HERMITE-HADAMARD TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS VIA STRONGLY h-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via strongly h-convex functions. Some midpoint type and trapezoid type estimates related to them for n-times differentiable functions are also obtained, which extend some known results.

1. Introduction

Let I be an interval in $\mathbb{R}$ and $h : [0,1] \to [0,\infty)$ be a given function. A function $f : I \to \mathbb{R}$ is called *h-convex* provided that

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in (0,1)$. This notation was introduced by Varošanec [40] and generalizes the classes of *convex functions, s-convex functions (in the second sence), Godunova-Levin functions* and *P-functions*, which are obtained by taking in (1.1) $h(t) = t$, $h(t) = t^s$ $(s \in (0,1))$, $h(t) = 1/t$ and $h(t) = 1$, respectively. Many properties of them can be found, for instance, in [7, 11, 13, 16, 18, 30, 31, 45].

A significant application of the convex function is the well-known Hermite-Hadamard inequality, if $f : [a,b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

This inequality was studied extensively and had been extended under various convex type functions. In 1995, Dragomier, Pecaric and Persson [12] established similar results for Godunova-Levin functions and P-functions. In 1999, Dragomir and Fitzpatrick [10] obtained an analogous inequality for s-convex functions (in the second sence). In 2008, Sarikaya, Saglam and Yildirim [35] extended it to h-convex functions.

Following Polyak [28], a function $f : I \to \mathbb{R}$ is said to be *strongly convex with modulus* $\beta > 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \beta t(1-t)(x-y)^2, \tag{1.3}$$

for all $x, y \in I$ and $t \in (0,1)$. The function played an important role in optimization theory and mathematical economics (see e.g. [19, 20, 21, 22, 28, 29, 33, 37, 41, 42]). In 2011, Angulo, Gimenez, Moros and Nikodem [3] introduced the strongly $h-$convex function,

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Key words and phrases. Hermite-Hadamard type inequalities; strongly h-convex functions; Riemann-Liouville fractional integrals.

2010 Mathematics Subject Classification. Primary 26A33; Secondary 26A51, 26D15.

The research was supported by the College Students' Science and Technology Innovation Project (Xinmiao Project) of Zhejiang Province and the National Natural Science Foundation of China (No. 11771358).

which unified the classes of strongly convex functions and h-convex functions. And then they extended (1.2) to these new functions.

Definition 2. [3] *Let $h : [0, 1] \to [0, \infty)$ be a given function and β be a positive constant. We say that $f : I \to \mathbb{R}$ is strongly h-convex with modulus β, or f belongs to the class $SX(h, \beta, I)$, if*

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) - \beta t(1-t)(x-y)^2, \tag{1.4}$$

for all $x, y \in I$ and $t \in (0, 1)$.

Particularly, if f satisfies (1.4) with $h(t) = t$, $h(t) = t^s$ $(s \in (0, 1))$, $h(t) = 1/t$ and $h(t) = 1$, then f is said to be *strongly convex functions, strongly s-convex functions, strongly Godunova-Levin functions* and *strongly P-function*, respectively. Moreover, it is not difficult to check that $h(1/2) > 0$ if $f \in SX(h, \beta, I)$ and $f \ge 0$. As an application, the authors [3] established the following Hermite-Hadamard inequality.

Theorem A. *Let $f \in SX(h, \beta, [a, b])$ and h be Lebesgue integrable on $(0, 1)$ with $h(1/2) > 0$. If f is Lebesgue integrable on $[a, b]$, then*

$$\begin{aligned} \frac{1}{2h(1/2)}\left[f\left(\frac{a+b}{2}\right) + \frac{\beta}{12}(b-a)^2\right] &\le \frac{1}{b-a}\int_a^b f(x)dx \\ &\le (f(a) + f(b))\int_0^1 h(t)dt - \frac{\beta}{6}(b-a)^2. \end{aligned} \tag{1.5}$$

It is notable that Theorem A reduces to the result in [35] with $\beta \to 0$.

