Stability for semilinear wave equation of variable coefficients with acoustic boundary conditions and general boundary memory feedback

Xiaomin CAO* Shugen CHAI

School of Mathematical Sciences, Shanxi university, Taiyuan,Shanxi, P.R. CHINA

Abstract This paper is concerned with the study of decay rates of the energy associated to a semilinear wave equation with variable coefficients in a smooth domain, subject to acoustic boundary conditions and dissipative boundary memory feedback, where a general Borel measure is involved. Under quite weak assumptions on this measure, we show the decay rates of the semilinear system are described by solutions to a first order nonlinear, dissipative ODE, which recovering and extending some of the results from the literature. The method we used are energy multiplier methods, geometric analysis and a standard integral inequality.

Keywords Stability, semilinear hyperbolic equation, Variable coefficients, Borel measure, Acoustic boundary conditions

AMS(MOS) subject classifications 35B35, 35L20, 35L05, 35L71

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$*$ Corresponding author.

E-mail addresses: caoxm@sxu.edu.cn (X.M. CAO), sgchai@sxu.edu.cn (S.G. CHAI).

1 Introduction

Let $\Omega \subset I\!R^n (n \geq 2)$ be an open, bounded, connected set having a boundary Γ of class C^2. We are concerned with the following initial boundary value problem

$$\begin{cases} u_{tt} - \operatorname{div} A(x)\nabla u + \rho(u_t) = 0, \quad \text{in} \quad \Omega \times (0, +\infty) \\ u = 0, \quad \text{on} \quad \Gamma_0 \times (0, +\infty) \\ -u_t = f(x)z_t + k(x)z, \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}} + F = h(x)z_t, \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad x \in \Omega \\ z(x, 0) = z_0(x), \quad x \in \Gamma_1, \end{cases} \tag{1.1}$$

where $A(x)$ be a symmetric and positive matrix for each $x \in R^n$ with smooth elements $a_{ij}(x), i, j = 1, 2, \cdots, n$, $\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \sum\limits_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \nu_i, \nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is the unit normal of Γ pointing toward the exterior of Ω, (Γ_0, Γ_1) is a partition of Γ; F is the feedback function which may depend on the state (u, u_t), position x and time t; ρ, f, k and h are given functions.

Let $I\!R^n$ have the usual topology and $x = (x_1, x_2, \cdots, x_n)$ be the natural coordinate system in $I\!R^n$. We define

$$\mathrm{g} = \mathrm{A}^{-1}(\mathrm{x}) \quad \text{for} \quad \mathrm{x} \in \mathrm{R}^{\mathrm{n}},$$

as a Riemannian metric on $I\!R^n$ and consider the couple $(I\!R^n, \mathrm{g})$ as a Riemannian manifold. Denote by $\langle \cdot, \cdot \rangle$ the Euclidean product of $I\!R^n$. Denote by $\mathrm{g} = \langle \cdot, \cdot \rangle_{\mathrm{g}}$ the inner product and by D the covariant differential of the metric g, respectively. Then

$$\langle X, Y \rangle_{\mathrm{g}} = \langle A^{-1}(x)X, Y \rangle \quad \text{for} \quad X, Y \in I\!R^n_x, x \in I\!R^n,$$

and the covariant differential DH of a vector field H is a tensor field of rank 2, defined by

$$DH(X, Y) = \mathrm{g}(\mathrm{D_Y H}, \mathrm{X}), \quad \text{for} \quad \mathrm{X}, \mathrm{Y} \in \mathrm{R}^{\mathrm{n}}_{\mathrm{x}}, \mathrm{x} \in \mathrm{R}^{\mathrm{n}}.$$

Let H be a vector field on Riemannian manifold $(I\!R^n, \mathrm{g})$ such that

$$DH(X, X) \geq \alpha |X|^2_{\mathrm{g}}, \quad \forall X \in I\!R^n_x, \ x \in \overline{\Omega}, \tag{1.2}$$

for some constant $\alpha > 0$, where $|\cdot|_{\mathrm{g}}$ is the norm of $I\!R^n_x$.

Remark 1.1 *About the existence of vector field H, see [36, 37] for details and examples. In particular, if $a_{ij} = \delta_{ij}$, then we can choose $H = x - x_0$ for fixed $x_0 \in I\!R^n$ and $\alpha = 1$.*

We consider a partition (Γ_0, Γ_1) of the boundary Γ such that

$$\Gamma_0 \neq \emptyset; \tag{1.3}$$

$$H \cdot \nu \leq 0 \quad \text{for} \quad x \in \Gamma_0; \tag{1.4}$$

$$H \cdot \nu \geq 0 \quad \text{for} \quad x \in \Gamma_1. \tag{1.5}$$

Furthermore, we assume

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset \quad \text{or} \quad H \cdot \mathbf{n} \leq 0 \quad \text{on} \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_1, \tag{1.6}$$

where $\mathbf{n}$ denotes the unit normal vector pointing outward at Γ_1 when considering Γ_1 as a sub-manifold of Γ.

Set

$$F = H \cdot \nu \Big(\mu_0 u_t + \int_0^t u_t(t-s) d\mu(s) \Big), \tag{1.7}$$

where μ_0 is some positive constant and μ is a Borel measure on $I\!R_+$.

When $\rho = 0$ and $A(x) = I$, the problem of proving uniform decay rates for wave equations with boundary dissipations but without acoustic boundary conditions has attracted a lot of attention in recent years, see [1, 2, 7, 10, 21, 28, 29, 30]. The problem (1.1) covers the case of a problem with memory type as studied in the references [1, 2, 7, 28], when the measure μ is given by $\mu(s) = k(s)ds$, where ds stands for Lebesgue measure and k is a nonnegative kernel. It also covers the case of a problem with a delay as studied for instance in the references [29, 30], when the measure μ is given by $\mu = \mu_1 \delta_\tau$, where μ_1 is a nonnegative constant and $\tau > 0$ represents the delay. An intermediate case treated in the reference [30] is the case when $d\mu(s) = k(s)\chi_{[\tau_1,\tau_2]}(s)ds$, where $0 < \tau_1 < \tau_2$, $\chi_{[\tau_1,\tau_2]}$ is the characteristic function of the interval $[\tau_1, \tau_2]$ and k is a nonnegative function in $L^\infty([\tau_1, \tau_2])$. We would like to highlight [10], where a general borelian measure is involved, the authors recovered and extended some of the results from the literature.

