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A local Picard iteration method with geometric stability for multispecies Lotka-Volterra models

Asghar Ghorbani*[1] | Praveen Agarwal[2,3]

[1]Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
[2]Department of Mathematics, Anand International College of Engineering, Jaipur, India
[3]International Centre for Basic and Applied Sciences, Jaipur, India

Correspondence
*Asghar Ghorbani, Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran.
Email: aghorbani@um.ac.ir

Summary

In this article, an efficient modification of the Picard iteration method for solving the multispecies Lotka-Volterra modelss (MLVMs) is firstly proposed. Then the convergence and stability of the modified method are discussed. In order to indicate the efficiency of the modified method, three cases of the MLVMs are given. The obtained results evidence that the developed approach is a useful semi-analytical scheme for the solution of the MLVMs.

KEYWORDS:
Local Picard method, Geometric stability, Lotka-Volterra Model

1 | INTRODUCTION

The multispecies Lotka-Volterra models (MLVMs) model the dynamic behaviour of an arbitrary number of competitors [1]. These equations originally have been formulated for describing the time history of a biological system. But they arise in different fields such as physical, simultaneous chemical, control and biological problems (see, for example, [2,3]). The one-predator one-prey Lotka-Volterra model is one of the most popular ones to evidence a simple nonlinear control system. It is noted that finding precise results of the LVMs can become a hard duty when the number of species is large [4].

Here, we investigate the semi-analytic solution of the MLVMs of the below type [5,6]

$$\frac{dP_i}{dt} = P_i\left[b_i + \sum_{j=1}^{m} a_{ij}P_j\right], \qquad i = 1, 2, \cdots, m, \tag{1}$$

subject to the conditions

$$P_i(0) = c_i, \qquad i = 1, 2, \cdots, m, \tag{2}$$

by using an effective improvement of the Picard method (PM). In equations (1) and (2), a_{ij}, b_i and c_i are real constants.

These equations are solved by means of approximate analytical methods such as the homotopy analysis method [7], homotopy perturbation method [8] and the variational iteration method [9]. However, the convergence region of the corresponding results is very small. For this reason, here, a new version of the PM is proposed to accurately simulate (1). The idea is inspired by the one given in [10]. The main differences with respect to that paper are the other approach, the convergence discussion and the newly geometric stability analysis.

2 | DESCRIPTION OF THE NEW METHOD

The Picard approach is described to solve the MLVMs of (1). First we consider its so-called integral associated equation as follows:

$$P_i(t) = c_i + \int_0^t P_i(s)\left(b_i + \sum_{j=1}^{m} a_{ij} P_j(s)\right) ds, \qquad i = 1, 2, \cdots, m. \tag{3}$$

The Picard iterative process consists of constructing a sequence of functions, which will get closer and closer to the desired/exact solution, i.e.,:

$$P_{i,n+1}(t) = c_i + \int_0^t P_{i,n}(s)\left(b_i + \sum_{j=1}^{m} a_{ij} P_{j,n}(s)\right) ds, \qquad i = 1, 2, \cdots, m, \tag{4}$$

where $P_{i,0}(t)$, $i = 1, 2, \cdots, m$ is the initial guess. Consequently, the exact solution can be gained by the following limit (the proof could be found in most all differential equations textbooks):

$$P_i(t) = \lim_{n\to\infty} P_{i,n}(t), \qquad i = 1, 2, \cdots, m. \tag{5}$$

As we found, in general, the use of the above PM for solving the MLVMs may produce waste calculations. The unneeded calculations can or can not conduce to rapid convergence. To fully remove these computations, we could apply the Taylor series around $t = 0$ to the integrand of the iterative procedure(1) (with this assumption that the integrand is an analytic function in each of iterations of the Picard process). So, we will obtain a new version of the PM for solving (1) as below:

$$\begin{cases} P_{i,0}(t) = c_i, \\ P_{i,1}(t) = P_{i,0}(t) + \int_0^t G_{i,0}(s) ds, \\ P_{i,n+1}(t) = P_{i,n}(t) + \int_0^t [G_{i,n}(s) - G_{i,n-1}(s)] ds, \qquad n \geq 1, \end{cases} \tag{6}$$

where $i = 1, 2, \cdots, m$ and

$$P_{i,n}(s)\left(b_i + \sum_{j=1}^{m} a_{ij} P_{j,n}(s)\right) = G_{i,n}(s) + O\left(s^{n+1}\right). \tag{7}$$