Let $f : [a, b] \to \mathbb{R}$ be integrable. The left and right Riemann-Liouville fractional integrals $J^\alpha_{a^+}f$ and $J^\alpha_{b^-}f$ of order $\alpha > 0$ are defined by

$$J^\alpha_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, x > a, \tag{1.6}$$

and

$$J^\alpha_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, x < b, \tag{1.7}$$

respectively, where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dx$ is the Gamma function.

In recent years, Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals are studied extensively, for instance, see [5, 9, 24, 27, 32, 36, 39, 43, 44] and the references therein. In this paper, we extend them to strongly $h-$convex functions and obtain some error estimates related to these inequalities.

In the sequel, we assume that the function h in the above definitions is always Lebesgue integrable on $[0, 1]$. Denote $L(I)$ be the set of Lebesgue integrable functions on the interval I and let $C^n(I)$ be the space of functions f with its derivatives $f^{(k)}$ continuous on I for all $0 \le k \le n$,

2. New Hermite-Hadamard inequality via Riemann-Liouville fractional integrals

In 2017, Sarikaya and Yildirim [34] first obtained a remarkable inequality of Hermite-Hadamard type involving the left and right Riemann-Liouville fractional integrals.

Theorem B.[34] *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $f \in L([a, b])$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right] \leq \frac{f(a)+f(b)}{2}.$$

In 2020, Budak, Ertuğral and Sarikaya [6] extended it to more generalized fractional integrals. In 2021, Zhang, Farid and Akbar [46] obtained an analogue inequality as Theorem B for strongly $(s, m)-$convex functions. In this section, we establish the following similar inequality for strongly $h-$convex functions.

Theorem 1. *Let $f \in L([a, b])$ and $f \in SX(h, \beta, [a, b])$ with $h(1/2) > 0$. Then*

$$\begin{aligned} & \frac{1}{2h(1/2)}\left[f\left(\frac{a+b}{2}\right) + \frac{\beta(b-a)^2}{2(\alpha+1)(\alpha+2)}\right] \\ \leq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right] \\ \leq\ & \frac{f(a)+f(b)}{2}\alpha\int_0^1 t^{\alpha-1}\left[h\left(\frac{t}{2}\right) + h\left(1-\frac{t}{2}\right)\right]dt - \frac{\beta\alpha(\alpha+3)(b-a)^2}{4(\alpha+1)(\alpha+2)}. \end{aligned}$$

Proof. Since f is a strongly $h-$covex function with modulus β, for any $t \in [0, 1]$ we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left[\frac{1}{2}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + \frac{1}{2}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right] \\ &\leq h(1/2)f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + h(1/2)f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) - \frac{\beta}{4}t^2(b-a)^2, \end{aligned}$$

with means that

$$\begin{aligned} & \frac{1}{\alpha h(1/2)}f\left(\frac{a+b}{2}\right) \\ \leq\ & \int_0^1 (1-t)^{\alpha-1}f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt + \int_0^1 (1-t)^{\alpha-1}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt \\ & -\frac{\beta(b-a)^2}{4h(1/2)}\int_0^1 (1-t)^{\alpha-1}t^2 dt \\ =\ & \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_{(a+b)/2}^{b}(b-u)^{\alpha-1}f(u)du + \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_a^{(a+b)/2}(u-a)^{\alpha-1}f(u)du \qquad (2.1) \\ & -\frac{\beta(b-a)^2}{4h(1/2)\alpha(\alpha+1)(\alpha+2)} \\ =\ & \frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right] - \frac{\beta(b-a)^2}{4h(1/2)\alpha(\alpha+1)(\alpha+2)}. \qquad (2.2) \end{aligned}$$

Therefore we finish the first inequality of the theorem.

