n \\ k-1 \end{array}\right]_{p,q}, \\
\left[\begin{array}{c} n+1 \\ k \end{array}\right]_{p,q} &= p^k \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} + p^{n+1-k} \left[\begin{array}{c} n \\ k-1 \end{array}\right]_{p,q} - (p^n - q^n) \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{p,q}
\end{aligned}$$

with

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{q/p} = \frac{(q/p;q/p)_n}{(q/p;q/p)_k (q/p;q/p)_{n-k}};$$

$(q/p;q/p)_n = (1-q/p)(1-q^2/p^2)\cdots(1-q^n/p^n)$; and the (p,q)-shifted factorial

$$((a,b);(p,q))_n := (a-b)(ap-bq)\cdots(ap^{n-1}-bq^{n-1})$$

or, equivalently,

$$((a,b);(p,q))_n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} (-1)^k p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} a^{n-k} b^k.$$

Finally, the algebra $\mathcal{A}_{p,q}$, generated by $\{1,\ A,\ A^\dagger,\ N\}$, associated with (p,q)-Janagathan-Srinivasa deformation, satisfies the following commutation relations:

$$\begin{array}{ll} A\,A^\dagger - pA^\dagger A = q^N, & A\,A^\dagger - qA^\dagger A = p^N \\ [N,\ A^\dagger] = A^\dagger, & [N,\ A] = -A. \end{array}$$

Finally, let us recall that the $\mathcal{R}(p,q)$-deformed exponential function [3]

$$Exp_{\mathcal{R}(p,q)}(z) = \sum_{n=0}^{\infty} \frac{1}{[n]_{\mathcal{R}(p,q)}!}\, z^n \tag{6}$$

obeys the $\mathcal{R}(p,q)$-difference equation

$$\mathcal{R}(P,Q)\, Exp_{\mathcal{R}(p,q)}(z) = z\, Exp_{\mathcal{R}(p,q)}(z). \tag{7}$$

Setting

$$\begin{aligned} F(z) &= \frac{z}{z - \mathcal{R}(1,0)} \\ G(P,Q) &= \frac{p(Q-P)\mathcal{R}(p^P,q^Q) + (p^P - q^Q)\mathcal{R}(1,0)}{pQ\,\mathcal{R}(p^P,q^Q)}, \quad \text{if} \quad l = 0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} F(z) &= z \\ G(P,Q) &= \frac{q^Q - p^P}{pQ\,\mathcal{R}(p^P,q^Q)}, \quad \text{if} \quad l > 0, \end{aligned} \tag{9}$$

where l is given in the the relation (1), leads to

$$Exp_{\mathcal{R}(p,q)}(z) = [1 - F(z)G(P,Q)]\, Exp_{\mathcal{R}(p,q)}\big(\frac{q}{p}z\big) \tag{10}$$

and

$$Exp_{\mathcal{R}(p,q)}(z) = \prod_{k=0}^{n-1}\big[1 - F\big(\frac{q^k}{p^k}z\big)G(P,Q)\big]\, Exp_{\mathcal{R}(p,q)}\big(\frac{q^n}{p^n}z\big). \tag{11}$$

3. $\mathcal{R}(p,q)$-deformed hypergeometric series

For $a,b\in\mathbb{C}$, let us consider the $\mathcal{R}(p,q)$-deformed shifted factorial:

$$\big(a,b;\mathcal{R}(p,q)\big)_0 = 1, \quad \big(a,b;\mathcal{R}(p,q)\big)_n := \prod_{k=0}^{n-1}\Big(a - F\big(\frac{q^k}{p^k}\,b\big)G(P,Q)\Big),\ n\in\mathbb{N}, \tag{12}$$

and

$$(a,b;\mathcal{R}(p,q))_\infty := \prod_{k=0}^{\infty}\Big(a - F\big(\frac{q^k}{p^k}\,b\big)G(P,Q)\Big), \tag{13}$$

and the $\mathcal{R}(p,q)$-deformed n-shifted factorial:

$$\begin{aligned} \big(a_1,\ldots a_s, b_1\ldots,b_t;\mathcal{R}(p,q)\big)_n &= \big(a_1,b_1;\mathcal{R}(p,q)\big)_n \ldots \big(a_s,b_t;\mathcal{R}(p,q)\big)_n \\ \big(a_1,\ldots a_s, b_1\ldots,b_t;\mathcal{R}(p,q)\big)_\infty &= \big(a_1,b_1;\mathcal{R}(p,q)\big)_\infty \ldots \big(a_s,b_t;\mathcal{R}(p,q)\big)_\infty. \end{aligned} \tag{14}$$

Lemma 1. *For $a \in \mathbb{C}$, the $\mathcal{R}(p,q)$-deformed shifted factorials satisfy the following useful identities:*

$$\Big(a, \frac{q^k}{p^k}b; \mathcal{R}(p,q)\Big)_{n-k} = \frac{\big(a,b;\mathcal{R}(p,q)\big)_n}{\big(a,b;\mathcal{R}(p,q)\big)_k},\ k \in \{0,1,2,\cdots,n\}. \tag{15}$$

For any $\alpha \in \mathbb{C}$,

$$\big(a,b;\mathcal{R}(p,q)\big)_\alpha = \frac{\big(a,b;\mathcal{R}(p,q)\big)_\infty}{\Big(a, \frac{q^\alpha}{p^\alpha}b; \mathcal{R}(p,q)\Big)_\infty}. \tag{16}$$

$$\begin{aligned}\big(a,b;\mathcal{R}(p,q)\big)_{n+k} &= \big(a,b;\mathcal{R}(p,q)\big)_n \big(a, \frac{q^n}{p^n}b;\mathcal{R}(p,q)\big)_k \\ &= \big(a,b;\mathcal{R}(p,q)\big)_k \big(a, \frac{q^k}{p^k}b;\mathcal{R}(p,q)\big)_n.\end{aligned} \tag{17}$$

$$\big(a, \frac{q^n}{p^n}b;\mathcal{R}(p,q)\big)_k = \frac{\big(a,b;\mathcal{R}(p,q)\big)_k \big(a, \frac{q^k}{p^k}b;\mathcal{R}(p,q)\big)_n}{\big(a,b;\mathcal{R}(p,q)\big)_n}. \tag{18}$$

Proof. It stems from a straightforward computation.

The $\mathcal{R}(p,q)$-deformed binomial coefficient (4) can be re-written in term of the $\mathcal{R}(p,q)$-factorial as follows:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)} := \frac{\big(a,b;\mathcal{R}(p,q)\big)_n}{\big(a,b;\mathcal{R}(p,q)\big)_k\big(a,b;\mathcal{R}(p,q)\big)_{n-k}}, \quad (n,k)\in\mathbb{N}\times\mathbb{N}, \quad n \geq k.$$

Then, the generalized $\mathcal{R}(p,q)$-deformed basic hypergeometric function is defined by:

$${}_r\phi_s\left(\begin{array}{c} (a_1,b_1),\ldots,(a_r,b_r) \\ (c_1,d_1),\ldots,(b_s,d_s)\end{array}\middle|\mathcal{R}(p,q);x\right) = \sum_{n=0}^{\infty}\frac{\big(a_1,b_1;\mathcal{R}(p,q)\big)_n\ldots\big(a_r,b_r;\mathcal{R}(p,q)\big)_n}{\big(c_1,d_1;\mathcal{R}(p,q)\big)_n\ldots\big(c_s,d_s;\mathcal{R}(p,q)\big)_n}\Big[-\big(\frac{q}{p}\big)^{\binom{n}{2}}\Big]^{1+s-r}x^n,$$

where $r > s+1$. The $\mathcal{R}(p,q)$-Cauchy identity is given by:

$$\sum_{k=0}^{\infty}\frac{\big(a,b;\mathcal{R}(p,q)\big)_k}{\big(p,q;\mathcal{R}(p,q)\big)_k}x^k = \frac{\big(a,bx;\mathcal{R}(p,q)\big)_k}{\big(a,x;\mathcal{R}(p,q)\big)_\infty} \tag{19}$$

affording the special cases

$$\sum_{k=0}^{\infty}\frac{x^k}{\big(p,q;\mathcal{R}(p,q)\big)_k} = \frac{1}{\big(1,x;\mathcal{R}(p,q)\big)_\infty} \tag{20}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{\big(p,q;\mathcal{R}(p,q)\big)_k} = \big(1,x;\mathcal{R}(p,q)\big)_\infty. \tag{21}$$

Remark 1. *The following special cases deserve attention:*

(a) Putting $\mathcal{R}(x,1) = \frac{x-1}{q-1}$, we recover the q-deformed hypergeometric series given in [13].

(b) Taking $\mathcal{R}(x,y) = \frac{x-y}{p-q}$, we obtain the hypergeometric series from the (p,q)-deformed quantum algebra investigated in [10]:

For $a, b \in \mathbb{C}$, the (p,q)-deformed shifted factorial is defined by:

$$\big(a,b;p,q\big)_0 = 1, \quad \big(a,b;p,q\big)_n := \prod_{k=0}^{n-1}\big(ap^k - bq^k\big),\ n \in \mathbb{N},$$

and,

$$(a,b;p,q)_\infty := \prod_{k=0}^{\infty}\big(ap^k - bq^k\big).$$

Furthermore, the (p,q)-deformed n-shifted factorial is given as follows:

$$\begin{aligned}\big(a_1,\ldots a_s, b_1\ldots,b_t;p,q\big)_n &= \big(a_1,b_1;p,q\big)_n\cdots\big(a_s,b_t;p,q\big)_n\\ \big(a_1,\ldots a_s, b_1\ldots,b_t;p,q\big)_\infty &= \big(a_1,b_1;p,q\big)_\infty\cdots\big(a_s,b_t;p,q\big)_\infty\end{aligned}$$

and for $a, b \in \mathbb{C}$ and $\alpha \in \mathbb{C}$, the (p,q)-deformed shifted factorials satisfy the following useful identities:

$$\Big(a, \frac{q^k}{p^k}b;p,q\Big)_{n-k} = \frac{\big(a,b;p,q\big)_n}{\big(a,b;p,q\big)_k},\ k \in \{0,1,2,\cdots,n\},$$

$$\big(a,b;p,q\big)_\alpha = \frac{\big(a,b;p,q\big)_\infty}{\Big(a,\frac{q^\alpha}{p^\alpha}b;p,q\Big)_\infty}$$

$$\begin{aligned}\big(a,b;p,q\big)_{n+k} &= \big(a,b;p,q\big)_n\big(a,\frac{q^n}{p^n}b;p,q\big)_k\\ &= \big(a,b;p,q\big)_k\big(a,\frac{q^k}{p^k}b;p,q\big)_n,\end{aligned}$$

$$\big(a,\frac{q^n}{p^n}b;p,q\big)_k = \frac{\big(a,b;p,q\big)_k\big(a,\frac{q^k}{p^k}b;p,q\big)_n}{\big(a,b;p,q\big)_n}.$$

Moreover, the (p,q)-deformed binomial coefficient is defined as:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} := \frac{\big(a,b;p,q\big)_n}{\big(a,b;p,q\big)_k\big(a,b;p,q\big)_{n-k}}, \quad (n,k) \in \mathbb{N}\times\mathbb{N}, \quad n \geq k$$

and the generalized (p,q)-deformed basic hypergeometric function yields:

$${}_r\phi_s\left(\begin{array}{c} (a_1,b_1),\ldots,(a_r,b_r) \\ (c_1,d_1),\ldots,(b_s,d_s)\end{array}\middle| p,q;x\right) := \sum_{n=0}^{\infty}\frac{\big(a_1,b_1;p,q\big)_n\cdots\big(a_r,b_r;p,q\big)_n}{\big(c_1,d_1;p,q\big)_n\cdots\big(c_s,d_s;p,q\big)_n}\Big[(-1)\big(\frac{q}{p}\big)^{\binom{n}{2}}\Big]^{1+s-r}x^n$$

where $r > s+1$. Note that

$${}_{r+1}\phi_s\left(\begin{array}{c} (a_1,b_1),\ldots,(a_{r+1},b_{r+1}) \\ (c_1,d_1),\ldots,(b_r,d_r)\end{array}\middle| p,q;x\right) := \sum_{n=0}^{\infty}\frac{\big(a_1,b_1;p,q\big)_n\cdots\big(a_{r+1},b_{r+1};p,q\big)_n}{\big(c_1,d_1;p,q\big)_n\cdots\big(c_r,d_r;p,q\big)_n}x^n.$$

Besides, the (p,q)-Cauchy identity is expressed by:

$$\sum_{k=0}^{\infty}\frac{\big(a,b;p,q\big)_k}{\big(p,q;p,q\big)_k}\,x^k=\frac{\big(a,bx;p,q\big)_k}{\big(a,x;p,q\big)_\infty} \tag{22}$$

providing the special cases

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{x^k}{\big(p,q;p,q\big)_k} &= \frac{1}{\big(1,x;p,q\big)_\infty}\\ \sum_{k=0}^{\infty}\frac{(-1)^k\,x^k}{\big(p,q;p,q\big)_k} &= \big(1,x;p,q\big)_\infty.\end{aligned}$$

4. $\mathcal{R}(p,q)$-deformed polynomials and properties

In this section, we construct some polynomials from the generalized quantum algebras [3]. Related properties are investigated and particular cases are deduced from the formalism established.