In the case of variable coefficients with a general $A(x)$, boundary stability of the wave equation was considered in the references [9, 12, 16, 25, 28] and many others. When delay exists, the problem (1.1) covers the case of a problem as studied for instance in the references [30, 31].

The wave equation with acoustic boundary conditions is a coupled system of second and first order partial differential equations in time, where the coupling is given on the portion of the boundary. It was introduced by Morse and Ingard [27] and developed by Beale and Rosencrans [5]. Since then, many authors have studied problems with acoustic boundary conditions. See, for instance, [4, 8, 13, 14, 18, 20, 23, 24] and references therein.

In [3], Abbas and Nicaise proved the asymptotic stability and nonuniform stability of a semigroup associated to a multidimensional wave equation with generalized acoustic boundary condition. Later, Graber and Said-Houari [18] studied the following semilinear problem with the porous acoustic boundary conditions:

$$\begin{cases} u_{tt} - \Delta u + \alpha(x)u + \phi(u_t) = j_1(u), \quad \text{in} \quad \Omega \times (0, +\infty) \\ u = 0, \quad \text{on} \quad \Gamma_0 \times (0, +\infty) \\ f(x)z_t + g(x)z = -u_t, \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \\ \partial_\nu u - h(x)\eta(z_t) + \rho(u_t) = j_2(u), \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega \\ z(x,0) = z_0(x), \quad x \in \Gamma_1, \end{cases} \tag{1.8}$$

where $\alpha : \Omega \to I\!R$ and $f, g, h : \Gamma \to I\!R$ are given functions.The existence and uniqueness of local solutions was proved by nonlinear semigroup theory. Introducing some restrictions on the source terms, the authors proved the local solution can be extended to be global. In addition, stability and blow up results were proved. However, these papers all dealt with constant coefficient cases.

In general, the coefficient matrices $A(x)$ in (1.1) are related to some property of materials in applications. The authors considered the uniform energy decay with nonlinear acoustic boundary conditions in [34] and energy decay with memory type acoustic boundary conditions in [35]. We would like to cite [24], where a variable-coefficient wave equation with acoustic boundary conditions and a time-varying delay in the boundary feedback was studied by Li and Chai. See also [17, 26] and references therein. In the all papers mentioned above, Riemannian geometry method was used. The method was first introduced by Yao [36] for controllability of the wave equation with variable coefficients. For a survey on the differential geometric methods, see Yao [37].

Summarizing, we shall consider the problem (1.1) where Γ_0, Γ_1 and F are defined by (1.2)-(1.7).

Here, we use the multiplier inequalities of the geometric version and Lasiecka and Tataru arguments [22] to derive some decay estimates of the energy for the variable coefficients problem (1.1).

The paper is organized as follows. In section 2, we present the assumptions and we enunciate the main results. In section 3, the proofs of the stability result are given. Finally, we made a collection of some preliminaries results which are used in the paper. We will use C to denote generic positive constants.

2 Main results

First, as in Cornilleau and Nicaise [10], we assume that there exists $\beta > 0$ such that

$$\mu_* := \int_0^{+\infty} e^{\beta s} d|\mu|(s) < \mu_0, \tag{2.1}$$

where $|\mu|$ is the absolute value of the measure μ.

We consider the following assumptions.

Assumption. The functions ρ, f, k and h satisfying the following:

(i) $\rho \in C(I\!R)$ is a increasing function with $\rho(0) = 0$ satisfying

$$s\rho(s) > 0 \text{ for all } \quad s \neq 0, \tag{2.2}$$

and there exist positive constant ρ such that

$$s\rho(s) \leq \rho s^2, \quad |s| \geq 1. \tag{2.3}$$

(ii) $f, k, h \in C(\overline{\Gamma}_1)$ are positive functions and there exist positive constant a_0 such that

$$\min_{x \in \overline{\Gamma}_1} \{f(x), k(x), h(x)\} > a_0.$$

Defining

$$H^1_{\Gamma_0}(\Omega) = \{ \ u \in H^1(\Omega) \mid u|_{\Gamma_0} = 0 \ \},$$

$$\mathcal{A}u = -\operatorname{div} A(x)\nabla u.$$

Next, we present well-posedness of the solution in the framework of hypotheses (1.2)-(1.7).

Proposition 2.1 *Suppose* $(u_0, u_1, z_0) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1)$. *Then* (1.1) *admits a unique solution*

$$u \in C([0, \infty); H^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$$

in the weak sense of Propst and Prüss. Moreover, if $u_0 \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ *and* $u_1 \in H^1_{\Gamma_0}(\Omega)$*, then*

$$u \in C^1([0, \infty); H^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega)),$$

and in addition

$$\mathcal{A}u(t) \in L^2(\Omega), \quad \frac{\partial u}{\partial \nu_{\mathcal{A}}}\Big|_{\Gamma_1}(t) \in H^{1/2}(\Gamma_1), \quad \forall t \geq 0.$$

Remark 2.1 *The proof of Proposition 2.1 can be easily given by an application of Theorem 4.4 in Propst and Prüss [32].*

In the next section, we assume that μ is supported in $[0, \tau](\tau > 0)$ and that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$.

Let $w(x, \theta, s, t) = u_t(x, t - \theta s), \quad x \in \Gamma_1, \theta \in (0, 1), s \in (0, \tau), t > \tau$.

Define Hilbert space $\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1 \times (0, 1) \times (0, \tau)) \times L^2(\Gamma_1)$ with the following inner product:

$$\begin{aligned} \langle (u, v, w, z)^T, (\hat{u}, \hat{v}, \hat{w}, \hat{z})^T \rangle \ &= \ \int_\Omega [\langle \nabla_{\mathrm{g}} u(x), \nabla_{\mathrm{g}} \hat{u}(x) \rangle_{\mathrm{g}} + v(x)\hat{v}(x)] dx \\ &+ \int_{\Gamma_1} \int_0^1 \int_0^\tau w(x, \theta, s)\hat{w}(x, \theta, s) d\mu(s) d\theta d\Gamma \\ &+ \int_{\Gamma_1} k(x)h(x)z(x)\hat{z}(x) d\Gamma. \end{aligned} \tag{2.4}$$

Define the operator $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{B}\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \mathcal{A}u - \rho(v) \\ -s^{-1}w_\rho \\ -\frac{1}{f(x)}[k(x)z + v] \end{pmatrix},$$

where

$$\begin{aligned} D(\mathcal{B}) &= \{(u,v,w,z)^T \in \mathcal{H} : \mathcal{A}u \in L^2(\Omega), \frac{\partial u}{\partial \nu_{\mathcal{A}}} = -H \cdot \nu(\mu_0 u_t + \int_0^\tau w(x,1,s)d\mu(s)) \quad \text{on} \quad \Gamma_1, \\ & \quad v(x) = w(x,0,s) \quad \text{on} \quad \Gamma_1 \times (0,\tau)\}. \end{aligned}$$

Proposition 2.2 *Assume that Assumption and (1.2)-(1.7) hold. Suppose* $(u_0, u_1, z_0) \in \mathcal{H}$*. Then* (1.1) *admits a unique solution* $(u, u_t, w, z)^T \in C([\tau,\infty); \mathcal{H})$*. Moreover, if* $(u_0, u_1, z_0) \in D(\mathcal{B})$ *, then the solution is regular with* $(u, u_t, w, z)^T \in C([\tau,\infty); D(\mathcal{B}))$.