Now, the procedure (6) could override calculating all the repeated and unneeded terms.

In practice, by applying the algorithm (6), we get a truncated series approximation, which is valid in a short interval t. Here, we divide the interval $I = [0, T]$ to the sub-intervals $I_k = [t_{k+1}, t_k]$ with $h_k = t_{k+1} - t_k$, $k = 0, 1, 2, \cdots, M-1$. Therefore, we could construct the following n_{k+1}-order piecewise approximation $P^{k+1}_{i,n_{k+1}}$ on I_k for (1) (we call it the local PM):

$$\begin{cases} P^{k+1}_{i,n+1}(t) = P^{k+1}_{i,n}(t) + \int_{t_k}^t [G^{k+1}_{i,n}(s) - G^{k+1}_{i,n-1}(s)] ds, \\ P^{k+1}_{i,0}(t) = P^{k}_{i,n_k}(t_k) = c_{i,k}, \quad n = 0, 1, \cdots, n_{k+1} - 1, \\ P^{k+1}_{i,n}(s)\left(b_i + \sum_{j=1}^{m} a_{ij} P^{k+1}_{j,n}(s)\right) = G^{k+1}_{i,n}(s) + O\left((s - t_k)^{n+1}\right), \end{cases} \tag{8}$$

where $P^0_{i,n_0}(0) = P_i(0) = c_i = c_{i,0}$ and $G^{k+1}_{i,-1}(s) \equiv 0$. Now, the n_{k+1}-order local PM approximation $P^{k+1}_{i,n_{k+1}}$ can be obtained on $[t_{k+1}, t_k]$ as well on the entire interval $[0, T]$. In the light of the above, the n_{k+1}-order semi-analytical local PM solution for (1) can be expressed as:

$$P^{k+1}_{i,n_{k+1}}(t) = \sum_{r=0}^{n_{k+1}} \frac{\gamma^k_{i,r}}{r!}(t - t_k)^r + O[(t - t_k)^{n_{k+1}+1}], \quad t \in I_k, \tag{9}$$

with an error of the order $h^{n_{k+1}+1}$ per step.

3 | CONVERGENCE OF THE METHOD

Here, the convergence of the methods (6) and (8) are discussed. We, therefore, have the following theorem.

Theorem 1. Assume that the sequences $\{G_{i,n}\}_0^\infty$ $(i = 1, ..., m)$ defined by (9) are uniformly convergent to the right hand (1) on an interval J such that $[0, T] \subseteq J$. Then $\{P_{i,n}\}$ $(i = 1, ..., m)$ produced by (8) is the exact solution of (1).

Proof. Put

$$f_i\left(t, P_1, ..., P_m\right) = P_i \left[b_i + \sum_{j=1}^{m} a_{ij} P_j\right], \qquad i = 1, 2, \cdots, m. \tag{10}$$