Let us start defining the generating function of the$\mathcal{R}(p,q)$-deformed Cauchy polynomials

$$P_n(x,y):=\prod_{k=0}^{n-1}\bigg(x-F(\frac{q^k}{p^k}\,y)G(P,Q)\bigg), \tag{23}$$

also called the homogeneous form of the $\mathcal{R}(p,q)$-Cauchy identity, by:

$$\sum_{k=0}^{\infty}P_k(x,y)\frac{t^k}{\big(p,q;\mathcal{R}(p,q)\big)_k}=\frac{\big(a,yt;\mathcal{R}(p,q)\big)_k}{\big(a,xt;\mathcal{R}(p,q)\big)_\infty}. \tag{24}$$

It gives the identity (20) for $y=0$ and $a=1$. As it has been highlighted from [14],[15],[16],[17],[18], the Rogers-Szego polynomials play an important role in the study of the Askey-Wilson polynomials.

Definition 1. *The $\mathcal{R}(p,q)$-deformed Rogers-Szegö polynomials are defined by:*

$$h_n\big(x|\mathcal{R}(p,q)\big):=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_{\mathcal{R}(p,q)}x^k,\quad n\in\mathbb{N}. \tag{25}$$

The basic identities for the $h_n\big(x|\mathcal{R}(p,q)\big)$ are developed in the following proposition.

Proposition 1. *The generating function for the $\mathcal{R}(p,q)$-deformed Rogers-Szegö polynomials is given by the expression:*

$$\sum_{n=0}^{\infty}h_n(x;\mathcal{R}(p,q))\frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n}=\frac{1}{\big(p,t,xt;\mathcal{R}(p,q)\big)_\infty} \tag{26}$$

with the Mehler's formula:

$$\sum_{n=0}^{\infty}h_n(x;\mathcal{R}(p,q))h_n(y|\mathcal{R}(p,q))\frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n}=\frac{\big(1,xyt^2;\mathcal{R}(p,q)\big)_\infty}{\big(1,t,xt,yt,xyt;\mathcal{R}(p,q)\big)_\infty}, \tag{27}$$

where $\max\{|t|,|xt|,|yt|,|xyt|\}<1$.

Furthermore, the Roger's formula for $h_n(x;\mathcal{R}(p,q))$ is given by:

$$\sum_{n=0}^{\infty} h_{n+m}(x|\mathcal{R}(p,q))\frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n}\frac{s^m}{\big(p,q;\mathcal{R}(p,q)\big)_m} = \frac{\big(1,xst;\mathcal{R}(p,q)\big)_\infty}{\big(1,t,s,xt,xs;\mathcal{R}(p,q)\big)_\infty}, \tag{28}$$

where $\max\{|s|,|t|,|xs|,|xt|\} < 1$.

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} h_n(x;\mathcal{R}(p,q))\frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n\\ k\end{array}\right]_{\mathcal{R}(p,q)}\frac{x^k t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^k t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n\big(p,q;\mathcal{R}(p,q)\big)_{n-k}}\\ &= \sum_{n=0}^{\infty}\frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n}\sum_{k=0}^{n}\frac{(xt)^k}{\big(p,q;\mathcal{R}(p,q)\big)_k}\\ &= \frac{1}{\big(p,t;\mathcal{R}(p,q)\big)_\infty}\frac{1}{\big(p,xt;\mathcal{R}(p,q)\big)_\infty}.\end{aligned}$$

Then, the relation (26) follows, and the remaining formulas are self-evident.

Remark 2. *Particular cases*

(a) *The Rogers-Szegö polynomials and relations associated to the q-deformed quantum algebra are given in [14],[15],[16], [17],[18].*

(a) *The (p,q)-deformed Rogers-Szegö polynomials and identities are deduced as follows:*

$$h_n\big(x|p,q\big) = \sum_{k=0}^{n}\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q} x^k, \quad n\in\mathbb{N},$$

with the generating function :

$$\sum_{n=0}^{\infty} h_n(x;p,q)\frac{t^n}{\big(p,q;p,q\big)_n} = \frac{1}{\big(p,t,xt;p,q\big)_\infty}$$

and the Mehler's formula:

$$\sum_{n=0}^{\infty} h_n(x;p,q)h_n(y|p,q)\frac{t^n}{\big(p,q;p,q\big)_n} = \frac{\big(1,xyt^2;p,q\big)_\infty}{\big(1,t,xt,yt,xyt;p,q\big)_\infty},$$

where $\max\{|t|,|xt|,|yt|,|xyt|\} < 1$. *Furthermore, the Roger's formula for $h_n(x;(p,q))$ is given by:*

$$\sum_{n=0}^{\infty} h_{n+m}(x|p,q)\frac{t^n}{\big(p,q;p,q\big)_n}\frac{s^m}{\big(p,q;p,q\big)_m} = \frac{\big(1,xst;p,q\big)_\infty}{\big(1,t,s,xt,xs;p,q\big)_\infty},$$

where $\max\{|s|,|t|,|xs|,|xt|\} < 1$.

Now, we investigate the $\mathcal{R}(p,q)$-deformed Hahn polynomials.

Definition 2. *The $\mathcal{R}(p,q)$-deformed Hahn polynomials are defined by:*

$$\phi_n^{(a,b)}(x;\mathcal{R}(p,q)) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)} \big(a,b;\mathcal{R}(p,q)\big)_k\, x^k. \tag{29}$$

Note that, for $a=1$ and $b=0$, we get $\phi_n^{1,0}(x) = h_n\big(x|\mathcal{R}(p,q)\big)$.

Proposition 2. *The generating function for $\phi_n^{(a,b)}(x;\mathcal{R}(p,q))$ is given by:*

$$\sum_{n=0}^{\infty} \phi_n^{(a,b)}(x;\mathcal{R}(p,q)) \frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} = \frac{\big(a,bxt;\mathcal{R}(p,q)\big)_\infty}{\big(p,t,xt;\mathcal{R}(p,q)\big)_\infty}, \quad \max\{|t|,|xt|\} < 1 \tag{30}$$

and the Mehler's formula is retrieved as:

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_n^{a,b}(x;\mathcal{R}(p,q))\phi_n^{c,b}(y;\mathcal{R}(p,q)) \frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} &= \frac{\big(p,axt,byt;\mathcal{R}(p,q)\big)_\infty}{\big(p,t,xt,yt;\mathcal{R}(p,q)\big)_\infty} \\ &\times {}_3\phi_2\left(\begin{array}{c} (p,a),(p,b),(p,t) \\ (p,axt),(p,byt) \end{array}\middle| \mathcal{R}(p,q); xyt\right),\end{aligned} \tag{31}$$

where $\max\{|t|,|xt|,|yt|,|xyt|\} < 1$.

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_n^{(a,b)}(x;\mathcal{R}(p,q)) \frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)} \big(a,b;\mathcal{R}(p,q)\big)_k x^k \sum_{n=0}^{\infty} \frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} \\ &= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \frac{\big(a,b;\mathcal{R}(p,q)\big)_k x^k}{\big(p,q;\mathcal{R}(p,q)\big)_k \big(p,q;\mathcal{R}(p,q)\big)_{n-k}} \\ &= \sum_{k=0}^{\infty} \frac{\big(a,b;\mathcal{R}(p,q)\big)_k (xt)^k}{\big(p,q;\mathcal{R}(p,q)\big)_k} \sum_{n=0}^{\infty} \frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} \\ &= \frac{\big(a,bxt;\mathcal{R}(p,q)\big)_\infty}{\big(p,xt;\mathcal{R}(p,q)\big)_\infty} \frac{1}{\big(p,t;\mathcal{R}(p,q)\big)_\infty} \\ &= \frac{\big(a,bxt;\mathcal{R}(p,q)\big)_\infty}{\big(p,t,xt;\mathcal{R}(p,q)\big)_\infty}\end{aligned}$$

proving the relation (30), and the rest follows.

Furthermore,

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{n+m}^{a}(x) \frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} \frac{s^m}{\big(p,q;\mathcal{R}(p,q)\big)_m} &= \frac{\big(p,axs;\mathcal{R}(p,q)\big)_\infty}{\big(p,s,xs,xt;\mathcal{R}(p,q)\big)_\infty} \\ &\times {}_2\phi_1\left(\begin{array}{c} (p,xa),(p,xs) \\ (p,axs) \end{array}\middle| \mathcal{R}(p,q); t\right),\end{aligned} \tag{32}$$

where $\max\{|s|,|t|,|xs|,|xt|\} < 1$. Note that

$$\phi_n^{y/x}(x) = h_n(x,y|\mathcal{R}(p,q)) \tag{33}$$

$$\phi_n^{y}(1/x) = x^{-n}\, h_n(x,y|\mathcal{R}(p,q)). \tag{34}$$

Remark 3. *Particular cases of Hahn polynomials and properties are deduced as:*

(a) The q-deformed Hahn polynomials and identities are determined in [19].

(b) The (p,q)-deformed Hahn polynomials are defined by:

$$\phi_n^{(a,b)}(x;p,q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} \big(a,b;p,q\big)_k\, x^k$$

giving, for $a=1$ and $b=0$, $\phi_n^{1,0}(x) = h_n\big(x|p,q\big)$. The generating function for $\phi_n^{(a,b)}(x;p,q)$ is given by:

$$\sum_{n=0}^{\infty} \phi_n^{(a,b)}(x;p,q)\frac{t^n}{\big(p,q;p,q\big)_n} = \frac{\big(a,bxt;p,q\big)_\infty}{\big(p,t,xt;p,q\big)_\infty}, \quad \max\{|t|,|xt|\} < 1$$

with the Mehler's formula:

$$\sum_{n=0}^{\infty} \phi_n^{a,b}(x;p,q)\phi_n^{c,b}(y;p,q)\frac{t^n}{\big(p,q;p,q\big)_n} = \frac{\big(p,axt,byt;p,q\big)_\infty}{\big(p,t,xt,yt;p,q\big)_\infty}\, {}_3\phi_2\left(\begin{array}{c} (p,a),(p,b),(p,t) \\ (p,axt),(p,byt) \end{array} \Bigg| p,q;xyt \right),$$

where $\max\{|t|,|xt|,|yt|,|xyt|\} < 1$.