In the sequel, we consider the measure λ obtained by the application of Proposition 4.1 to $|\mu|$.

Inspired by Cornilleau et al.[10], we define the energy of the solution (1.1) by the following formula:

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\int_\Omega |\nabla_{\mathrm{g}} u|_{\mathrm{g}}^2 dx + \frac{1}{2}\int_{\Gamma_1} khz^2 d\Gamma \\ &+ \frac{1}{2}\int_{\Gamma_1} H \cdot \nu \int_0^t \Big(\int_0^s u_t^2(x, t-\tau)d\tau\Big) d\lambda(s) d\Gamma \\ &+ \frac{1}{2}\int_{\Gamma_1} H \cdot \nu \int_t^\infty \Big(\int_0^s u_t^2(x, s-\tau)d\tau\Big) d\lambda(s) d\Gamma. \end{aligned} \tag{2.5}$$

So, we have the following stabilization result.

Theorem 2.1 *Assume that Assumption and (1.2)-(1.7) hold. Then, there exist constant* $T_0 > 0$ *such that the energy* $E(t)$*, associated to the solution* u *of (1.1) satisfies*

$$E(t) \le S\Big(\frac{t}{T_0} - 1\Big), \quad \textit{for all } t > T_0, \tag{2.6}$$

with $S(t)$ *decays uniformly to zero, where* $S(t)$ *is a solution of the differential equation (4.5).*

3 Proofs of the Main Results

To prove Theorem 2.1, we need the following several lemmas.

Lemma 3.1 *There exists a constant $C > 0$, such that, for any solution of (1.1) and any $T \geq S \geq 0$,*

$$E(S) - E(T) \geq C \int_S^T \int_{\Gamma_1} H \cdot \nu \Big(u_t^2 + \int_0^t u_t^2(t-s) d\lambda(s) \Big) d\Gamma dt + \int_S^T \int_{\Gamma_1} fhz_t^2 d\Gamma dt + \int_S^T \int_\Omega u_t \rho(u_t) dx dt. \tag{3.1}$$

That is to say, the energy is a non-increasing function of time.

Proof. Set

$$E_0(t) = \frac{1}{2} \int_\Omega u_t^2 dx + \frac{1}{2} \int_\Omega |\nabla_{\mathrm{g}} u|_{\mathrm{g}}^2 dx + \frac{1}{2} \int_{\Gamma_1} khz^2 d\Gamma. \tag{3.2}$$

It follows that

$$E_0(S) - E_0(T) = -\int_S^T \int_{\Gamma_1} u_t \frac{\partial u}{\partial \nu_{\mathcal{A}}} d\Gamma dt - \int_S^T \int_{\Gamma_1} khzz_t d\Gamma dt + \int_S^T \int_\Omega u_t \rho(u_t) dx dt. \tag{3.3}$$

Using boundary condition (1.7) and Young's inequality, for any $\varepsilon > 0$, we obtain

$$\begin{aligned} E_0(S) - E_0(T) &= \int_S^T \int_{\Gamma_1} H \cdot \nu \Big(\mu_0 u_t^2 + u_t \int_0^t u_t(t-s) d\mu(s) \Big) d\Gamma dt \\ &\quad + \int_S^T \int_{\Gamma_1} fhz_t^2 d\Gamma dt + \int_S^T \int_\Omega u_t \rho(u_t) dx dt \\ &\geq \int_S^T \int_{\Gamma_1} H \cdot \nu \Big((\mu_0 - \frac{\varepsilon}{2}) u_t^2 - \frac{1}{2\varepsilon} (\int_0^t u_t(t-s) d\mu(s))^2 \Big) d\Gamma dt \\ &\quad + \int_S^T \int_{\Gamma_1} fhz_t^2 d\Gamma dt + \int_S^T \int_\Omega u_t \rho(u_t) dx dt. \end{aligned} \tag{3.4}$$

By Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} E_0(S) - E_0(T) &\geq \int_{\Gamma_1} H \cdot \nu \Big((\mu_0 - \frac{\varepsilon}{2}) \int_S^T u_t^2 dt - \frac{\mu_*}{2\varepsilon} \int_S^T \int_0^t u_t^2(t-s) d|\mu|(s) dt \Big) d\Gamma \\ &\quad + \int_S^T \int_{\Gamma_1} fhz_t^2 d\Gamma dt + \int_S^T \int_\Omega u_t \rho(u_t) dx dt. \end{aligned} \tag{3.5}$$

We split $E - E_0$ into two terms:

$$[E - E_0]|_T^S = \frac{1}{2} \int_{\Gamma_1} H \cdot \nu [\varphi - \psi]|_T^S d\sigma, \tag{3.6}$$

where

$$\varphi = \int_0^t \int_0^s u_t^2(t-\tau) d\tau d\lambda(s), \quad \psi = \int_0^t \int_0^s u_t^2(\tau) d\tau d\lambda(s).$$

Variable replacement gives

$$\varphi = \int_0^t \int_0^t u_t^2(\tau) d\tau d\lambda(s) - \int_0^t \int_0^{t-s} u_t^2(\tau) d\tau d\lambda(s).$$

And by Fubini's theorem, we get

$$\psi = \int_0^t u_t^2(\tau) \int_\tau^t d\lambda(s) d\tau.$$

Hence,

$$\begin{aligned}
[\varphi - \psi]|_T^S &= \Big[\int_0^t u_t^2(\tau)\lambda([0,\tau]) d\tau - \int_0^t \int_0^{t-s} u_t^2(\tau) d\tau d\lambda(s) \Big] \Big|_T^S \\
&= \Big[\int_0^t \int_0^{t-s} u_t^2(\tau) d\tau d\lambda(s) \Big] \Big|_S^T - \Big[\int_0^t u_t^2(\tau)\lambda([0,\tau]) d\tau \Big] \Big|_S^T \\
&\geq \int_S^T \int_0^t u_t^2(t-s) d\lambda(s) dt - \mu_0 \int_S^T u_t^2(t) dt,
\end{aligned} \tag{3.7}$$

where Fubini's theorem and $\lambda([0,\tau]) \leq \lambda(I\!R_+) < \mu_0$ were used .