First, we prove the relation

$$P_{i,n} = P_{i,1} + \int_0^t [G_{i,n-1}(s) - G_{i,0}(s)]ds, \tag{11}$$

by using the induction on n for $n > 1$. We have from (8) that

$$P_{i,2} = P_{i,1} + \int_0^t [G_{i,1}(s) - G_{i,0}(s)]ds. \tag{12}$$

Now, assume that

$$P_{i,n-1} = P_{i,1} + \int_0^t [G_{i,n-2}(s) - G_{i,0}(s)]ds, \tag{13}$$

holds. Then, according to (8) and (13), we will have

$$\begin{aligned} P_{i,n} &= P_{i,n-1} + \int_0^t [G_{i,n-1}(s) - G_{i,n-2}(s)]ds \\ &= P_{i,1} + \int_0^t [G_{i,n-2}(s) - G_{i,0}(s)]ds + \int_0^t [G_{i,n-1}(s) - G_{i,n-2}(s)]ds \\ &= P_{i,1} + \int_0^t [G_{i,n-1}(s) - G_{i,0}(s)]ds. \end{aligned}$$

Since by the above assumption, the sequence $G_{i,n}$ is uniformly convergent to f_i, then according to [11], we have

$$\begin{aligned} \lim_{n\to\infty} P_{i,n} &= P_{i,1} + \lim_{n\to\infty} \int_0^t [G_{i,n-1}(s) - G_{i,0}(s)]ds \\ &= P_{i,1} + \int_0^t \lim_{n\to\infty} [G_{i,n-1}(s) - G_{i,0}(s)]ds \\ &= P_{i,1} + \int_0^t \lim_{n\to\infty} G_{i,n-1}(s)ds - \int_0^t G_{i,0}(s)]ds \\ &= c_i + \int_0^t f_i\left(s, P_1, ..., P_m\right) ds \\ &= P_i, \end{aligned}$$

which is the solution of the system (1). $\square$

It is noted that one can give the proof of the convergence for the procedure (8) in a similar way.

4 | GEOMETRIC STABILITY

In the next section, we will implement the local PM using a fixed step size h. Thus the convergence of the numerical result of the proposed algorithm is dependent on choosing the value of the step size h.

If we apply the above local PM (h, n_k) to the test equation $P' = \alpha P,\ P(t_0) = c_0$, we will get the following iteration scheme:

$$P_{k,n_k} = \Big[R_h(\omega)\Big]^k c_0, \qquad \omega = \alpha h, \tag{14}$$

where

$$R_h(\omega) = \sum_{j=0}^{n_k} \frac{\omega^j}{j!}, \tag{15}$$

is called the stability function [12], which depends on h. What is our interest is the set of all h such that (14) is numerically stable.

The stability region of a local PM for given h is defined as

$$\mathcal{S}_h := \{\omega \in \mathbb{C} \ : \ \mid R_h(\omega) \mid \le 1\}. \tag{16}$$

Except for some special cases [13], in general, the stability region usually is difficult to characterize.

It could be observed that the output of the above local PM for the fixed order and t (i.e., $c_{i,k}$) is always as a finite series in h. As will be seen later in this paper, a practical scheme to detect the valid segment h is this that we plot the curve of $c_{i,k}$ with respect to h. In case of series convergence, there will be a segment in its figure. We call it the valid segment and display it with $\mathcal{S}_h$. Now, if one chooses the value h in $\mathcal{S}_h$, then the local PM will be numerically stable.

5 | NUMERICAL IMPLEMENTATION

In order to exhibit the efficiency and accuracy of the present local PM, we will test three modeling cases of the MLVMs.

One species: For $m = 1$, we have the following one species [9,8]:

$$\frac{dP_1}{dt} = P_1(b + aP_1), \quad a < 0, \quad b > 0, \quad P_1(0) > 0, \tag{17}$$

where $a = -3$ and $b = 1$. According to (8), we get the following PM approximations in the sub-intervals I_k:

$$\begin{aligned} P_{1,1}^{k+1}(t) &= c_{1,k} - c_{1,k}(3c_{1,k} - 1)(t - t_k), \\ P_{1,2}^{k+1}(t) &= P_{1,1}^{k+1}(t) + \tfrac{1}{2}c_{1,k}(6c_{1,k} - 1)(3c_{1,k} - 1)(t - t_k)^2, \\ &\vdots \end{aligned} \tag{18}$$

where $c_{1,0} = P_1(0) = 0.1$ and

$$c_{1,k+1} = P_{1,n_{k+1}}^{k+1}(t_{k+1}), \quad k = 0, 1, \cdots, M-1, \quad n_{k+1} = 1, 2, \cdots. \tag{19}$$