The $\mathcal{R}(p,q)$-differential operator defined by:

$$D_{\mathcal{R}(p,q)}f(x) = \frac{p-q}{pP-qQ}\mathcal{R}(pP,qQ)\,\frac{f(p\,x)-f(q\,x)}{x} \tag{35}$$

satisfies the following identities:

$$\begin{aligned} D^k_{\mathcal{R}(p,q)}\left(\frac{1}{\big(x,yt;\mathcal{R}(p,q)\big)_\infty}\right) &= \frac{t^k}{\big(x,yt;\mathcal{R}(p,q)\big)_\infty} \\ D^n_{\mathcal{R}(p,q)}\left(\frac{\big(x,yv;\mathcal{R}(p,q)\big)_\infty}{\big(x,yt;\mathcal{R}(p,q)\big)_\infty}\right) &= t^n\big(x,v/t;\mathcal{R}(p,q)\big)_n\,\frac{\big(x,yvq^n;\mathcal{R}(p,q)\big)_\infty}{\big(x,yt;\mathcal{R}(p,q)\big)_\infty}. \end{aligned} \tag{36}$$

The $\mathcal{R}(p,q)$-deformed exponential operator

$$T\big(z\,D_{\mathcal{R}(p,q)}\big) := \sum_{k=0}^{\infty}\frac{\big(z\,D_{\mathcal{R}(p,q)}\big)^n}{\big(p,q;\mathcal{R}(p,q)\big)_n}. \tag{37}$$

is used to perform the $\mathcal{R}(p,q)$-deformed Rogers-Szegö polynomials $h_n\big(x|\mathcal{R}(p,q)\big)$, to deduce Mehler's formula and Rogers formula for $h_n\big(x|\mathcal{R}(p,q)\big)$.

The homogeneous $\mathcal{R}(p,q)$-deformed operator D_{xy} on functions in two variables

$$D_{xy}\big(f(x,y)\big) := \frac{p-q}{pP-qQ}\mathcal{R}(pP,qQ)\,\frac{f(x,p\,q^{-1}\,y)-g(p^{-1}\,q\,x,y)}{p^{-1}\,x-q^{-1}\,y}, \tag{38}$$

turns out to be suitable to derive the homogeneous form of the binomial theorem and the Leibniz formula for this operator. From the homogeneous $\mathcal{R}(p,q)$-difference operator, we deduce the homogeneous $\mathcal{R}(p,q)$-shift operator $E(Dxy)$ as follows:

$$E(Dxy) = \sum_{k=0}^{\infty}\frac{D^k_{xy}}{\big(p,q;\mathcal{R}(p,q)\big)_k}. \tag{39}$$

Remark 4. *The differential operators and quantum algebras known in the literature are easily recovered from the above results as follows:*

(a) The results concerning the operators related to the q-deformed quantum algebra are reitrived in [1, 20].

(b) The (p,q)-differential operator defined by:

$$D_{p,q}f(x) = \frac{f(p\,x)-f(q\,x)}{x}.$$

satisfies the following identities:

$$D^k_{p,q}\bigg(\frac{1}{\big(x,yt;p,q\big)_\infty}\bigg) = \frac{t^k}{\big(x,yt;p,q\big)_\infty}$$
$$D^n_{p,q}\bigg(\frac{\big(x,yv;p,q\big)_\infty}{\big(x,yt;p,q\big)_\infty}\bigg) = t^n\big(x,v/t;p,q\big)_n\frac{\big(x,yvq^n;p,q\big)_\infty}{\big(x,yt;p,q\big)_\infty}.$$

The (p,q)-deformed exponential operator is defined as follows:

$$T\big(z\,D_{\mathcal{R}(p,q)}\big) = \sum_{k=0}^{\infty}\frac{\big(z\,D_{p,q}\big)^n}{\big(p,q;p,q\big)_n}.$$

Moreover, the homogeneous (p,q)-deformed operator D_{xy} on functions in two variables is defined by:

$$D_{xy}\big(f(x,y)\big) = \frac{f(x,p\,q^{-1}\,y)-g(p^{-1}\,q\,x,y)}{p^{-1}\,x-q^{-1}\,y}$$

with the homogeneous (p,q)-shift operator $E(Dxy)$ given by:

$$E(Dxy) = \sum_{k=0}^{\infty}\frac{D^k_{xy}}{\big(p,q;p,q\big)_k}.$$

Definition 3. *The homogeneous $\mathcal{R}(p,q)$-deformed Roger's-Szegö polynomials are given as:*

$$h_n\big(x,y|\mathcal{R}(p,q)\big) := \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{R}(p,q)} P_k(x,y). \tag{40}$$

Note that, when $y=0$, we obtain the $\mathcal{R}(p,q)$-deformed Rogers-Szegö $h_n\big(x|\mathcal{R}(p,q)\big)$.

From the $\mathcal{R}(p,q)$-deformed exponential operator $T(D_{\mathcal{R}(p,q)})$ and the homogeneous $\mathcal{R}(p,q)$-deformed oprator D_{xy}, we obtain the following results:

Proposition 3. *The generating function for the $\mathcal{R}(p,q)$-deformed Rogers-Szegö polynomials is given as follows:*

$$\sum_{n=0}^{\infty}h_n(x,y|\mathcal{R}(p,q))\frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} = \frac{\big(p,yt;\mathcal{R}(p,q)\big)_\infty}{\big(p,t,xt;\mathcal{R}(p,q)\big)_\infty},\quad \max\{|t|,|xt|\}<1 \tag{41}$$

with the Mehler's formula :

$$\sum_{n=0}^{\infty}h_n\big(x,y|\mathcal{R}(p,q)\big)h_n\big(u,v|\mathcal{R}(p,q)\big)\frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n} = \frac{\big(p,yt,xvt;\mathcal{R}(p,q)\big)_\infty}{\big(p,t,xt,xut;\mathcal{R}(p,q)\big)_\infty}$$

$$\times\ {}_3\phi_2\left(\begin{array}{c}(p,y),(p,xt),(p,v/u)\\(p,yt),(p,xvt)\end{array}\middle|\mathcal{R}(p,q);ut\right), \tag{42}$$

where $\max\{|t|,|xt|,|ut|,|xut|\}<1$. *Moreover, the Rogers formula for* $h_n\big(x,y|\mathcal{R}(p,q)\big)$ *yields:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}\big(x,y|\mathcal{R}(p,q)\big)\frac{t^n}{\big(p,q;\mathcal{R}(p,q)\big)_n}\frac{s^m}{\big(p,q;\mathcal{R}(p,q)\big)_m}=\frac{\big(p,ys;\mathcal{R}(p,q)\big)_\infty}{\big(p,s,xs,xt;\mathcal{R}(p,q)\big)_\infty}$$
$$\times\ {}_2\phi_1\left(\begin{array}{c}(p,y),(p,xs)\\(p,ys)\end{array}\middle|\mathcal{R}(p,q);t\right), \tag{43}$$

where $\max\{|t|,|s|,|xt|,|xs|\}<1$.

Remark 5. *Let us emphasize the following:*

(a) The Mehler and Rogers formulae for $h_n(x|q)$ *were obtained by Chen et al in [20].*

(b) The homogeneous (p,q)*-deformed Roger's-Szegö polynomials are given by:*

$$h_n\big(x,y|p,q\big)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}P_k(x,y)$$

with the corresponding generating function:

$$\sum_{n=0}^{\infty}h_n(x,y|p,q)\frac{t^n}{\big(p,q;p,q\big)_n}=\frac{\big(p,yt;p,q\big)_\infty}{\big(p,t,xt;p,q\big)_\infty},\quad \max\{|t|,|xt|\}<1$$

the Mehler's formula:

$$\sum_{n=0}^{\infty}h_n\big(x,y|p,q\big)h_n\big(u,v|p,q\big)\frac{t^n}{\big(p,q;p,q\big)_n}=\frac{\big(p,yt,xvt;p,q\big)_\infty}{\big(p,t,xt,xut;p,q\big)_\infty}$$
$$\times\ {}_3\phi_2\left(\begin{array}{c}(p,y),(p,xt),(p,v/u)\\(p,yt),(p,xvt)\end{array}\middle|p,q;ut\right),$$

where $\max\{|t|,|xt|,|ut|,|xut|\}<1$, *and the Rogers formula:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}\big(x,y|p,q\big)\frac{t^n}{\big(p,q;p,q\big)_n}\frac{s^m}{\big(p,q;p,q\big)_m}=\frac{\big(p,ys;p,q\big)_\infty}{\big(p,s,xs,xt;p,q\big)_\infty}$$
$$\times\ {}_2\phi_1\left(\begin{array}{c}(p,y),(p,xs)\\(p,ys)\end{array}\middle|p,q;t\right),$$

where $\max\{|t|,|s|,|xt|,|xs|\}<1$.

5. Generalized Cauchy operator and some identities

In this section, we introduce the Cauchy operator for basic hypergeometric series induced from the quantum deformed algebras [3]. Furthermore, we derive some operator identities and deduce relevant particular cases from quantum algebras existing in the literature.

Definition 4. *The $\mathcal{R}(p,q)$-deformed Cauchy operator can be defined as follows:*

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big) := \sum_{n=0}^{\infty}\frac{\big(x,y;\mathcal{R}(p,q)\big)_n}{\big(p,q;\mathcal{R}(p,q)\big)_n}\big(z\,D_{\mathcal{R}(p,q)}\big)^n. \tag{44}$$

It reduces to the $\mathcal{R}(p,q)$-deformed exponential operator $T\big(zD_{\mathcal{R}(p,q)}\big)$ when $x=1$ and $y=0$. It generates the following identities:

Theorem 1.

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)(c^n) = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)}\big(x,y;\mathcal{R}(p,q)\big)_k\,z^k c^{n-k}, \tag{45}$$

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\left\{\frac{1}{\big(x,ct;\mathcal{R}(p,q)\big)_\infty}\right\} = \frac{\big(x,yzt;\mathcal{R}(p,q)\big)_\infty}{\big(x,ct,zt;\mathcal{R}(p,q)\big)_\infty}, \quad |zt|<1, \tag{46}$$

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\left\{\frac{1}{\big(x,cs,ct;\mathcal{R}(p,q)\big)_\infty}\right\} = \frac{(x,yzt;\mathcal{R}(p,q))_\infty}{(x,cs,ct,zt;\mathcal{R}(p,q))_\infty}\,{}_2\phi_1\left(\begin{array}{c}(x,ct),(x,y)\\(x,yzt)\end{array}\middle|\mathcal{R}(p,q);zs\right), \tag{47}$$

$$\begin{aligned}T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\left\{\frac{\big(x,cv;\mathcal{R}(p,q)\big)_\infty}{\big(x,cs,ct;\mathcal{R}(p,q)\big)_\infty}\right\} &= \frac{(x,yzs,cv;\mathcal{R}(p,q))_\infty}{(x,zs,ct,cs;\mathcal{R}(p,q))_\infty}\\ &\quad {}_3\phi_2\left(\begin{array}{c}(x,y),(x,cs),(x,v/t)\\(x,yzs),(x,cv)\end{array}\middle|\mathcal{R}(p,q);zt\right),\end{aligned} \tag{48}$$

where $\max\{|zs|,|zt|\}<1$.