It follows from (3.4)-(3.7) that

$$\begin{aligned}
E(S) - E(T) &= [E - E_0]|_T^S + E_0(S) - E_0(T) \\
&\geq \int_S^T \int_{\Gamma_1} H \cdot \nu \Big((\mu_0 - \frac{\mu_0 + \varepsilon}{2}) u_t^2(t) + \frac{1}{2}(1 - \frac{\mu_*}{\varepsilon}) \int_0^t u_t^2(t-s) d\lambda(s) \Big) d\sigma dt \\
&\quad + \int_S^T \int_{\Gamma_1} f h z_t^2 d\Gamma dt + \int_S^T \int_\Omega u_t \rho(u_t) dx dt.
\end{aligned} \tag{3.8}$$

The proof completed by choosing $\mu_* < \varepsilon < \mu_0$.

Remark 3.1 *In fact, we have E_0 is non-increasing. Twice using Fubini's theorem, we obtain the following identities:*

$$\begin{aligned}
\int_S^T \int_0^t u_t^2(t-s) d|\mu|(s) dt &= \int_S^T \int_s^T u_t^2(t-s) dt d|\mu|(s) \\
&= \int_S^T \int_0^{T-s} u_t^2(t) dt d|\mu|(s) \\
&= \int_S^T \Big(\int_0^{T-t} d|\mu|(s) \Big) u_t^2(t) dt,
\end{aligned}$$

for any $T \geq S \geq 0$. Using (3.5) and the fact that $|\mu|([0, T-t]) \leq \mu_0$ and the choice of $\varepsilon = \mu_*$, it follows that

$$\begin{aligned}
E_0(S) - E_0(T) &\geq \int_{\Gamma_1} H \cdot \nu \Big(\mu_0 - \frac{\varepsilon}{2} - \frac{\mu_0 \mu_*}{2\varepsilon} \Big) \Big(\int_S^T u_t^2 dt \Big) d\Gamma \\
&\quad + \int_S^T \int_{\Gamma_1} f h z_t^2 d\Gamma dt + \int_S^T \int_\Omega u_t \rho(u_t) dx dt \\
&= \int_{\Gamma_1} H \cdot \nu \Big(\frac{\mu_0 - \mu_*}{2} \Big) \Big(\int_S^T u_t^2 dt \Big) d\Gamma \\
&\quad + \int_S^T \int_{\Gamma_1} f h z_t^2 d\Gamma dt + \int_S^T \int_\Omega u_t \rho(u_t) dx dt \\
&\geq 0.
\end{aligned} \tag{3.9}$$

In the context of singularities, using Proposition 3 of [10], the following Rellich inequality is useful.

Lemma 3.2 *Suppose $u \in H^1(\Omega)$ such that*

$$\mathcal{A}u \in L^2(\Omega), \;\; u|_{\Gamma_0} \in H^{\frac{3}{2}}(\Gamma_0) \quad \text{and} \quad \frac{\partial u}{\partial \nu_{\mathcal{A}}}\Big|_{\Gamma_1} \in H^{\frac{1}{2}}(\Gamma_1).$$

Then u satisfies $2\frac{\partial u}{\partial \nu_{\mathcal{A}}}H(u) - |\nabla_{\rm g} u|_{\rm g}^2 H \cdot \nu \in L^1(\Gamma)$ and we have the following inequality:

$$\begin{aligned} -2\int_{\Omega} \mathcal{A}uH(u)dx \le& \int_{\Omega} \Big(|\nabla_{\rm g} u|_{\rm g}^2 \operatorname{div} H - 2DH(\nabla_{\rm g} u, \nabla_{\rm g} u)\Big)dx \\ &+ \int_{\Gamma} \Big(2\frac{\partial u}{\partial \nu_{\mathcal{A}}}H(u) - |\nabla_{\rm g} u|_{\rm g}^2 H \cdot \nu\Big)d\Gamma. \end{aligned} \tag{3.10}$$

Remark 3.2 *If $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, (3.10) is an identity, see Yao [36]. If $n \le 3$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is not empty set, in Komornik [19], an application of Theorem 6.10 in Grisvard [15] gives (3.10). For any dimensions, Bey et al. [6] extended and proved the Rellich inequality.*

With this result, we prove the following observability for the problem (1.1).

Lemma 3.3 *Let u be a solution to the system* (1.1). *Then there exists a time $T_0 > 0$ such that, for $T > T_0$ and any $\delta (0 < \delta < \frac{1}{2})$, there exists $C > 0$ for which*

$$\begin{aligned} E(T) \le& C\int_0^T \int_{\Gamma_1} \Big(fhz_t^2 + H \cdot \nu[u_t^2 + \int_0^t u_t^2(t-s)d\mu(s)]\Big)d\Gamma dt \\ &+C\int_0^T \int_{\Omega} [u_t\rho(u_t) + \rho^2(u_t)]dxdt + C\|u\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}. \end{aligned} \tag{3.11}$$

Proof. Set $P = \operatorname{div} H - \alpha$. Multiplying (1.1) by $2H(u) + Pu$, an integration by parts gives

$$\begin{aligned} 0 =& \int_0^T \int_{\Omega} [u_{tt} + \mathcal{A}u + \rho(u_t)][2H(u) + Pu]dxdt \\ =& \Big(\int_{\Omega} u_t[2H(u) + Pu]dx\Big)\Big|_0^T - \int_0^T \int_{\Omega} \Big(u_t[2H(u_t) + Pu_t] - [\mathcal{A}u + \rho(u_t)][2H(u) + Pu]\Big)dxdt. \end{aligned} \tag{3.12}$$