In order to get the valid segment of the second order PM, i.e., $n_{k+1} = 2,\ k = 0, 1, \cdots, M-1$, the curves of $c_{1,100}$ and $c_{1,1000}$ are plotted in Figures 1 and 2 . In view of those curves, it is simple to see the stable segment of (19) i.e., $h \in (0, 2.2)$.

The absolute error of the second order PM solution for $h = 0.1$ and $M = 1000$ ($E_2(t) = |P_{1,PM}(t) - P_{1,RK78}|$) can be observed in Figure 3 . Here `RK78` is the optimal Maple solver of ODEs.

Moreover, the numerical outputs of the second-order PM for different step size h can be observed in Table 1 .

Two species: For $m = 2$, (1) presents the competing for a common ecological niche [9,8]:

$$\begin{cases} \frac{dP_1}{dt} = P_1(b_1 + a_{11}P_1 + a_{12}P_2), \\ \frac{dP_2}{dt} = P_2(b_2 + a_{21}P_1 + a_{22}P_2), \end{cases} \tag{20}$$

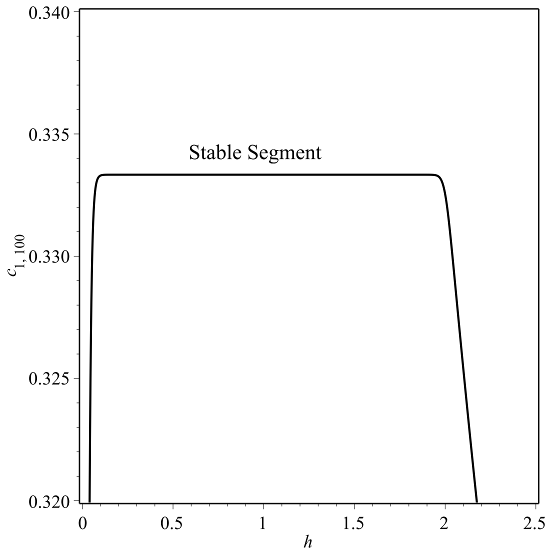

FIGURE 1 Stable segment of the second-order PM for $k = 100$ (i.e., $c_{1,100}$) for Case 1.

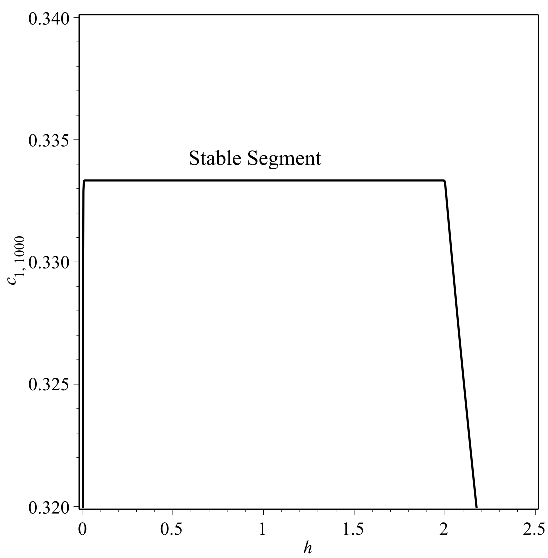

FIGURE 2 Stable segment of the second order PM for $k = 1000$ (i.e., $c_{1,1000}$) for Case 1.