Proof. We have

$$\begin{aligned}T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\{c^n\} &= \sum_{k=0}^{\infty}\frac{\big(x,y;\mathcal{R}(p,q)\big)_k}{\big(p,q;\mathcal{R}(p,q)\big)_k}z^k\,D^k_{\mathcal{R}(p,q)}\,c^n\\ &= \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)}\big(x,y;\mathcal{R}(p,q)\big)_k z^k\,c^{n-k}.\end{aligned} \tag{49}$$

Moreover,

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\left\{\frac{1}{\big(x,ct;\mathcal{R}(p,q)\big)_\infty}\right\} = \frac{1}{\big(x,ct;\mathcal{R}(p,q)\big)_\infty}\sum_{n=0}^{\infty}\frac{\big(x,y;\mathcal{R}(p,q)\big)_n(zt)^n}{\big(p,q;\mathcal{R}(p,q)\big)_n}.$$

Using the Cauchy $\mathcal{R}(p,q)$-binomial theorem (19), we obtain (46). Furthermore, from the $\mathcal{R}(p,q)$-Leibniz rule for the operator $D_{\mathcal{R}(p,q)}$:

$$D^n_{\mathcal{R}(p,q)}\Big\{f(x)g(x)\Big\} := \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)}\big(p\,q\big)^{-k(n-k)}D^k_{\mathcal{R}(p,q)}f\big(p^{n-k}\,x\big)D^{n-k}_{\mathcal{R}(p,q)}g\big(q^k\,x\big),$$

we have

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\left\{\frac{1}{\big(x,cs,ct;\mathcal{R}(p,q)\big)_\infty}\right\} = \sum_{n=0}^{\infty}\frac{(x,y;\mathcal{R}(p,q))_n}{(p,q;\mathcal{R}(p,q))_n}z^n\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)}\big(p\,q\big)^{-k(n-k)}$$

$$
\begin{aligned}
&\times D^k_{\mathcal{R}(p,q)}\bigg\{\frac{1}{(x,csp^{n-k};\mathcal{R}(p,q))_\infty}\bigg\}D^{n-k}_{\mathcal{R}(p,q)}\bigg\{\frac{1}{(x,ctq^k;\mathcal{R}(p,q))_\infty}\bigg\}\\
&=\frac{1}{(x,cs,ct;\mathcal{R}(p,q))_\infty}\sum_{k=0}^\infty\frac{(x,ct,y;\mathcal{R}(p,q))_k(zs)^k}{(p,q;\mathcal{R}(p,q))_k}\\
&\times\frac{\big(x,\frac{q^k}{p^k}yzt;\mathcal{R}(p,q)\big)_\infty}{(x,zt;\mathcal{R}(p,q))_\infty}\\
&=\frac{1}{(x,cs,ct;\mathcal{R}(p,q))_\infty}\frac{\big(x,yzt;\mathcal{R}(p,q)\big)_\infty}{(x,zt;\mathcal{R}(p,q))_\infty}\\
&\times\sum_{k=0}^\infty\frac{(x,ct,y;\mathcal{R}(p,q))_k(zs)^k}{\big(x,yzt;\mathcal{R}(p,q)\big)_k(p,q;\mathcal{R}(p,q))_k}\\
&=\frac{(x,yzt;\mathcal{R}(p,q))_\infty}{(x,cs,ct,zt;\mathcal{R}(p,q))_\infty}{}_2\phi_1\left(\begin{array}{c}(x,ct),(x,y)\\(x,yzt)\end{array}\bigg|\mathcal{R}(p,q);zs\right).
\end{aligned}
$$

Furthermore, in light of Leibniz's formula, the left-hand side of (48) gives

$$
\begin{aligned}
T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\bigg\{\frac{(x,cv;\mathcal{R}(p,q))_\infty}{\big(x,cs,ct;\mathcal{R}(p,q)\big)_\infty}\bigg\}&=\sum_{k=0}^\infty\frac{(x,y,v/tp^{n-k};\mathcal{R}(p,q))_k}{(p,q;\mathcal{R}(p,q))_k)_\infty}\frac{(x,cvq^k;\mathcal{R}(p,q))_\infty(zt)^k}{(x,ctp^{n-k};\mathcal{R}(p,q)}\\
&\times T_x\left(\frac{q^k}{p^k}y,z\,q^{-k};D_{\mathcal{R}(p,q)}\right)\bigg\{\frac{1}{(x,csq^k;\mathcal{R}(p,q))_\infty}\bigg\}.
\end{aligned}
\tag{50}
$$

Using the relation (46), the above sum takes the form:

$$
\begin{aligned}
T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\bigg\{\frac{(x,cv;\mathcal{R}(p,q))_\infty}{\big(x,cs,ct;\mathcal{R}(p,q)\big)_\infty}\bigg\}&=\frac{(x,yzs,cv;\mathcal{R}(p,q))_\infty}{(x,zs,ct,cs;\mathcal{R}(p,q))_\infty}\\
&\times\sum_{k=0}^\infty\frac{(x,y,cs,v/tp^{n-k};\mathcal{R}(p,q))_k}{(p,q;\mathcal{R}(p,q))_k(x,cv;\mathcal{R}(p,q))_k}\frac{(zt)^k}{(x,yzs;\mathcal{R}(p,q))_k}
\end{aligned}
$$

and the relation (48) follows, and the proof is achieved.

Remark 6. *The next particular cases of the Cauchy operator deserve attention:*

(i) The q-deformed Cauchy operator

$$T_x(y,z,D_q)=\sum_{n=0}^\infty\frac{(y;q)_n}{(q;q)_n}(z\,D_q)^n,$$

where

$$D_qf(z)=\frac{f(qz)-f(q^{-1}z)}{(q-q^{-1})z},$$

verifies the following identities:

$$T_x(y,z,D_q)(c^n)=\sum_{k=0}^n\left[\begin{array}{c}n\\k\end{array}\right]_q(x,y;q)_k\,z^kc^{n-k},$$

$$T_x(y,z,D_q)\bigg\{\frac{1}{(x,ct;q)_\infty}\bigg\}=\frac{(x,yzt;q)_\infty}{(x,ct,zt;q)_\infty},\quad |zt|<1,$$

$$T_x(y,z,D_q)\bigg\{\frac{1}{(x,cs,ct;q)_\infty}\bigg\}=\frac{(x,yzt;q)_\infty}{(x,cs,ct,zt;q)_\infty}{}_2\phi_1\left(\begin{array}{c}(x,ct),(x,y)\\(x,yzt)\end{array}\bigg|q;zs\right)$$

$$T_x(y,z,D_q)\left\{\frac{(x,cv;q)_\infty}{(x,cs,ct;q)_\infty}\right\} = \frac{(x,yzs,cv;q)_\infty}{(x,zs,ct,cs;q)_\infty}\,{}_3\phi_2\left(\begin{array}{c}(x,y),(x,cs),(x,v/t)\\(x,yzs),(x,cv)\end{array}\middle| q;zt\right),$$

where $\max\{|zs|,|zt|\}<1$.

(ii) The (p,q)*-deformed Cauchy operator :*

$$T_x(y,z,D_{p,q}) = \sum_{n=0}^{\infty}\frac{(x,y;p,q)_n}{(p,q;p,q)_n}(z\,D_{p,q})^n$$

obeys the identities:

$$T_x(y,z,D_{p,q})(c^n) = \sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}(x,y;p,q)_k\,z^k c^{n-k},$$

$$T_x(y,z,D_{p,q})\left\{\frac{1}{(x,ct;p,q)_\infty}\right\} = \frac{(x,yzt;p,q)_\infty}{(x,ct,zt;p,q)_\infty},\quad |zt|<1,$$

$$T_x(y,z,D_{p,q})\left\{\frac{1}{(x,cs,ct;p,q)_\infty}\right\} = \frac{(x,yzt;p,q)_\infty}{(x,cs,ct,zt;p,q)_\infty}\,{}_2\phi_1\left(\begin{array}{c}(x,ct),(x,y)\\(x,yzt)\end{array}\middle| p,q;zs\right)$$

$$T_x(y,z,D_{p,q})\left\{\frac{(x,cv;p,q)_\infty}{(x,cs,ct;p,q)_\infty}\right\} = \frac{(x,yzs,cv;p,q)_\infty}{(x,zs,ct,cs;p,q)_\infty}\,{}_3\phi_2\left(\begin{array}{c}(x,y),(x,cs),(x,v/t)\\(x,yzs),(x,cv)\end{array}\middle| p,q;zt\right), \tag{51}$$

where $\max\{|zs|,|zt|\}<1$.

The homogeneous Rogers- Szegö polynomials $h_n(x,y|\mathcal{R}(p,q))$ are obtained from the Cauchy operator as a limit case:

$$\lim_{c\longrightarrow 1} T\big(y/x,x,D_{\mathcal{R}(p,q)}\big)(c^n) = h_n(x,y|\mathcal{R}(p,q)). \tag{52}$$

The relation (52) allows to determine the Mehler and Rogers formulae for $h_n(x,y|\mathcal{R}(p,q))$. The particular case induced from the q-deformation was introduced by Chen and Gu [1].

Let us now highlight some identities relating to $\mathcal{R}(p,q)$-deformed Cauchy operator $T_x(y,z;D_{\mathcal{R}(p,q)})$, which are very important to investigate the extended identities for the homogeneous $\mathcal{R}(p,q)$-Rogers-Szegö polynomials $h_n(x,y|\mathcal{R}(p,q))$.

Theorem 2. *For* $n\in\mathbb{N}$*, the following identity holds.*

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\left\{\frac{c^n}{\big(x,cs,ct;\mathcal{R}(p,q)\big)_\infty}\right\} = \frac{(x,yzt;\mathcal{R}(p,q))_\infty}{(x,zt,cs,ct;\mathcal{R}(p,q))_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\left[\begin{array}{c}n\\j\end{array}\right]_{\mathcal{R}(p,q)}$$
$$\times\frac{(x,y,ct;\mathcal{R}(p,q))_{j+l}(x,cs;\mathcal{R}(p,q))_j}{(x,yzt;\mathcal{R}(p,q))_{j+l}(p,q;\mathcal{R}(p,q))_l}c^{n-j}z^{j+l}s^l, \tag{53}$$

where $\max\{|zs|,|zt|\}<1$.

Proof. Using the Leibniz formula, we have:

$$\begin{aligned}
T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\Big\{\frac{c^n}{\big(p,cs,ct;\mathcal{R}(p,q)\big)_\infty}\Big\} &= \sum_{k=0}^{\infty}\frac{\big(x,y,\mathcal{R}(p,q)\big)_k z^k}{\big(p,q;\mathcal{R}(p,q)\big)_k}D^k_{\mathcal{R}(p,q)}\Big\{\frac{c^n}{\big(p,cs,ct;\mathcal{R}(p,q)\big)_\infty}\Big\}\\
&= \sum_{k=0}^{\infty}\frac{\big(x,y,\mathcal{R}(p,q)\big)_k z^k}{\big(p,q;\mathcal{R}(p,q)\big)_k}\sum_{j=0}^{k}\left[\begin{array}{c} k\\ j\end{array}\right]_{\mathcal{R}(p,q)}(p\,q)^{-j(k-j)}\\
&\times p^{j(k-j)}D^j_{\mathcal{R}(p,q)}c^n D^{k-j}_{\mathcal{R}(p,q)}\Big\{\frac{1}{\big(p,csq^j,ctq^j;\mathcal{R}(p,q)\big)_\infty}\Big\}\\
&= \sum_{j=0}^{n}\sum_{k=0}^{\infty}\frac{\big(x,y,\mathcal{R}(p,q)\big)_{k+j} z^{k+j}}{\big(p,q;\mathcal{R}(p,q)\big)_k}\left[\begin{array}{c} n\\ j\end{array}\right]_{\mathcal{R}(p,q)}q^{-jk}c^{n-j}\\
&\times D^k_{\mathcal{R}(p,q)}\Big\{\frac{1}{\big(p,csq^j,ctq^j;\mathcal{R}(p,q)\big)_\infty}\Big\}. \qquad (54)
\end{aligned}$$

Exploiting the relation (17) and the operator (44), we get

$$\begin{aligned}
T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\Big\{\frac{c^n}{\big(x,cs,ct;\mathcal{R}(p,q)\big)_\infty}\Big\} &= \sum_{j=0}^{n}\left[\begin{array}{c} n\\ j\end{array}\right]_{\mathcal{R}(p,q)}\big(x,y,\mathcal{R}(p,q)\big)_j c^{n-j}z^j\\
&\times T_x\left(\frac{q^j}{p^j}y, zq^{-j}, D_{\mathcal{R}(p,q)}\right)\Big\{\frac{1}{\big(x,csq^j,ctq^j;\mathcal{R}(p,q)\big)_\infty}\Big\}. \qquad (55)
\end{aligned}$$

Since

$$\begin{aligned}
T_x\left(\frac{q^j}{p^j}y, zq^{-j}, D_{\mathcal{R}(p,q)}\right)\Big\{\frac{1}{\big(x,csq^j,ctq^j;\mathcal{R}(p,q)\big)_\infty}\Big\} &= \frac{(x,p^{-j}q^jyzt;\mathcal{R}(p,q))_\infty}{(x,zt,csq^j,ctq^j;\mathcal{R}(p,q))_\infty}\\
&\times {}_2\phi_1\left(\begin{array}{c}(x,yq^jp^{-j}),(x,ctq^j)\\ (x,yztp^{-j}q^j)\end{array}\Big|\mathcal{R}(p,q);zs\right),
\end{aligned}$$

after some algebra, the result follows.