It follows from Green formula that

$$\begin{aligned} -\int_{\Omega} \mathcal{A}uPudx =& \int_{\Omega} \operatorname{div} A(x)\nabla u \cdot Pudx \\ =& \int_{\Gamma} Pu\frac{\partial u}{\partial \nu_{\mathcal{A}}}d\Gamma - \int_{\Omega} P|\nabla_{\rm g} u|_{\rm g}^2 dx - \frac{1}{2}\int_{\Gamma} u^2\frac{\partial P}{\partial \nu_{\mathcal{A}}}d\Gamma - \frac{1}{2}\int_{\Omega} u^2 \mathcal{A}Pdx. \end{aligned} \tag{3.13}$$

Using Lemma 3.2 and (3.13), if (1.2) holds, it follows that

$$
\begin{aligned}
&-\int_\Omega \mathcal{A}u(2H(u)+Pu)dx\\
&\leq -\alpha\int_\Omega |\nabla_{\mathrm{g}}u|^2_{\mathrm{g}}dx+\int_\Gamma\Big(\frac{\partial u}{\partial\nu_{\mathcal{A}}}[2H(u)+Pu]-|\nabla_{\mathrm{g}}u|^2_{\mathrm{g}}H\cdot\nu\Big)d\Gamma\\
&-\frac{1}{2}\int_\Gamma u^2\frac{\partial P}{\partial\nu_{\mathcal{A}}}d\Gamma-\frac{1}{2}\int_\Omega u^2\mathcal{A}Pdx. \qquad (3.14)
\end{aligned}
$$

Remark that

$$
\begin{aligned}
\int_\Gamma\Big(\frac{\partial u}{\partial\nu_{\mathcal{A}}}[2H(u)+Pu]-|\nabla_{\mathrm{g}}u|^2_{\mathrm{g}}H\cdot\nu\Big)d\Gamma &= \int_\Gamma\Big\{\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big(2[\frac{\partial u}{\partial\nu_{\mathcal{A}}}\frac{1}{|\nu_{\mathcal{A}}|^2_{\mathrm{g}}}H\cdot\nu+\langle H,\nabla_{\mathrm{g}_\tau}u\rangle_{\mathrm{g}}]+Pu\Big)\\
&-\Big(|\nabla_{\mathrm{g}_\tau}u|^2_{\mathrm{g}}+\Big|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big|^2\frac{1}{|\nu_{\mathcal{A}}|^2_{\mathrm{g}}}\Big)H\cdot\nu\Big\}d\Gamma, \qquad (3.15)
\end{aligned}
$$

where $\nabla_{\mathrm{g}_\tau}u\in\Gamma_x$, the tangent space of Γ at x, and

$$
H(u)=\frac{\partial u}{\partial\nu_{\mathcal{A}}}\frac{1}{|\nu_{\mathcal{A}}|^2_{\mathrm{g}}}H\cdot\nu+\langle H,\nabla_{\mathrm{g}_\tau}u\rangle_{\mathrm{g}},
$$

$$
|\nabla_{\mathrm{g}}u|^2=|\nabla_{\mathrm{g}_\tau}u|^2_{\mathrm{g}}+\Big|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big|^2\frac{1}{|\nu_{\mathcal{A}}|^2_{\mathrm{g}}}
$$

is used.

Moreover, since $u=0$ and $H\cdot\nu\leq 0$ on Γ_0, we observe that

$$
\begin{aligned}
&-\int_\Omega\mathcal{A}u[2H(u)+Pu]dx\\
&\leq-\alpha\int_\Omega|\nabla_{\mathrm{g}}u|^2_{\mathrm{g}}dx+\int_{\Gamma_1}\Big\{\Big(\Big|\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big|^2\frac{1}{|\nu_{\mathcal{A}}|^2_{\mathrm{g}}}-|\nabla_{\mathrm{g}_\tau}u|^2_{\mathrm{g}}\Big)H\cdot\nu+\Big(\langle H,\nabla_{\mathrm{g}_\tau}u\rangle_{\mathrm{g}}+Pu\Big)\frac{\partial u}{\partial\nu_{\mathcal{A}}}\Big\}d\Gamma\\
&-\frac{1}{2}\int_{\Gamma_1}u^2\frac{\partial P}{\partial\nu_{\mathcal{A}}}d\Gamma-\frac{1}{2}\int_\Omega u^2\mathcal{A}Pdx. \qquad (3.16)
\end{aligned}
$$

It follows from divergence theorem and $u|_{\Gamma_0}=0$ that

$$
\begin{aligned}
\int_\Omega u_t[2H(u_t)+Pu_t]dx &= \int_\Omega(P-\operatorname{div}H)u_t^2dx+\int_\Gamma u_t^2H\cdot\nu d\Gamma\\
&=-\int_\Omega\alpha u_t^2dx+\int_{\Gamma_1}u_t^2H\cdot\nu d\Gamma. \qquad (3.17)
\end{aligned}
$$

Furthermore, for $\varepsilon>0$ sufficiently small, we have

$$
\begin{aligned}
\int_0^T\int_\Omega\rho(u_t)[2H(u)+Pu]dxdt &\leq C_\varepsilon\int_0^T\int_\Omega\rho^2(u_t)dxdt+\varepsilon\int_0^T\int_\Omega|\nabla_{\mathrm{g}}u|^2_{\mathrm{g}}dxdt\\
&+C\int_0^T\int_\Omega\Big(\rho^2(u_t)+u^2\Big)dxdt. \qquad (3.18)
\end{aligned}
$$

Substituting (3.16) - (3.18) into (3.12), we obtain that

$$
\begin{aligned}
\alpha\int_0^T\int_\Omega(u_t^2+|\nabla_{\mathrm{g}}u|^2_{\mathrm{g}})dxdt &\leq C[E(0)+E(T)]+C\int_0^T\int_{\Gamma_1}\Big(|\frac{\partial u}{\partial\nu_{\mathcal{A}}}|^2+|\nabla_{\mathrm{g}_\tau}u|^2_{\mathrm{g}}+u_t^2H\cdot\nu\Big)d\Gamma dt\\
&+C\int_0^T\int_{\Gamma_1}u^2d\Gamma dt+C\int_0^T\int_\Omega[u^2+\rho^2(u_t)]dxdt. \qquad (3.19)
\end{aligned}
$$

Using Lemma 7.2 in Lasiecka and Triggiani [21] to (1.1), it follows that: for any $\epsilon > 0$ and $0 < \delta < 1/2$ small enough, arbitrary but fixed, there exists a constant $C_{T,\delta,\epsilon} > 0$ such that

$$\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_1} |\nabla_{g_\tau} u|_g^2 d\Gamma dt \le C_{T,\delta,\epsilon}\{\int_0^T\int_{\Gamma_1}(|\frac{\partial u}{\partial \nu_{\mathcal{A}}}|^2 + u_t^2)d\Gamma dt + \int_0^T\int_\Omega \rho^2(u_t)dxdt + \|u\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\}. \tag{3.20}$$