where $P_1(0) = 4$ and $P_2(0) = 10$, and the constants are selected as $a_{11} = -0.0014$, $a_{12} = -0.0012$, $a_{21} = -0.0009$, $a_{22} = -0.001$, $b_1 = 0.1$ and $b_2 = 0.08$. Proceeding as before, we can gain the following PM approximations in the sub-intervals I_k:

$$\begin{cases} P_{1,1}^{k+1}(t) = c_{1,k} + \frac{1}{5000} c_{1,k}(-500 + 7c_{1,k} + 6c_{2,k})(t - t_k), \\ P_{2,1}^{k+1}(t) = c_{2,k} + \frac{1}{10000} c_{2,k}(-800 + 9c_{1,k} + 10c_{2,k})(t - t_k), \end{cases} \tag{21}$$

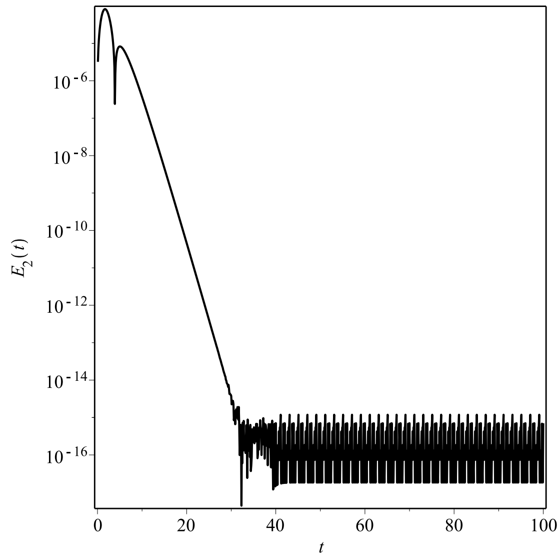

FIGURE 3 The absolute error of the second order PM with $h = 0.1$ for Case 1.

TABLE 1 The numerical results obtained from solving Eq. (17) using the 2th-order PM for $k = 1000$ and several big step sizes.

Step Size	T	**Absolute Error**
$h = 0.5$	500	4×10^{-16}
$h = 0.75$	750	4×10^{-16}
$h = 1$	1000	5×10^{-16}
$h = 1.5$	1500	6×10^{-16}
$h = 2$	2000	8×10^{-5}
$h = 2.1$	2100	8×10^{-3}
$h = 2.3$	2500	$Float(undefined)$

$$\begin{cases} P_{1,2}^{k+1}(t) = P_{1,1}^{k+1}(t) + \frac{1}{5\times10^7} c_{1,k}(98c_{1,k}^2 + 153c_{1,k}c_{2,k} \\ \qquad +66c_{2,k}^2 - 10500c_{1,k} - 8400c_{2,k} + 250000)(t - t_k)^2, \\ P_{2,2}^{k+1}(t) = P_{2,1}^{k+1}(t) + \frac{1}{2\times10^8} c_{2,k}(207c_{1,k}^2 + 378c_{1,k}c_{2,k} \\ \qquad +200c_{2,k}^2 - 23400c_{1,k} - 24000c_{2,k} + 640000)(t - t_k)^2, \\ \vdots \end{cases} \tag{22}$$

To determine the stable segment of the second order PM solution, i.e., $n_{k+1} = 2,\ k = 0, 1, \cdots, M - 1$, here we plot the curves of $c_{1,5000}$ and $c_{2,5000}$ w.r.t h, as shown in Figures 4 and 5 . These curves indicate the valid segment for the above parameters and conditions, i.e., $h \in (0, 25)$.

In the view of Figures 4 -5 and the stability region $h \in (0, 25)$, the numerical results of the second-order PM when $h = 24$ (Convergent) and $h = 25$ (Divergent) can be observed in Figures 6 and 7 , respectively.