Putting $n=0$ in the relation (53), we obtain (47). Setting $s=0$, we obtain the following result:

Corollary 1. *For* $n\in\mathbb{N}$,

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\Big\{\frac{c^n}{\big(x,ct;\mathcal{R}(p,q)\big)_\infty}\Big\} = \frac{(x,yzt;\mathcal{R}(p,q))_\infty}{(x,zt,ct;\mathcal{R}(p,q))_\infty}\sum_{j=0}^{n}\left[\begin{array}{c} n\\ j\end{array}\right]_{\mathcal{R}(p,q)}\frac{(x,y,ct;\mathcal{R}(p,q))_j}{(x,yzt;\mathcal{R}(p,q))_j}c^{n-j}b^j, \qquad (56)$$

where $\max\{|bt|\}<1$.

Taking $n=0$ in the relation (56), we get (45).

Theorem 3. *For* $n\in\mathbb{N}$, *the following relations hold:*

(a)

$$T_x\big(y,z,D_{\mathcal{R}(p,q)}\big)\Big\{\frac{c^n(x,cv;\mathcal{R}(p,q))_\infty}{\big(x,cs,ct;\mathcal{R}(p,q)\big)_\infty}\Big\} = \frac{(x,yzs,cv;\mathcal{R}(p,q))_\infty}{(x,zs,cs,ct;\mathcal{R}(p,q))_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\left[\begin{array}{c} n\\ j\end{array}\right]_{\mathcal{R}(p,q)}c^{n-j}z^{j+l}t^l$$

$$\times \frac{(x, ct; \mathcal{R}(p,q))_j}{(p,q;\mathcal{R}(p,q))_l} \frac{(x, v/t; \mathcal{R}(p,q))_l (x, y, cs; \mathcal{R}(p,q))_{j+l}}{(x, yzs, cv; \mathcal{R}(p,q))_{j+l}}, \qquad (57)$$

where $\max\{|zs|, |zt|\} < 1$. *Setting* $v = 0$ *in the relation* (57), *we obtain* (53). *Putting* $v = s = 0$ *in the relation* (57), *we get* (56) *and taking* $n = 0$ *in* (57), *we have* (48).

(b)

$$T_x\big(y, z, D_{\mathcal{R}(p,q)}\big)\bigg\{\frac{1}{\big(x, cs, ct, cv; \mathcal{R}(p,q)\big)_\infty}\bigg\} = \frac{(x, yzt; \mathcal{R}(p,q))_\infty}{(x, zt, cs, ct; \mathcal{R}(p,q))_\infty} \sum_{k,j=0}^{\infty} \begin{bmatrix} n \\ j \end{bmatrix}_{\mathcal{R}(p,q)}$$
$$\times \frac{(x, y, ct; \mathcal{R}(p,q))_{j+k} (x, cs; \mathcal{R}(p,q))_k}{(x, yzt; \mathcal{R}(p,q))_{j+k}}$$
$$\times \frac{(zv)^k}{(p,q;\mathcal{R}(p,q))_k} \frac{(zs)^j}{(p,q;\mathcal{R}(p,q))_j}, \qquad (58)$$

where $\max\{|zs|, |zt|\} < 1$. *Setting* $v = 0$ *in the relation* (58), *we get* (47).

(c)

$$T_x\big(y, z, D_{\mathcal{R}(p,q)}\big)\bigg\{\frac{(x, cv; \mathcal{R}(p,q))_\infty}{\big(x, cs, ct, cu; \mathcal{R}(p,q)\big)_\infty}\bigg\} = \frac{(x, yzs, cv; \mathcal{R}(p,q))_\infty}{(x, zs, cs, ct, cu; \mathcal{R}(p,q))_\infty} \sum_{k,j=0}^{\infty} \frac{(x, ct; \mathcal{R}(p,q))_k}{(p,q;\mathcal{R}(p,q))_j}$$
$$\times \frac{(x, y, cs; \mathcal{R}(p,q))_{j+k} (x, v/t; \mathcal{R}(p,q))_j}{(x, yzs, cv; \mathcal{R}(p,q))_{j+k}} \frac{(zt)^j\,(zu)^k}{(p,q;\mathcal{R}(p,q))_k}, \qquad (59)$$

where $\max\{|zs|, |zt|\} < 1$. *Putting* $v = 0$ *and replacing* s *by* t *in the realtion* (59), *we obtain* (58). *Setting* $v = u = 0$ *in* (59), *we get* (47).

(d)

$$T_x\big(y, z, D_{\mathcal{R}(p,q)}\big)\bigg\{\frac{(x, cv, cw; \mathcal{R}(p,q))_\infty}{\big(x, cs, ct, cu, cz; \mathcal{R}(p,q)\big)_\infty}\bigg\} = \frac{(x, yzs; \mathcal{R}(p,q))_\infty}{(x, zs, cs; \mathcal{R}(p,q))_\infty} \frac{(cv, cw; \mathcal{R}(p,q))_\infty}{(ct, cu, cz; \mathcal{R}(p,q))_\infty}$$
$$\times \sum_{k,j,l=0}^{\infty} \frac{(x, y, cs; \mathcal{R}(p,q))_{j+k+l} (x, ct; \mathcal{R}(p,q))_{k+l}}{(x, yzs, cv; \mathcal{R}(p,q))_{j+k+l}}$$
$$\times \frac{(x, v/t; \mathcal{R}(p,q))_j (x, w/z; \mathcal{R}(p,q))_k}{(x, cw; \mathcal{R}(p,q))_k}$$
$$\times \frac{(zt)^j}{(p,q;\mathcal{R}(p,q))_j} \frac{(zs)^k}{(p,q;\mathcal{R}(p,q))_k} \frac{(zu)^l}{(p,q;\mathcal{R}(p,q))_l}, \qquad (60)$$

where $\max\{|zs|, |zt|\} < 1$. *Putting* $w = 0$ *in the relation* (60), *we obtain* (59). *Setting* $v = w = z = 0$ *in* (60) *and changing* s *with* t *in* (60), *we obtain* (58), *and finally setting* $v = w = u = z = 0$ *in* (60), *we get* (47).

Proof. The proof is similar to that of theorem (2).

Remark 7. *(i) For* $n \in \mathbb{N}$, *we deduce the identites associated to the* q*-deformation as follows:*

$$T_x(y, z, D_q)\bigg\{\frac{c^n}{(x, cs, ct; q)_\infty}\bigg\} = \frac{(x, yzt; q)_\infty}{(x, zt, cs, ct; q)_\infty}$$

$$\times \sum_{l=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(x,y,ct;q)_{j+l}(x,cs;q)_j}{(x,yzt;q)_{j+l}(q;q)_l}\, c^{n-j} z^{j+l} s^l,$$

$$T_x(y,z,D_q)\left\{\frac{c^n}{(x,ct;q)_\infty}\right\} = \frac{(x,yzt;q)_\infty}{(x,zt,ct;q)_\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(x,z,ct;q)_j}{(x,yzt;q)_j} c^{n-j} z^j,$$

$$T_x(y,z,D_q)\left\{\frac{c^n(x,cv;q)_\infty}{(x,cs,ct;q)_\infty}\right\} = \frac{(x,yzs,cv;q)_\infty}{(x,zs,cs,ct;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q c^{n-j} z^{j+l} t^l$$
$$\times \frac{(x,y,cs;q)_{j+l}(x,ct;q)_j(x,v/t;q)_l}{(x,yzs,cv;q)_{j+l}(q;q)_l},$$

$$T_x(y,z,D_q)\left\{\frac{1}{(x,cs,ct,cv;q)_\infty}\right\} = \frac{(x,yzt;q)_\infty}{(x,zt,cs,ct;q)_\infty}\sum_{k,j=0}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(zv)^k}{(q;q)_k}\frac{(zs)^j}{(q;q)_j}$$
$$\times \frac{(x,y,ct;q)_{j+k}(x,cs;q)_k}{(x,yzt;q)_{j+k}},$$

$$T_x\big(y,z,D_q\big)\left\{\frac{(x,cv;q)_\infty}{(x,cs,ct,cu;q)_\infty}\right\} = \frac{(x,yzs,cv;q)_\infty}{(x,zs,cs,ct,cu;q)_\infty}\sum_{k,j=0}^{\infty}\frac{(zt)^j}{(q;q)_j}\frac{(zu)^k}{(q;q)_k}$$
$$\times \frac{(x,y,cs;q)_{j+k}(x,ct;q)_k(x,v/t;q)_j}{(x,yzs,cv;q)_{j+k}},$$

$$T_x(y,z,D_q)\left\{\frac{(x,cv,cw;q)_\infty}{(x,cs,ct,cu,cz;q)_\infty}\right\} = \frac{(x,yzs,cv,cw;q)_\infty}{(x,zs,cs,ct,cu,cz;q)_\infty}\sum_{k,j,l=0}^{\infty}\frac{(x,y,cs;q)_{j+k+l}(x,ct;q)_{k+l}}{(x,yzs,cv;q)_{j+k+l}}$$
$$\times \frac{(x,v/t;q)_j(x,w/z;q)_k}{(x,cw;q)_k}\frac{(zt)^j}{(q;q)_j}\frac{(zs)^k}{(q;q)_k}\frac{(zu)^l}{(q;q)_l},$$

where $\max\{|zs|,|zt|\}<1$.

(ii) For $n\in\mathbb{N}$, *we obtain the following identities corresponding to the* (p,q)*-deformation:*

$$T_x\big(y,z,D_{p,q}\big)\left\{\frac{c^n}{(x,cs,ct;p,q)_\infty}\right\} = \frac{(x,yzt;p,q)_\infty}{(x,zt,cs,ct;p,q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} c^{n-j} z^{j+l} s^l$$
$$\times \frac{(x,y,ct;p,q)_{j+l}(x,cs;p,q)_j}{(x,yzt;p,q)_{j+l}(p,q;p,q)_l},$$

$$T_x(y,z,D_{p,q})\left\{\frac{c^n}{(x,ct;p,q)_\infty}\right\} = \frac{(x,yzt;p,q)_\infty}{(x,zt,ct;p,q)_\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \frac{(x,y,ct;p,q)_j}{(x,yzt;p,q)_j} c^{n-j} z^j,$$

$$T_x\big(y,z,D_{p,q}\big)\left\{\frac{c^n(x,cv;p,q)_\infty}{(x,cs,ct;p,q)_\infty}\right\} = \frac{(x,yzs,cv;p,q)_\infty}{(x,zs,cs,ct;p,q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} c^{n-j} z^{j+l} t^l$$

$$\times \frac{(x,y,cs;p,q)_{j+l}(x,ct;p,q)_j(x,v/t;p,q)_l}{(x,yzs,cv;p,q)_{j+l}(p,q;p,q)_l},$$

$$T_x\big(y,z,D_{p,q}\big)\bigg\{\frac{1}{(x,cs,ct,cv;p,q)_\infty}\bigg\} = \frac{(x,yzt;p,q)_\infty}{(x,zt,cs,ct;p,q)_\infty}\sum_{k,j=0}^{\infty}\left[\begin{array}{c} n \\ j \end{array}\right]_{p,q}\frac{(x,y,ct;p,q)_{j+k}}{(x,yzt;p,q)_{j+k}}$$
$$\times \frac{(x,cs;p,q)_k(zv)^k}{(p,q;p,q)_k}\frac{(zs)^j}{(p,q;p,q)_j},$$

$$T_x\big(y,z,D_{p,q}\big)\bigg\{\frac{(x,cv;p,q)_\infty}{(x,cs,ct,cu;p,q)_\infty}\bigg\} = \frac{(x,yzs,cv;p,q)_\infty}{(x,zs,cs,ct,cu;p,q)_\infty}\sum_{k,j=0}^{\infty}\frac{(x,y,cs;p,q)_{j+k}}{(x,yzs,cv;p,q)_{j+k}}$$
$$\times \frac{(x,ct;p,q)_k(x,v/t;p,q)_j(zt)^j}{(p,q;p,q)_j}\frac{(zu)^k}{(p,q;p,q)_k},$$

and

$$T_x(y,z,D_{p,q})\bigg\{\frac{(x,cv,cw;p,q)_\infty}{(x,cs,ct,cu,cz;p,q)_\infty}\bigg\} = \frac{(x,yzs,cv,cw;p,q)_\infty}{(x,zs,cs,ct,cu,cz;p,q)_\infty}\sum_{k,j,l=0}^{\infty}\frac{(x,y,cs;p,q)_{j+k+l}}{(x,yzs,cv;p,q)_{j+k+l}}$$
$$\times \frac{(x,ct;p,q)_{k+l}(x,v/t;p,q)_j}{(x,cw;p,q)_k}\frac{(x,w/z;p,q)_k(zt)^j}{(p,q;p,q)_j}$$
$$\times \frac{(zs)^k}{(p,q;p,q)_k}\frac{(zu)^l}{(p,q;p,q)_l},$$

where $\max\{|zs|,|zt|\} < 1$.