On the other hand, it is easy to see that

$$\begin{aligned} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) &\leq h\left(\frac{1-t}{2}\right)f(a) + h\left(\frac{1+t}{2}\right)f(b) - \frac{\beta(1-t)(1+t)}{4}(b-a)^2, \\ f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) &\leq h\left(\frac{1+t}{2}\right)f(a) + h\left(\frac{1-t}{2}\right)f(b) - \frac{\beta(1-t)(1+t)}{4}(b-a)^2, \end{aligned}$$

which, combing with (2.1) and (2.2), imply that

$$\begin{aligned}
&\frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\\
= &\int_0^1(1-t)^{\alpha-1}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt+\int_0^1(1-t)^{\alpha-1}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
\le &[f(a)+f(b)]\int_0^1(1-t)^{\alpha-1}\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt-\frac{\beta(b-a)^2}{2}\frac{\alpha+3}{(\alpha+1)(\alpha+2)}.
\end{aligned}$$

Thus we finish the proof of Theorem 1. □

If taking $h(t)=t$ and $h(t)=t^s$, then Theorem 1 reduces to Corollary 3 and Corollary 4 in [46], respectively. And, it is not difficult to see that the theorem is Theorem A for $\alpha=1$. Letting $\beta\to 0$ in Theorem 1, we have the following result.

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be an $h-$covex function with $h(1/2)>0$ and $f\in L([a,b])$. Then*

$$\begin{aligned}
\frac{1}{2h(1/2)}f\left(\frac{a+b}{2}\right) &\le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\\
&\le \frac{f(a)+f(b)}{2}\alpha\int_0^1 t^{\alpha-1}\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt.
\end{aligned}$$

Especially, if $h(t)=t$, Corollary 1 becomes Theorem B.

3. Midpoint type inequalities for n times differentiable functions

In the past few decades, various applications are studied extensively with respect to the inequality of (1.2). In 2000, Pearce and Pecaric[30] proved an important equality connect with the right part of Hermite-Hadamard inequality.

Lemma A. [30] *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$. Then*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\\
= &\frac{1}{b-a}\left[\int_{(a+b)/2}^b(b-x)f'(x)dx-\int_a^{(a+b)/2}(x-a)f'(x)dx\right]\\
= &(b-a)\left[\int_0^{1/2}tf'(ta+(1-t)b)dt-\int_{1/2}^1(1-t)f'(ta+(1-t)b)dt\right].
\end{aligned}$$

By the lemma, the authors showed the following result.

Theorem C. [30] *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function. If $|f'|^q$ is convex on $[a,b]$ with $1\le q<\infty$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{3.1}$$

Furthermore, some estimates for concave functions are also achieved in [17] and [30]. In 2004, Kirmaci [15] rediscovered Lemma A and established some other estimates similar to Theorem C. In 2011, Alomori, Darus and Kirmaci[2] obtained analogue results for $s-$convex functions. Meanwhile, there are large number of works dedicated to study the difference estimates connected with the right part of (1.2), for instance, in [4, 8, 14, 16, 27, 30] and the references therein.

In 2017, Sarikaya and Yildirim [34] found an important identity related to Riemann-Liouville integrals as follows.

Lemma B. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$. Then*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\\ =\ &\frac{b-a}{4}\int_0^1(1-t)^\alpha\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

It is not difficult to check that Lemma B becomes Lemma A with $\alpha=1$. As a consequence, they obtained the following midpoint type inequalities for differentiable functions.

Theorem D.[34] *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function. Suppose that $|f'|^q$ is convex on $[a,b]$ for $1\le q<\infty$.*

(i) *If $1\le q<\infty$, then*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\ \le\ &\frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{1/q}\left\{\left[(\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q\right]^{1/q}\right.\\ &\left.+\left[(\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q\right]^{1/q}\right\}.\end{aligned}$$

(ii) *If $1<q<\infty$, then*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\ \le\ &\frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left[\left(\frac{|f'(a)|+3|f'(b)|}{4}\right)^{1/q}+\left(\frac{3|f'(a)|+4|f'(b)|}{4}\right)^{1/q}\right]\\ \le\ &\frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{1/p}\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

where $1/p+1/q=1$.

It is notable that Theorem D reduces to the theorems in [15]. In 2016, Set, Sarikaya and Gözpinart [38] generalized the proceeding theorem to conformal fractional integrals. In 2020, the authors [6] extended them to more generalized fractional integrals. In 2021, the authors [46] gained similar inequalities for strongly $(s,m)-$convex functions. On the other hand, Noor and Awan [23] proved an equality for twice differentiable functions.