Three species: For $m = 3$, we have the follwong three species of the Lotka-Volterra equation[9]:

$$\begin{cases} \frac{dP_1}{dt} = P_1(1 - P_1 - aP_2 - bP_3), \\ \frac{dP_2}{dt} = P_2(1 - bP_1 - P_2 - aP_3), \\ \frac{dP_3}{dt} = P_3(1 - aP_1 - bP_2 - P_3), \end{cases} \tag{23}$$

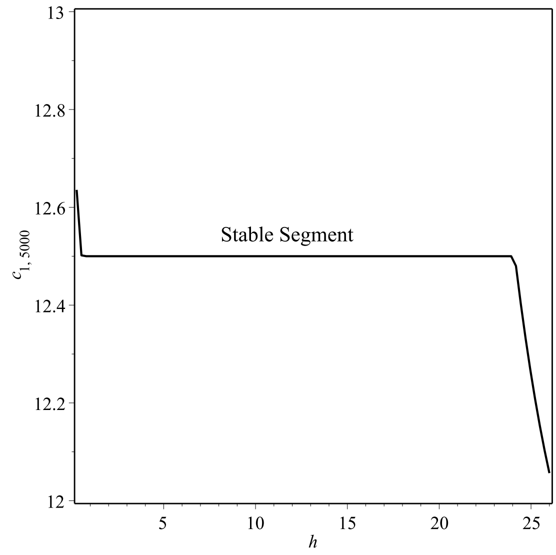

FIGURE 4 The stable segment of the 2th-order PM for $k = 5000$ (i.e., $c_{1,5000}$) for solving the equation (20) of Case 2.

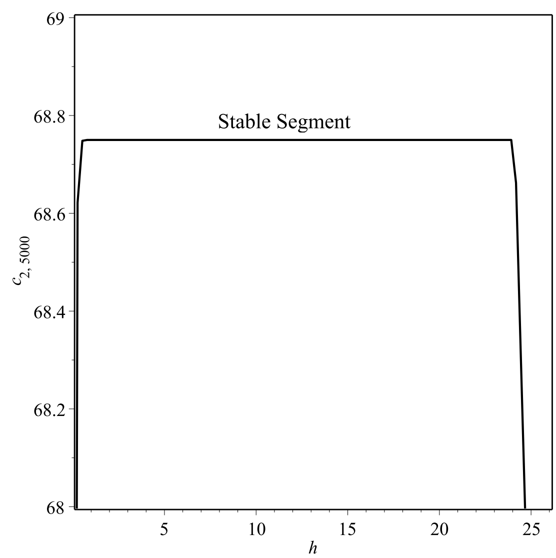

FIGURE 5 The stable segment of the 2th-order PM for $k = 5000$ (i.e., $c_{2,5000}$) for solving the equation (20) of Case 2.

where $P_1(0) = 0.2$, $P_2(0) = 0.3$, $P_3(0) = 0.5$, $a = 0.1$ and $b = 0.1$,[9,8].

Figure 8 discloses the stable segment of the second order PM, i.e., $h \in (0, 2]$.

Moreover, the numerical outputs of the second order PM for different step size h can be seen in Table 2 (the errors were reported in end points).

In closing our analysis, we point out that several modeling cases of the MLVEs were tested by using the local PM algorithm proposed in this paper, and the obtained results have shown satisfactory performance.

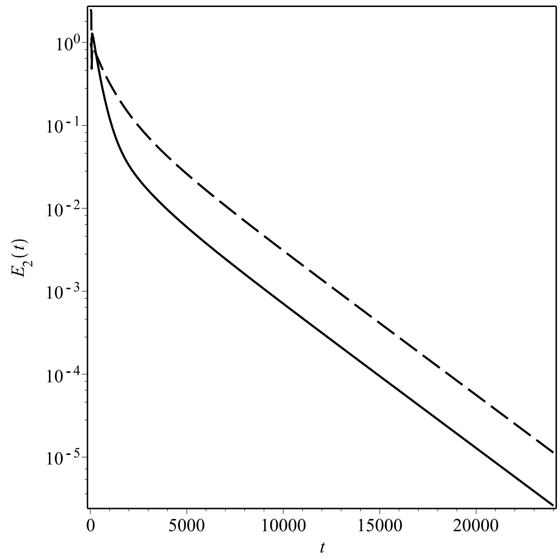

FIGURE 6 The convergent result of the 2th-order PM with $h = 24 \in \mathcal{S}_h$ for solving the equation (20) of Case 2.