6. The identities for $h_n^a(x,y;\mathcal{R}(p,q))$-polynomials

In this section, the generating function, Mehler's formula, Rogers formula, and their extended versions for the generalized homogeneous Rogers-Szegö polynomials induced from $\mathcal{R}(p,q)$-deformed algebra are discussed. Identities related to the quantum algebras known in the literature are derived as particular cases.

The generalized homogeneous Rogers-Szegö polynomials are given by the following relation:

$$h_n^a\big(x,y|\mathcal{R}(p,q)\big) = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)}\big(a,y;\mathcal{R}(p,q)\big)_k\, x^{n-k}. \tag{61}$$

Note that, taking $\mathcal{R}(x,1) = \frac{x-1}{x}$ and $a = p = 1$, we recover the homogeneous q-deformed Rogers-Szegö polynomials determined by Saad and Sukhi [21]:

$$h_n(x,y|q) = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q (y;q)_k\, x^{n-k}.$$

Using the representation of the $\mathcal{R}(p,q)$-deformed homogeneous Rogers-Szegö polynomials in the form (61) by the $\mathcal{R}(p,q)$-Cauchy operator, i.e.:

$$T_a\big(y,1,D_{\mathcal{R}(p,q)}\big)\{x^n\} = h_n^a(x,y;\mathcal{R}(p,q)), \tag{62}$$

we compute the basic identities for $h_n^a(x,y;\mathcal{R}(p,q))$ as summarized in the following theorems.

Theorem 4. *The generating function for $h_n^a(x,y;\mathcal{R}(p,q))$ gives:*

$$\sum_{n=0}^{\infty} h_n^a(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))} = \frac{(a,yt;\mathcal{R}(p,q))_\infty}{(a,t,xt;\mathcal{R}(p,q))_\infty}, \ \max\{|t|,|xt|\} < 1. \tag{63}$$

Proof. From the relations (62) and (46), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n^a(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))} &= \sum_{n=0}^{\infty} T_a(y,1;D_{\mathcal{R}(p,q)})\{x^n\}\frac{t^n}{(p,q;\mathcal{R}(p,q))_n} \\
&= T_a(y,1;D_{\mathcal{R}(p,q)})\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(p,q;\mathcal{R}(p,q))_n}\right\} \\
&= T_a(y,1;D_{\mathcal{R}(p,q)})\left\{\frac{1}{(1,xt;\mathcal{R}(p,q))_\infty}\right\} \\
&= \frac{(a,yt;\mathcal{R}(p,q))}{(a,t,xt;\mathcal{R}(p,q))}
\end{aligned}$$

and the proof is achieved.

Theorem 5. *The extended generating function for $h_n^a(x,y;\mathcal{R}(p,q))$ is determined by:*

$$\sum_{n=0}^{\infty} h_{n+k}^a(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))} = \frac{(a,yt;\mathcal{R}(p,q))_\infty}{(a,t,xt;\mathcal{R}(p,q))_\infty}\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_{\mathcal{R}(p,q)}\frac{(a,y,xt;\mathcal{R}(p,q))_j}{(a,yt;\mathcal{R}(p,q))_j}x^{k-j}, \tag{64}$$

where $\max\{|t|,|xt|\} < 1$.

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} h_{n+k}^a(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))} &= \sum_{n=0}^{\infty} T_a(y,1;D_{\mathcal{R}(p,q)})\{x^{n+k}\}\frac{t^n}{(p,q;\mathcal{R}(p,q))_n} \\
&= T_a(y,1;D_{\mathcal{R}(p,q)})\left\{x^k\sum_{n=0}^{\infty}\frac{(xt)^n}{(p,q;\mathcal{R}(p,q))_n}\right\} \\
&= T_a(y,1;D_{\mathcal{R}(p,q)})\left\{\frac{x^k}{(1,xt;\mathcal{R}(p,q))_\infty}\right\}
\end{aligned}$$

and the result follows.

Putting $k=0$ in the relation (64), we obtain the generating function (63) for the generalized homogeneous Rogers-Szegö polynomials.

Theorem 6. *The Mehler's formula for $h_n^a(x,y;\mathcal{R}(p,q))$ yields:*

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n^a(x,y;\mathcal{R}(p,q))h_n^a(u,v;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))} &= \frac{(a,yt,xvt;\mathcal{R}(p,q))_\infty}{(a,t,xt,xut;\mathcal{R}(p,q))_\infty} \\
&\times {}_3\phi_2\left(\begin{array}{c} (a,y),(a,xt),(a,v/u) \\ (a,yt),(a,xvt) \end{array}\middle|\mathcal{R}(p,q);ut\right),
\end{aligned} \tag{65}$$

where $\max\{|t|,|xt|,|xut|,|ut|\} < 1$.

Proof. We get:

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^a(x,y;\mathcal{R}(p,q))h_n^a(u,v;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))} &= \sum_{n=0}^{\infty} T_a(y,1;D_{\mathcal{R}(p,q)})\{x^n\}\\ &\times h_n^a(u,v;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\\ &= T_a(y,1;D_{\mathcal{R}(p,q)})\Big\{\frac{(a,xvt;\mathcal{R}(p,q))}{(a,xt,xut;\mathcal{R}(p,q))}\Big\}\\ &= \frac{(a,yt,xvt;\mathcal{R}(p,q))_\infty}{(a,t,xt,xut;\mathcal{R}(p,q))_\infty}\\ &\times {}_3\phi_2\left(\begin{array}{c}(a,y),(a,xt),(a,v/u)\\(a,yt),(a,xvt)\end{array}\Big|\mathcal{R}(p,q);ut\right).\end{aligned}$$

Then the proof is achieved.

Taking $u=v$ in the relation (65), we get the generating function (63) for the generalized homogeneous Rogers-Szegö polynomials $h_n^a(x,y,\mathcal{R}(p,q))$. Besides, setting $y=v=0$ in the relation (65), we get Mehler's formula (27) for the Rogers-Szegö polynomials.

Theorem 7. *The extended Mehler's formula for $h_n^a(x,y,\mathcal{R}(p,q))$ is given in the form:*

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^a(x,y;\mathcal{R}(p,q))\frac{h_{n+k}^a(u,v;\mathcal{R}(p,q))\,t^n}{(p,q;\mathcal{R}(p,q))_n} &= \frac{(a,yut,vt;\mathcal{R}(p,q))_\infty}{(a,t,ut,xut;\mathcal{R}(p,q))_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_{\mathcal{R}(p,q)} u^{k-j}(xt)^l\\ &\times\frac{(a,y/x;\mathcal{R}(p,q))_l}{(p,q;\mathcal{R}(p,q))_l}\frac{(a,v,ut;\mathcal{R}(p,q))_{j+l}(a,xut;\mathcal{R}(p,q))_j}{(a,yut,vt;\mathcal{R}(p,q))_{j+l}},\end{aligned}\tag{66}$$

where $\max\{|t|,|xt|,|ut|,|xut|\}<1$.

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^a(x,y;\mathcal{R}(p,q))h_{n+k}^a(u,v;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))_n} &= \sum_{n=0}^{\infty} h_n^a(x,y;\mathcal{R}(p,q))\\ &\times T_a(v,1;D_{\mathcal{R}(p,q)})\Big\{u^k\frac{(ut)^n}{(p,q;\mathcal{R}(p,q))_n}\Big\}\\ &= T_a(v,1;D_{\mathcal{R}(p,q)})\Big\{u^k\frac{(a,yut;\mathcal{R}(p,q))_\infty}{(a,ut,xut;\mathcal{R}(p,q))_\infty}\Big\}\\ &= \frac{(a,yut,vt;\mathcal{R}(p,q))_\infty}{(a,t,ut,xut;\mathcal{R}(p,q))_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_{\mathcal{R}(p,q)} u^{k-j}\,(xt)^l\\ &\times\frac{(a,v,ut;\mathcal{R}(p,q))_{j+l}(a,xut;\mathcal{R}(p,q))_j}{(a,yut,vt;\mathcal{R}(p,q))_{j+l}}\frac{(a,y/x;\mathcal{R}(p,q))_l}{(p,q;\mathcal{R}(p,q))_l}.\end{aligned}$$

Taking $k=0$ in the relation (66), we obtain the Mehler's formula (65).

The results for the Rogers formula for the generalized homogeneous Rogers-Szegö polynomials from $\mathcal{R}(p,q)$-deformed algebras are summarized in the following theorems.

Theorem 8. *The Rogers formula for $h_n^a(x,y,\mathcal{R}(p,q))$ is given by:*

$$\sum_{n,m=0}^{\infty} h_{n+m}^a(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m} = \frac{(a;\mathcal{R}(p,q))_\infty}{(a,s;\mathcal{R}(p,q))_\infty}\frac{(ys;\mathcal{R}(p,q))_\infty}{(xs,xt;\mathcal{R}(p,q))_\infty}$$

$$\times\ {}_2\phi_1\left(\begin{array}{c}(a,y),(a,xs)\\(a,ys)\end{array}\middle|\mathcal{R}(p,q);t\right), \qquad (67)$$

where $\max\{|t|,|s|,|xt|,|xs|\}<1$.

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h^a_{n+m}(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m} &= \sum_{n=0}^{\infty}T_a(y,1;D_{\mathcal{R}(p,q)})\\ &\times\frac{\{x^{n+m}\}t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m}\\ &=\frac{(a,ys;\mathcal{R}(p,q))}{(a,s,xs,xt;\mathcal{R}(p,q))}\\ &\times\ {}_2\phi_1\left(\begin{array}{c}(a,y),(a,xs)\\(a,ys)\end{array}\middle|\mathcal{R}(p,q);t\right).\end{aligned}$$

Setting $y=0$ and $a=1$ in the relation (67), we obtain the Rogers formula (28) for the Rogers-Szegö polynomials.

Theorem 9. *The extended Rogers formula for* $h^a_n(x,y,\mathcal{R}(p,q))$ *is determined as follows:*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h^a_{n+m+k}(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m} &= \frac{(a,yt;\mathcal{R}(p,q))_\infty}{(a,t,xt,xs;\mathcal{R}(p,q))_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_{\mathcal{R}(p,q)}\\ &\times\frac{(a,y,xt;\mathcal{R}(p,q))_{j+l}(a,xs;\mathcal{R}(p,q))_j}{(a,yt;\mathcal{R}(p,q))_{j+l}(p,q;\mathcal{R}(p,q))_l}x^{k-j}s^l \qquad (68)\end{aligned}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h^a_{n+m+k}(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}T_a(y,1;D_{\mathcal{R}(p,q)})\{x^{n+m+k}\}\\ &\times\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m}\\ &=T_a(y,1;D_{\mathcal{R}(p,q)})\Bigg\{x^k\sum_{n=0}^{\infty}\frac{(xt)^n}{(p,q;\mathcal{R}(p,q))_n}\\ &\times\sum_{m=0}^{\infty}\frac{(xs)^m}{(p,q;\mathcal{R}(p,q))_m}\Bigg\}\\ &=T_a(y,1;D_{\mathcal{R}(p,q)})\left\{\frac{x^k}{(1,xs,xt;\mathcal{R}(p,q))_\infty}\right\}\\ &=\frac{(a,yt;\mathcal{R}(p,q))_\infty}{(a,t,xt,xs;\mathcal{R}(p,q))_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_{\mathcal{R}(p,q)}\\ &\times\frac{(a,y,xt;\mathcal{R}(p,q))_{j+l}(a,xs;\mathcal{R}(p,q))_j}{(a,yt;\mathcal{R}(p,q))_{j+l}(p,q;\mathcal{R}(p,q))_l}x^{k-j}s^l\end{aligned}$$

Taking $k=0$ in the relation (68), we obtain the Rogers formula (67) for generalized homogeneous Rogers-Szegö polynomials.