Using the Sobolev embedding theorem and the trace theory, it is obvious that

$$\|u\|_{L^2(\Omega)} \le C\|u\|_{H^{1/2+\delta}(\Omega)}, \tag{3.21}$$

$$\|u\|_{L^2(\Gamma_1)} \le C\|u\|_{H^{1/2}(\Omega)} \le C\|u\|_{H^{1/2+\delta}(\Omega)}. \tag{3.22}$$

Applying inequality (3.19) over the interval $[\epsilon, T-\epsilon]$ rather than over $[0,T]$, and combining (3.20)-(3.22), we have

$$\alpha\int_{\epsilon}^{T-\epsilon}\int_\Omega (u_t^2 + |\nabla_g u|_g^2)dxdt \le C[E(0)+E(T)] + C\Big\{\int_0^T\int_{\Gamma_1}\Big(|\frac{\partial u}{\partial \nu_{\mathcal{A}}}|^2 + u_t^2 H\cdot\nu\Big)d\Gamma dt + \int_0^T\int_\Omega \rho^2(u_t)dxdt + \|u\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\Big\}. \tag{3.23}$$

On the other hand, we have

$$\begin{aligned}\int_0^T\int_{\Gamma_1} khz^2 d\Gamma dt &= \int_0^T\int_{\Gamma_1}(-u_t - fz_t)hzd\Gamma dt \\ &= -\Big(\int_{\Gamma_1} hzud\Gamma\Big)\Big|_0^T + \int_0^T\int_{\Gamma_1} hz_t ud\Gamma dt - \int_0^T\int_{\Gamma_1} fhzz_t d\Gamma dt \\ &\le CE(0) + \varepsilon\int_0^T E_0(t)dt + C_\varepsilon\int_0^T\int_{\Gamma_1} fhz_t^2 d\Gamma dt,\end{aligned} \tag{3.24}$$

where $\varepsilon > 0$ is constant.

On the another hand, from the boundary condition and Young's inequality, we have

$$\int_{\Gamma_1}|\frac{\partial u}{\partial \nu_{\mathcal{A}}}|^2 d\Gamma \le C\{\int_{\Gamma_1} fhz_t^2 d\Gamma + \int_{\Gamma_1} H\cdot\nu(u_t^2 + \int_0^t u_t^2(t-s)d\mu(s))d\Gamma\}. \tag{3.25}$$

Observe that

$$(\int_0^\epsilon + \int_{T-\epsilon}^T)\int_\Omega (u_t^2 + |\nabla_g u|_g^2)dxdt \le 2\epsilon E(0), \tag{3.26}$$

taking ε small enough, it follows that

$$\int_0^T E_0(t)dt \le CE(0) + C\Big\{\int_0^T\int_{\Gamma_1}\Big(fhz_t^2 + H\cdot\nu[u_t^2 + \int_0^t u_t^2(t-s)d\mu(s)]\Big)d\Gamma dt\Big\} + C\Big\{\int_0^T\int_\Omega \rho^2(u_t)dxdt + \|u\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\Big\}. \tag{3.27}$$

Furthermore, Lemma 5 in Cornilleau[10] and Lemma 3.1 give that

$$\int_0^T \int_{\Gamma_1} H\cdot\nu\Big(\int_0^t [\int_0^s u_t^2(x,t-\tau)d\tau]d\lambda(s) + \int_t^\infty [\int_0^s u_t^2(x,s-\tau)d\tau]d\lambda(s)\Big)d\Gamma dt$$
$$\leq C(\int_0^T \int_{\Gamma_1} H\cdot\nu \int_0^t u_t^2(x,t-s)d\lambda(s)d\Gamma dt + \int_0^\infty \int_{\Gamma_1} H\cdot\nu u_t^2 d\Gamma dt) \leq CE(0). \quad (3.28)$$

Whence

$$\int_0^T E(t)dt \leq \int_0^T E_0(t)dt + CE(0). \quad (3.29)$$

Next, for $E(0)$, we have

$$E(0) = E(T) + \int_0^T \int_{\Gamma_1} H\cdot\nu\Big(\mu_0 u_t^2 + u_t\int_0^t u_t(t-s)d\mu(s)\Big)d\Gamma dt$$
$$+\int_0^T \int_{\Gamma_1} fhz_t^2 d\Gamma dt + \int_0^T \int_\Omega u_t\rho(u_t)dxdt$$
$$+\frac{1}{2}\Big[\int_{\Gamma_1} H\cdot\nu\Big(\int_0^t [\int_0^s u_t^2(x,t-\tau)d\tau]d\lambda(s) + \int_t^\infty [\int_0^s u_t^2(x,s-\tau)d\tau]d\lambda(s)\Big)d\Gamma\Big]\Big|_T^0. \quad (3.30)$$

Since

$$\Big[\int_{\Gamma_1} H\cdot\nu\Big(\int_0^t [\int_0^s u_t^2(x,t-\tau)d\tau]d\lambda(s) + \int_t^\infty [\int_0^s u_t^2(x,s-\tau)d\tau]d\lambda(s)\Big)d\Gamma\Big]\Big|_T^0$$
$$= \int_{\Gamma_1} H\cdot\nu\Big(\int_0^\infty [\int_0^s u_t^2(x,s-\tau)d\tau]d\lambda(s) - \int_0^T [\int_0^s u_t^2(x,T-\tau)d\tau]d\lambda(s)$$
$$-\int_T^\infty [\int_0^s u_t^2(x,s-\tau)d\tau]d\lambda(s)\Big)d\Gamma$$
$$= \int_{\Gamma_1} H\cdot\nu\Big(\int_0^T [\int_0^s u_t^2(x,s-\tau)d\tau]d\lambda(s) - \int_0^T [\int_0^s u_t^2(x,T-\tau)d\tau]d\lambda(s)\Big)d\Gamma$$
$$\leq \int_{\Gamma_1} H\cdot\nu \int_0^T \Big(\int_0^s u_t^2(x,s-\tau)d\tau\Big)d\lambda(s)d\Gamma$$
$$= \int_{\Gamma_1} H\cdot\nu \int_0^T u_t^2(x,\theta)\int_\theta^T d\lambda(s)d\theta d\Gamma$$
$$\leq \mu_0 \int_{\Gamma_1} H\cdot\nu \int_0^T u_t^2 dt d\Gamma, \quad (3.31)$$

considering (3.27)-(3.30), we obtain that

$$\int_0^T E(t)dt \leq CE(T) + C\Big\{\int_0^T \int_{\Gamma_1} \Big(fhz_t^2 + H\cdot\nu[u_t^2 + \int_0^t u_t^2(t-s)d\mu(s)]\Big)d\Gamma dt\Big\}$$
$$+C\Big\{\int_0^T \int_\Omega [u_t\rho(u_t) + \rho^2(u_t)]dxdt + \|u\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\Big\}. \quad (3.32)$$