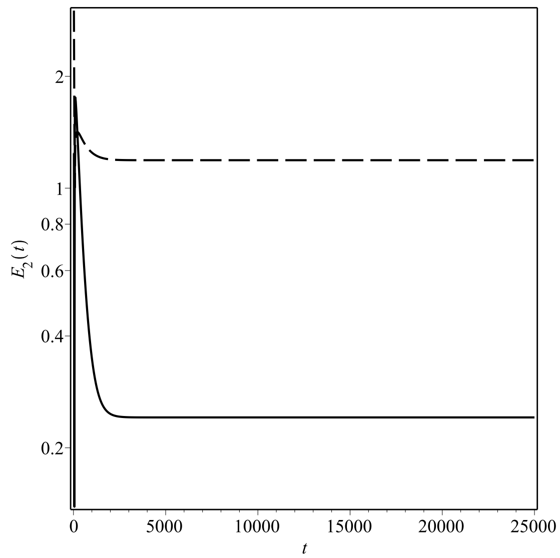

FIGURE 7 The divergent result of the 2th-order PM with $h = 25 \notin \mathcal{S}_h$ for solving the equation (20) of Case 2.

6 | CONCLUSIONS

According to the localazition, in this article, a local Picard method was proposed to simulate the solution of the multispecies Lotka-Volterra models. Also the convergence of the developed approach was discussed. Moreover, a geometric scheme was presented for discovering the so-called stability region of the method. Finally, three cases of the MLVMs were given to illustrate the efficiency and accuracy of the method. The numerical results were satisfactory. The present procedure can be further applied for other nonlinear population models.

Conflict of interest

The authors declare no potential conflict of interests.

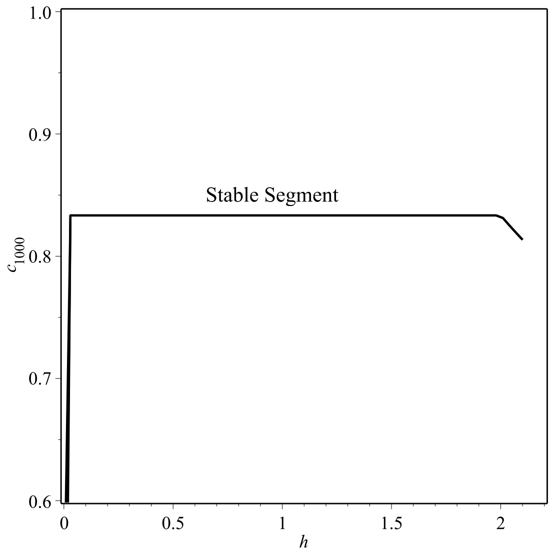

FIGURE 8 The stability region of the 2th-order PM for $k = 1000$ (i.e., $c_{1,1000}$, $c_{2,1000}$, $c_{3,1000}$) for solving the equation (23) of Case 3.

TABLE 2 The numerical results obtained from solving (23) using the 2th-order PM for $k = 1000$ and several big step sizes.

Step Size	T	Err. of P_1	Err. of P_2	Err. of P_3
$h = 0.5$	500	9.6×10^{-16}	9.6×10^{-16}	9.6×10^{-16}
$h = 1$	1000	3.7×10^{-17}	3.7×10^{-17}	3.7×10^{-17}
$h = 1.5$	1500	1.1×10^{-15}	1.1×10^{-15}	1.1×10^{-15}
$h = 2$	2000	2.1×10^{-4}	2.1×10^{-4}	2.1×10^{-4}
$h = 2.25$	2250	$Float(undefined)$	$Float(undefined)$	$Float(undefined)$

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