Remark 8. *Particular cases are derived as follows:*

(a) *The results corresponding to the q-deformation are recovered by taking* $\mathcal{R}(x,1) = \frac{x-1}{x}$ *in [2].*

(b) *Putting* $\mathcal{R}(x,y) = \frac{x-y}{p-q}$*, we obtain the results corresponding to the Jaganathan-Srinivasa algebra, i.e. the generating function, Mehler's formula and their extended versions for* $h_n^a(x,y;p,q)$ *:*

$$\sum_{n=0}^{\infty} h_n^a(x,y;p,q)\frac{t^n}{(p,q;p,q)} = \frac{(a,yt;p,q)_\infty}{(a,t,xt;p,q)_\infty}, \max\{|t|,|xt|\} < 1,$$

and

$$\sum_{n=0}^{\infty} h_{n+k}^a(x,y;p,q)\frac{t^n}{(p,q;p,q)} = \frac{(a,yt;p,q)_\infty}{(a,t,xt;p,q)_\infty}\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_{p,q}\frac{(a,y,xt;p,q)_j}{(a,yt;p,q)_j}x^{k-j},$$

where $\max\{|t|,|xt|\} < 1$. *Furthermore,*

$$\sum_{n=0}^{\infty} h_n^a(x,y;p,q)h_n^a(u,v;p,q)\frac{t^n}{(p,q;p,q)} = \frac{(a,yt,xvt;p,q)}{(a,t,xt,xut;p,q)}{}_3\phi_2\left(\begin{array}{c} (a,y),(a,xt),(a,v/u) \\ (a,yt),(a,xvt)\end{array}\Bigg| p,q;ut\right),$$

where max$\{|t|,|xt|\} < 1,$ *and*

$$\sum_{n=0}^{\infty} h_n^a(x,y;p,q)h_{n+k}^a(u,v;p,q)\frac{t^n}{(p,q;p,q)_n} = \frac{(a,yut,vt;p,q)_\infty}{(a,t,ut,xut;p,q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_{p,q}$$
$$\times\frac{(a,v,ut;p,q)_{j+l}(a,xut;p,q)_j}{(a,yut,vt;p,q)_{j+l}}\frac{(a,y/x;p,q)_l}{(p,q;p,q)_l}u^{k-j}(xt)^l,$$

where $\max\{|t|,|xt|,|ut|,|xut|\} < 1$. *Moreover, the Rogers formula and its extended version for* $h_n^a(x,y;p,q)$ *are deduced as:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}^a(x,y;p,q)\frac{t^n}{(p,q;p,q)_n}\frac{s^m}{(p,q;p,q)_m} = \frac{(a,ys;p,q)}{(a,s,xs,xt;p,q)}$$
$$\times {}_2\phi_1\left(\begin{array}{c} (a,y),(a,xs) \\ (a,ys)\end{array}\Bigg| p,q;t\right),$$

where $\max\{|t|,|s|,|xt|,|xs|\} < 1$ *and*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m+k}^a(x,y;p,q)\frac{t^n}{(p,q;p,q)_n}\frac{s^m}{(p,q;p,q)_m} = \frac{(a,yt;p,q)_\infty}{(a,t,xt,xs;p,q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_{p,q}$$
$$\times\frac{(a,y,xt;p,q)_{j+l}(a,xs;p,q)_j}{(a,yt;p,q)_{j+l}(p,q;p,q)_l}x^{k-j}s^l.$$

7. Extended formulae for $h_n^a(x,y;\mathcal{R}(p,q))$-polynomials

In this section, we use the Cauchy operator $T_x(y,z;D_{\mathcal{R}(p,q)})$ to derive other extended identities for the homogeneous Rogers-Szegö polynomials $h_n^a(x,y;\mathcal{R}(p,q))$ with the help of the relation (57).

Theorem 10.

$$\sum_{k=0}^{\infty} h^a_{m+k}(x,y;\mathcal{R}(p,q))\frac{h^a_{n+k}(u,v;\mathcal{R}(p,q))t^k}{(p,q;\mathcal{R}(p,q))_k} = \frac{(a,vt,uyt;\mathcal{R}(p,q))_\infty}{(a,t,ut,uxt;\mathcal{R}(p,q))_\infty}$$
$$\times \sum_{l=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{(a,v,ut;\mathcal{R}(p,q))_{i+l}}{(a,uyt;\mathcal{R}(p,q))_{i+l+j}}$$
$$\times \frac{(a,uxt;\mathcal{R}(p,q))_{i+j}(a,y/x;\mathcal{R}(p,q))_l}{(a,vt;\mathcal{R}(p,q))_{i+l}(p,q;\mathcal{R}(p,q))_l}$$
$$\times (a,y;\mathcal{R}(p,q))_j x^{l+m-j}u^{n-i}(tq^j)^l, \tag{69}$$

where $\max\{|t|,|ut|,|xt|,|xut|\}<1$.

Proof. We have

$$\sum_{k=0}^{\infty} h^a_{m+k}(x,y;\mathcal{R}(p,q))h^a_{n+k}(u,v;\mathcal{R}(p,q))\frac{t^k}{(p,q;\mathcal{R}(p,q))_k} = T_a(v,1;D_{\mathcal{R}(p,q)})\{u^{n+k}\}$$
$$\times \sum_{k=0}^{\infty} h^a_{m+k}(x,y;\mathcal{R}(p,q))\frac{t^k}{(p,q;\mathcal{R}(p,q))_k}$$
$$= T_a(v,1;D_{\mathcal{R}(p,q)})\Big\{u^n\frac{(a,uyt;\mathcal{R}(p,q))_\infty}{(a,ut,uxt;\mathcal{R}(p,q))_\infty}$$
$$\times \sum_{j=0}^{m}\begin{bmatrix} m \\ j\end{bmatrix}_{\mathcal{R}(p,q)}\frac{(a,y,uxt;\mathcal{R}(p,q))_j}{(a,uyt;\mathcal{R}(p,q))_j}x^{m-j}\Big\}$$
$$= \sum_{j=0}^{m}\begin{bmatrix} m \\ j\end{bmatrix}_{\mathcal{R}(p,q)}(a,y;\mathcal{R}(p,q))_j x^{m-j}$$
$$\times T_a(v,1;D_{\mathcal{R}(p,q)})\Big\{u^n\frac{(a,uyt(q/p)^j;\mathcal{R}(p,q))_\infty}{(a,ut,uxt(q/p)^j;\mathcal{R}(p,q))_\infty}\Big\}.$$

Since

$$T_a(v,1;D_{\mathcal{R}(p,q)})\Big\{u^n\frac{(a,uyt(q/p)^j;\mathcal{R}(p,q))_\infty}{(a,ut,uxt(q/p)^j;\mathcal{R}(p,q))_\infty}\Big\} = \frac{(a,vt;\mathcal{R}(p,q))_\infty}{(a,t;\mathcal{R}(p,q))_\infty}\frac{(uyt(q/p)^j;\mathcal{R}(p,q))_\infty}{(ut,uxt(q/p)^j;\mathcal{R}(p,q))_\infty}$$
$$\times \sum_{l=0}^{\infty}\sum_{i=0}^{n}\frac{(a,v,ut;\mathcal{R}(p,q))_{i+l}}{(a,vt,uyt(q/p)^j;\mathcal{R}(p,q))_{i+j+l}}$$
$$\times \frac{(a,uxt(q/p)^j;\mathcal{R}(p,q))_i(a,y/x;\mathcal{R}(p,q))_l}{(p,q;\mathcal{R}(p,q))_{i+j+l}}u^{n-i}(xtq^j)l,$$

after computation, the result follows.

Taking $m=n=0$ in the relation (69), we obtain Mehler's formula (65) for $h^a_n(x,y;\mathcal{R}(p,q))$. Setting $m=0$ in (69), we get extended Mehler's formula (66) for $h^a_n(x,y;\mathcal{R}(p,q))$.

From the relation (58), we obtain the following identity for $h^a_n(x,y;\mathcal{R}(p,q))$.

Theorem 11.

$$\sum_{n,m,k=0}^{\infty} h^a_{n+m+k}(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m}\frac{v^k}{(p,q;\mathcal{R}(p,q))_k} = \frac{(a,yt;\mathcal{R}(p,q))_\infty}{(a,t,xt,xs,xv;\mathcal{R}(p,q))_\infty}$$

$$\times \sum_{i,j=0}^{\infty} \frac{(a,y,xt;\mathcal{R}(p,q))_{i+j}(a,xs;\mathcal{R}(p,q))_i}{(a,yt;\mathcal{R}(p,q))_{i+j}}$$
$$\times \frac{v^i}{(p,q;\mathcal{R}(p,q))_i}\frac{s^j}{(p,q;\mathcal{R}(p,q))_j}, \tag{70}$$

where $\max\{|s|,|t|,|xs|,|xt|,|xv|\}<1$.

Proof. We have

$$\sum_{n,m,k=0}^{\infty} h^a_{n+m+k}(x,y;\mathcal{R}(p,q))\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m}\frac{v^k}{(p,q;\mathcal{R}(p,q))_k}$$
$$= \sum_{n,m,k=0}^{\infty} T_a(y,1;D_{\mathcal{R}(p,q)})\{x^{n+m+k}\}\frac{t^n}{(p,q;\mathcal{R}(p,q))_n}\frac{s^m}{(p,q;\mathcal{R}(p,q))_m}\frac{v^k}{(p,q;\mathcal{R}(p,q))_k}$$
$$= T_a(y,1;D_{\mathcal{R}(p,q)})\bigg\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(p,q;\mathcal{R}(p,q))_n}\sum_{m=0}^{\infty}\frac{(xs)^m}{(p,q;\mathcal{R}(p,q))_m}\sum_{k=0}^{\infty}\frac{(xv)^k}{(p,q;\mathcal{R}(p,q))_k}\bigg\}$$
$$= T_a(y,1;D_{\mathcal{R}(p,q)})\bigg\{\frac{1}{(a,xt,xs,xv;\mathcal{R}(p,q))_\infty}\bigg\}.$$

Then, the result follows.

Putting $v=0$ in the relation (70), we obtain the Rogers formula (67) for $h^a_n(x,y;\mathcal{R}(p,q))$.

Theorem 12.

$$\sum_{m,k=0}^{\infty} h^a_{m+k}(x,y;\mathcal{R}(p,q))h^a_{n+k}(u,v;\mathcal{R}(p,q))\frac{t^m}{(p,q;\mathcal{R}(p,q))_m}\frac{s^k}{(p,q;\mathcal{R}(p,q))_k}$$
$$= \frac{(a,ys,xvs,;\mathcal{R}(p,q))_\infty}{(a,s,xt,xs,xus,;\mathcal{R}(p,q))_\infty}\sum_{l,j=0}^{\infty}\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_{\mathcal{R}(p,q)}\frac{(a,y,xs;\mathcal{R}(p,q))_{j+l}}{(a,ys;\mathcal{R}(p,q))_{j+l}}$$
$$\times\frac{(a,xus;\mathcal{R}(p,q))_{i+l}(a,v/u;\mathcal{R}(p,q))_j(a,v;\mathcal{R}(p,q))_i}{(a,xvs;\mathcal{R}(p,q))_{i+j+l}(p,q;\mathcal{R}(p,q))_j(p,q;\mathcal{R}(p,q))_l}\,u^{j+n-i}(sq^i)^j t^l, \tag{71}$$

where $\max\{|s|,|xs|,|us|,|xus|,|xt|\}<1$.