Recalling that $E(t)$ is monotone decreasing, from (3.32) we obtain

$$TE(T) \leq \int_0^T E(t)dt$$

$$\leq CE(T)+C\int_0^T\int_{\Gamma_1}\Big(fhz_t^2+H\cdot\nu[u_t^2+\int_0^t u_t^2(t-s)d\mu(s)]\Big)d\Gamma dt$$
$$+C\int_0^T\int_\Omega[u_t\rho(u_t)+\rho^2(u_t)]dxdt+C\|u\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}. \tag{3.33}$$

Taking T large enough, dependent of H,α,ϵ, the estimate (3.11) follows from the inequality (3.33). The proof is complete.

Next, the lower order term in the inequality (3.11) can be absorbed by a compactness-uniqueness argument in the following lemma.

Lemma 3.4 *Suppose that all assumption (1.2)-(1.7) hold true. Let u be a solution of the system* (1.1). *Then there exists $T_0>0$ such that for all $T\geq T_0$, there exists $C_T>0$ for which*

$$E(T)\leq C_T\int_0^T\int_{\Gamma_1}\Big(fhz_t^2+H\cdot\nu[u_t^2+\int_0^t u_t^2(t-s)d\mu(s)]\Big)d\Gamma dt$$
$$+C_T\int_0^T\int_\Omega[u_t\rho(u_t)+a(x)\rho^2(u_t)]dxdt. \tag{3.34}$$

Proof. The lower order term in the inequality (3.11) can be absorbed by a compactness-uniqueness argument as follows, that is, the following is true:

$$\|u\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\leq C\Big\{\int_0^T\int_{\Gamma_1}\Big(fhz_t^2+H\cdot\nu[u_t^2+\int_0^t u_t^2(t-s)d\mu(s)]\Big)d\Gamma dt$$
$$+\int_0^T\int_\Omega[u_t\rho(u_t)+\rho^2(u_t)]dxdt\Big\}. \tag{3.35}$$

Suppose that (3.35) is not true. Then there exists a sequence $\{u_k,z_k\}$ of solutions of the problem (1.1) such that

$$\|u_k\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\geq k\Big\{\int_0^T\int_{\Gamma_1}\Big(fhz_{kt}^2+H\cdot\nu[u_{kt}^2+\int_0^t u_{kt}^2(t-s)d\mu(s)]\Big)d\Gamma dt$$
$$+\int_0^T\int_\Omega[u_{kt}\rho(u_{kt})+\rho^2(u_{kt})]dxdt\Big\}. \tag{3.36}$$

Hence, if

$$\|u_k\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}=1, \tag{3.37}$$

it follows that

$$\lim_{k\to\infty}\Big\{\int_0^T\int_{\Gamma_1}\Big(fhz_{kt}^2+H\cdot\nu[u_{kt}^2+\int_0^t u_{kt}^2(t-s)d\mu(s)]\Big)d\Gamma dt$$
$$+\int_0^T\int_\Omega[u_{kt}\rho(u_{kt})+\rho^2(u_{kt})]dxdt\Big\}=0. \tag{3.38}$$

Thus, $E_k(t)$ is bounded uniformly for every t. Then there exists a subsequence of $\{u_k,z_k\}$, still denoted by $\{u_k,z_k\}$, such that

$$u_k(0)\rightharpoonup\overline{u}_0\quad\text{in}\quad H^1_{\Gamma_0}(\Omega),$$

$$u_{kt}(0) \rightharpoonup \overline{u}_1 \quad \text{in} \quad L^2(\Omega),$$

$$z_k(0) \rightharpoonup \overline{z}_0 \quad \text{in} \quad L^2(\Gamma_1).$$

Denote by $\overline{u}$ the solution corresponding to the initial data $(\overline{u}_0, \overline{u}_1)$.

Then

$$\{u_k, u_{kt}\} \overset{*}{\rightharpoonup} \{\overline{u}, \overline{u}_t\} \quad \text{in} \quad L^\infty(0,T; H^1(\Omega) \times L^2\Omega).$$

It follows from Aubin-Lions compactness theorem that

$$u_k \to \overline{u} \quad \text{in} \quad L^2(0,T; H^{1/2+\delta}(\Omega))).$$

It is obvious that $\overline{u} \neq 0$.

By (3.38) and passing to the limit in the problem (1.1), we obtain $\overline{u} \in H^1(\Omega \times (0,T))$ which satisfies

$$\begin{cases} \overline{u}_{tt} - \operatorname{div} A(x)\nabla\overline{u} = 0, \quad \text{in} \quad \Omega \times (0,T), \\ \overline{u} = 0, \quad \text{on} \quad \Gamma_0 \times (0,T), \\ \overline{u}_t = 0, \ \frac{\partial \overline{u}}{\partial \nu_{\mathcal{A}}} = 0, \quad \text{on} \quad \Gamma_1 \times (0,T). \end{cases} \tag{3.39}$$

Let $v = \overline{u}_t$. Differentiating (3.39), we have

$$\begin{cases} v_{tt} - \operatorname{div} A(x)\nabla v = 0, \quad \text{in} \quad \Omega \times (0,T), \\ v = 0, \quad \text{on} \quad \Gamma_0 \times (0,T), \\ v = 0, \ \frac{\partial v}{\partial \nu_{\mathcal{A}}} = 0, \quad \text{on} \quad \Gamma_1 \times (0,T). \end{cases} \tag{3.40}$$

By standard uniqueness results for the wave equation, we have for T large enough, $\overline{u}_t = v \equiv 0$, which, together with (3.39), then we have $\bar{u}(x,t) = \bar{u}(x)$ which is independent of time t, and $\operatorname{div} \mathcal{A}(x)\nabla\overline{u} = 0$. Since $\overline{u}|_{\Gamma_0} = \frac{\partial \overline{u}}{\partial \nu_{\mathcal{A}}}|_{\Gamma_1} = 0$, it is easy to prove that $\overline{u} \equiv 0$, which contradicts the assumption $\|\overline{u}\|_{L^2(0,T;H^{1/2+\delta}(\Omega))} = 1$. The proof is complete.

Proof of Theorem 2.1 Firstly, we are going to prove that

$$\psi(E(T)) + E(T) \leq E(0), \tag{3.41}$$

where ψ is an appropriate positive, continuous and strictly increasing function with $\psi(0) = 0$.

If Assumption (i) holds, as in [22], let $p : [0, +\infty) \to I\!R$ be concave, strictly increasing functions satisfying $p(0) = 0$ and

$$p(s\rho(s)) \geq s^2 + \rho^2(s) \qquad \text{for all} |s| \leq 1. \tag{3.42}$$

We define

$$\Lambda = \{(x,t) \in \Omega \times (0,T); |u_t(x,t)| > 1\}.$$

We have

$$\int_\Lambda \rho^2(u_t) dxdt \leq \rho \int_0^T \int_\Omega u_t \rho(u_t) dxdt. \tag{3.43}$$

and

$$\int_{(\Omega\times(0,T))\setminus\Lambda}\rho^2(u_t)dxdt \le \int_{(\Omega\times(0,T))\setminus\Lambda} p(u_t\rho(u_t))dxdt. \tag{3.44}$$

It follows from Jensen's inequality that

$$\begin{aligned}
&\int_{(\Omega\times(0,T))\setminus\Lambda}\rho^2(u_t)dxdt\\
&\le \operatorname{meas}(\Omega\times(0,T))p(\frac{1}{\operatorname{meas}(\Omega\times(0,T))}\int_0^T\int_\Omega u_t\rho(u_t)dxdt)\\
&\le \operatorname{meas}(\Omega\times(0,T))\tilde{p}(\int_0^T\int_\Omega u_t\rho(u_t)dxdt),
\end{aligned} \tag{3.45}$$

where

$$\tilde{p}(s) = p(\frac{s}{\operatorname{meas}(\Omega\times(0,T))}). \tag{3.46}$$

Thus,

$$\begin{aligned}
&\int_0^T\int_\Omega(u_t\rho(u_t)+\rho^2(u_t))dxdt\\
&\le (1+\rho)\int_0^T\int_\Omega u_t\rho(u_t)dxdt + \operatorname{meas}(\Omega\times(0,T))\tilde{p}(\int_0^T\int_\Omega u_t\rho(u_t)dxdt).
\end{aligned} \tag{3.47}$$

Combining (3.34) with (3.47), we obtain

$$\begin{aligned}
E(T) \le CM\{&\frac{M_0}{M}(\int_0^T\int_\Omega u_t\rho(u_t)dxdt + \tilde{p}(\int_0^T\int_\Omega u_t\rho(u_t)dxdt))\\
&+\frac{1}{M}(\int_0^T\int_{\Gamma_1} H\cdot\nu\Big(u_t^2+\int_0^t u_t^2(t-s)d\mu(s)\Big)d\Gamma dt + \int_0^T\int_{\Gamma_1}(u_t^2+fh|z_t|^2)d\Gamma dt)\},
\end{aligned} \tag{3.48}$$

where $M_0 = 1+\rho+\operatorname{meas}(\Omega\times(0,T))$, $M=\max\{\operatorname{meas}(\Omega\times(0,T)),\operatorname{meas}(\Gamma_1\times(0,T))\}$.

Set

$$L = \frac{1}{CM},$$

$$c = \max\{\frac{M_0}{M},\frac{1}{M}\}.$$

As $\tilde{p}$ is increasing, the function $cI+\tilde{p}$ is invertible. This allows us to define

$$\psi(s) = (cI+\tilde{p})^{-1}(Ls) \tag{3.49}$$

and to conclude that

$$\psi(E(T)) \le E(0)-E(T). \tag{3.50}$$

Then the inequality (3.41) holds.

As in [22], the final conclusion

$$E(t) \le S\Big(\frac{t}{T_0}-1\Big), \quad \text{for all } t>T_0, \tag{3.51}$$

follows from Lemma 4.1, which completes the proof of Theorem 2.1.

Remark 3.3 *Specificially, taking $\rho(s) = s|s|^{\theta-1}$, $0 < \theta < 1$, then Assumption (i) holds. Thus the decay rates of system (1.1) is as follows:*

$$E(t) \le C(E(0))\Big[E^{-\frac{m}{2}}(0) + mt\Big]^{-\frac{2}{m}},\ m = \frac{1-\theta}{\theta}. \tag{3.52}$$

4 Appendix

The assumption (2.1) implies the following equivalence about Borel measure:

Proposition 4.1 *([10]) Let μ be a* Borel *positive measure on $I\!R_+$ and $\mu_0 > 0$. The following properties are equivalent:*

(1) $\exists\beta > 0$ *such that*

$$\int_0^{+\infty} e^{\beta s} d\mu(s) < \mu_0. \tag{4.1}$$

(2) *There exists a Borel measure λ on $I\!R_+$ such that*

$$\lambda(I\!R_+) < \mu_0, \quad \mu \le \lambda, \tag{4.2}$$

and for some constants $\gamma > 0$, for all measure set $\mathcal{B}$,

$$\int_{\mathcal{B}} \lambda([s, +\infty))ds \le \gamma^{-1}\lambda(\mathcal{B}). \tag{4.3}$$

Remark 4.1 *If μ is some Borel measure such that*

$$|\mu|(I\!R_+) < \mu_0, \quad \int_0^{+\infty} e^{as} d|\mu|s < +\infty,$$

for some constants $a > 0$, then μ fulfils the assumption (2.1) for β small enough.

Lemma 4.1 *([22]) Let ψ be a positive, increasing function such that $\psi(0) = 0$. Since ψ is increasing, it is possible to define an increasing function q, $q(x) = x - (I + \psi)^{-1}(x)$. Consider a sequence $(s_n)_{n\in N}$ of positive numbers which satisfies*

$$s_{m+1} + \psi(s_{m+1}) \le s_m. \tag{4.4}$$

Then $s_m \le S(m)$, where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0,\ S(0) = s_0. \tag{4.5}$$

Moreover, $S(t)$ is monotone decreasing with $\lim_{t\to\infty} S(t) = 0$.

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