Proof. We get

$$\sum_{m,k=0}^{\infty} h^a_{m+k}(x,y;\mathcal{R}(p,q))\frac{h^a_{n+k}(u,v;\mathcal{R}(p,q))t^m}{(p,q;\mathcal{R}(p,q))_m}\frac{s^k}{(p,q;\mathcal{R}(p,q))_k} = T_a(y,1;D_{\mathcal{R}(p,q)})$$
$$\times\bigg\{\sum_{m=0}^{\infty}\frac{(xt)^m}{(p,q;\mathcal{R}(p,q))_m}\sum_{k=0}^{\infty}h^a_{n+k}(u,v;\mathcal{R}(p,q))\frac{(xs)^k}{(p,q;\mathcal{R}(p,q))_k}\bigg\}$$
$$= T_a(y,1;D_{\mathcal{R}(p,q)})\bigg\{\frac{1}{(a,xt;\mathcal{R}(p,q))_\infty}\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_{\mathcal{R}(p,q)}$$
$$\times (a,v;\mathcal{R}(p,q))_i u^{n-i}\frac{(a,xvs(q/p)^i;\mathcal{R}(p,q))_\infty}{(a,xs,xus(q/p)^i;\mathcal{R}(p,q))_\infty}\bigg\}$$
$$= \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_{\mathcal{R}(p,q)}(a,v;\mathcal{R}(p,q))_i u^{n-i}T_a(y,1;D_{\mathcal{R}(p,q)})$$

$$\times \left\{ \frac{(a, xvs(q/p)^i; \mathcal{R}(p,q))_\infty}{(a, xt, xt, xs, xus(q/p)^i; \mathcal{R}(p,q))_\infty} \right\},$$

and the result follows.

Setting $n = t = 0$ in the relation (71), we obtain the Mehler's formula (65) for $h_n^a(x, y; \mathcal{R}(p,q))$.

Theorem 13.

$$\begin{aligned}
\sum_{n,m=0}^{\infty} h_{n+m}^a(x, y;\mathcal{R}(p,q)) \frac{h_n^a(u, v; \mathcal{R}(p,q)) t^n}{(p,q; \mathcal{R}(p,q))_n} \frac{h_m^a(z, w; \mathcal{R}(p,q)) s^m}{(p,q; \mathcal{R}(p,q))_m} = \\
\frac{(a, yt, xvt, xws; \mathcal{R}(p,q))_\infty}{(a, t, xt, xut, xs, xzs, ; \mathcal{R}(p,q))_\infty} \\
\times \sum_{j,k,l=0}^{\infty} \frac{(a, y, xt; \mathcal{R}(p,q))_{j+k+l} (a, xut; \mathcal{R}(p,q))_{k+l}}{(a, yt, xvt; \mathcal{R}(p,q))_{j+l+k} (a, xws; \mathcal{R}(p,q))_k} \\
\times \frac{(a, v/u; \mathcal{R}(p,q))_j (a, xs, w/z; \mathcal{R}(p,q))_k}{(p,q; \mathcal{R}(p,q))_j (p,q; \mathcal{R}(p,q))_k (p,q; \mathcal{R}(p,q))_l} (ut)^j (sz)^k s^l,
\end{aligned} \tag{72}$$

where $\max\{|t|, |xt|, |ut|, |xut|, |xs|, |xzs|\} < 1$.

Proof. We have

$$\begin{aligned}
\sum_{n,m=0}^{\infty} h_{n+m}^a(x, y; \mathcal{R}(p,q)) \frac{h_n^a(u, v; \mathcal{R}(p,q)) t^n}{(p,q; \mathcal{R}(p,q))_n} \frac{h_m^a(z, w; \mathcal{R}(p,q)) s^m}{(p,q; \mathcal{R}(p,q))_m} &= T_a\big(y, 1; D_{\mathcal{R}(p,q)}\big) \\
&\times \left\{ \sum_{n=0}^{\infty} h_n^a(u, v; \mathcal{R}(p,q)) \frac{(xt)^n}{(p,q; \mathcal{R}(p,q))_n} \right. \\
&\times \left. \sum_{m,k=0}^{\infty} h_m^a(z, w; \mathcal{R}(p,q)) \frac{(xs)^m}{(p,q; \mathcal{R}(p,q))_m} \right\} \\
&= T_a\big(y, 1; D_{\mathcal{R}(p,q)}\big) \left\{ \frac{(a, xvt, xws; \mathcal{R}(p,q))_\infty}{(a, xt, xut, xs, xzs; \mathcal{R}(p,q))_\infty} \right\}.
\end{aligned}$$

Taking $s = 0$ in the relation (72), we obtain the Mehler's formula (65) for $h_n^a(x, y; \mathcal{R}(p,q))$.

Theorem 14.

$$\begin{aligned}
\sum_{n,m,k=0}^{\infty} h_{n+k}^a(x, y; \mathcal{R}(p,q)) \frac{t^n}{(p,q; \mathcal{R}(p,q))_n} h_{m+k}^a(u, v; \mathcal{R}(p,q)) \frac{s^m}{(p,q; \mathcal{R}(p,q))_m} \frac{w^k}{(p,q; \mathcal{R}(p,q))_k} \\
= \frac{(a, yw, xvw; \mathcal{R}(p,q))_\infty}{(a, w, us, xt, xw, xuw; \mathcal{R}(p,q))_\infty} \\
\times \sum_{i,j,l=0}^{\infty} \frac{(a, y, xw; \mathcal{R}(p,q))_{i+l} (a, xuw; \mathcal{R}(p,q))_{i+j}}{(a, yw; \mathcal{R}(p,q))_{i+l} (a, xvw; \mathcal{R}(p,q))_{i+j+l}} \\
\times \frac{(a, v/u; \mathcal{R}(p,q))_l (a, v; \mathcal{R}(p,q))_j}{(p,q; \mathcal{R}(p,q))_j (p,q; \mathcal{R}(p,q))_i (p,q; \mathcal{R}(p,q))_l} t^i\, s^j\, (uwq^j)^l,
\end{aligned} \tag{73}$$

where $\max\{|s|, |w|, |us|, |xw|, |uw|, |xt|, |xuw|\} < 1$.

Setting $t = s = 0$ in the relation (73), we obtain the Mehler's formula (65) for $h_n^a(x, y; \mathcal{R}(p,q))$.

Remark 9. *Taking $\mathcal{R}(x,y)=\frac{x-y}{p-q}$, we obtain the results associated to the Jaganathan-Srinivasa algebra. Other extended formulae for $h_n^a(x,y;p,q)$ are provided by :*

$$\begin{aligned}\sum_{k=0}^{\infty} h^a_{m+k}(x,y;p,q)h^a_{n+k}(u,v;p,q)\frac{t^k}{(p,q;p,q)_k} &= \frac{(a,vt,uyt;p,q)_\infty}{(a,t,ut,uxt;p,q)_\infty}\\ &\times \sum_{l=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{(a,v,ut;p,q)_{i+l}}{(a,uyt;p,q)_{i+l+j}}\\ &\times \frac{(a,uxt;p,q)_{i+j}(a,y/x;p,q)_l}{(a,vt;p,q)_{i+l}(p,q;p,q)_l}\\ &\times (a,y;p,q)_j x^{l+m-j}u^{n-i}(tq^j)^l,\end{aligned}$$

where $\max\{|t|,|ut|,|xt|,|xut|\}<1,$ *and*

$$\begin{aligned}\sum_{n,m,k=0}^{\infty} h^a_{n+m+k}(x,y;p,q)\frac{t^n}{(p,q;p,q)_n}\frac{s^m}{(p,q;p,q)_m}\frac{v^k}{(p,q;p,q)_k} &= \frac{(a,yt;p,q)_\infty}{(a,t,xt,xs,xv;p,q)_\infty}\sum_{i,j=0}^{\infty}\frac{(a,xs;p,q)_i}{(a,yt;p,q)_{i+j}}\\ &\times \frac{(a,y,xt;p,q)_{i+j}\,v^i}{(p,q;p,q)_i}\frac{s^j}{(p,q;p,q)_j},\end{aligned}$$

where $\max\{|s|,|t|,|xs|,|xt|,|xv|\}<1.$

Finally,

$$\begin{aligned}\sum_{m,k=0}^{\infty} h^a_{m+k}(x,y;p,q)h^a_{n+k}(u,v;p,q)\frac{t^m}{(p,q;p,q)_m}\frac{s^k}{(p,q;p,q)_k} &= \frac{(a,ys,xvs,;p,q)_\infty}{(a,s,xt,xs,xus,;p,q)_\infty}\sum_{l,j=0}^{\infty}\sum_{i=0}^{n}\begin{bmatrix} n\\ i\end{bmatrix}_{p,q}\\ &\times \frac{(a,y,xs;p,q)_{j+l}}{(a,ys;p,q)_{j+l}}u^{j+n-i}(sq^i)^j\,t^l\\ &\times \frac{(a,xus;p,q)_{i+l}(a,v/u;p,q)_j(a,v;p,q)_i}{(a,xvs;p,q)_{i+j+l}(p,q;p,q)_j(p,q;p,q)_l},\end{aligned}$$

where $\max\{|s|,|xs|,|us|,|xus|,|xt|\}<1,$

$$\begin{aligned}\sum_{n,m=0}^{\infty} h^a_{n+m}(x,y;p,q)\frac{h^a_n(u,v;p,q)t^n}{(p,q;p,q)_n}\frac{h^a_m(z,w;p,q)s^m}{(p,q;p,q)_m} &= \frac{(a,yt,xvt,xws;p,q)_\infty}{(a,t,xt,xut,xs,xzs;p,q)_\infty}\\ &\times \sum_{j,k,l=0}^{\infty}\frac{(a,y,xt;p,q)_{j+k+l}(a,xut;p,q)_{k+l}}{(a,yt,xvt;p,q)_{j+l+k}(a,xws;p,q)_k}\\ &\times \frac{(a,v/u;p,q)_j(a,xs,w/z;p,q)_k}{(p,q;p,q)_j(p,q;p,q)_k(p,q;p,q)_l}\,(ut)^j(sz)^k s^l,\end{aligned}$$

where $\max\{|t|,|xt|,|ut|,|xut|,|xs|,|xzs|\}<1,$ *and*

$$\begin{aligned}\sum_{n,m,k=0}^{\infty} h^a_{n+k}(x,y;p,q)h^a_{m+k}(u,v;p,q)\frac{t^n}{(p,q;p,q)_n}\frac{s^m}{(p,q;p,q)_m}\frac{w^k}{(p,q;p,q)_k} &= \frac{(a,yw,xvw;p,q)_\infty}{(a,w,us,xt,xw,xuw;p,q)_\infty}\\ &\times \sum_{i,j,l=0}^{\infty}\frac{(a,y,xw;p,q)_{i+l}(a,xuw;p,q)_{i+j}}{(a,yw;p,q)_{i+l}(a,xvw;p,q)_{i+j+l}}\\ &\times \frac{(a,v/u;p,q)_l(a,v;p,q)_j}{(p,q;p,q)_j(p,q;p,q)_i(p,q;p,q)_l}\,t^i\,s^j\,(uwq^j)^l,\end{aligned}$$

where $\max\{|s|,|w|,|us|,|xw|,|uw|,|xt|,|xuw|\}<1.$

8. Concluding remarks

The Cauchy operators and identities in the framework of the $\mathcal{R}(p,q)$-deformed quantum algebras have been constructed. Moreover, the generating function, Mehler and Rogers formulae, and their extended identities for the homogeneous Rogers-Szegö polynomials have been computed and discussed. Particular identities from known quantum algebras have been deduced.

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Conflict of Interest

This work does not have any conflicts of interest.

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