Lemma C.[23] *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function and $f''\in L([a,b])$. Then*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\\ =\ &\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left[f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

Consequently, they established the following inequalities for $s-$convex functions.

Theorem E.[23] *Let $f \in C^2([a,b])$ and $f'' \in L([a,b])$. Suppose that $|f''|^q$ $(1 \le q < \infty)$ is an s-convex function (in the second sense).*

(i) *If $1 \le q < \infty$, then*

$$\begin{aligned} &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\ \le\ & \frac{(b-a)^2}{2^{3+s/q}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-1/q}\left\{\left(\int_0^1(1-t)^{\alpha+1}(1+t)^s dt\left|f''(a)\right|^q+\frac{1}{s+\alpha+2}\left|f''(b)\right|^q\right)^{1/q}\right. \\ &\left.+\left(\frac{1}{s+\alpha+2}\left|f''(a)\right|^q+\int_0^1(1-t)^{\alpha+1}(1+t)^s dt\left|f''(b)\right|^q\right)^{1/q}\right\}. \end{aligned}$$

(ii) *If $1 < q < \infty$, then*

$$\begin{aligned} &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\ \le\ & \frac{(b-a)^2}{2^{3+s/q}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{1/q}\left\{\left[\left(2^{s+1}-1\right)\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right. \\ &\left.+\left[\left|f''(a)\right|^q+\left(2^{s+1}-1\right)\left|f''(b)\right|^q\right]^{1/q}\right\}, \end{aligned}$$

where $1/p+1/q=1$.

In this section, we will establish some analogue results for strongly $h-$convex functions with n order derivatives. For the sake of convenience, if $f:[a,b]\to\mathbb{R}$ is an $n-$times differentiable function, we denote

$$\begin{aligned} \mathfrak{L}\left(f,\frac{a+b}{2}\right) &= \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right) \\ &\quad -\sum_{j=1}^{n-1}\frac{\left[1+(-1)^j\right](b-a)^j}{2^{j+1}\prod_{k=1}^{j}(\alpha+k)}f^{(j)}\left(\frac{a+b}{2}\right). \end{aligned} \tag{3.2}$$

It is easy to see that if $n=1$ or 2, $\mathfrak{L}\left(f,\frac{a+b}{2}\right)$ have the same concise form:

$$\mathfrak{L}\left(f,\frac{a+b}{2}\right)=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right). \tag{3.3}$$

Now we introduce the following key lemma.

Lemma 1 *Let $f \in C^n([a,b])$ and $f^{(n)} \in L([a,b])$, $n \in \mathbb{Z}^+$. Then*

$$\begin{aligned} \mathfrak{L}\left(f,\frac{a+b}{2}\right) &= \frac{(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}\left[(-1)^n\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right. \\ &\quad\left.+\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right], \end{aligned}$$

here we denote $\prod_{k=1}^{0}(\alpha+k)\equiv 1$.

It is not difficult to check that Lemma 1 reduces to Lemma B and Lemma C for $n=1$ and $n=2$, respectively.

Proof Without loss of generality, we may assume $n \geq 2$. Integration by parts n times show that

$$\begin{aligned}
&\int_0^1 (1-t)^{\alpha+n-1} f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \\
= \;& \frac{2}{b-a}f^{(n-1)}\left(\frac{a+b}{2}\right)-\frac{2(\alpha+n-1)}{b-a}\int_0^1 (1-t)^{\alpha+n-2}f^{(n-1)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \\
= \;& \frac{2}{b-a}f^{(n-1)}\left(\frac{a+b}{2}\right)-\frac{2^2(\alpha+n-1)}{(b-a)^2}f^{(n-2)}\left(\frac{a+b}{2}\right) \\
&+\frac{2^2(\alpha+n-1)(\alpha+n-2)}{(b-a)^2}\int_0^1 (1-t)^{\alpha+n-3}f^{(n-2)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \\
= \;& \cdots \\
= \;& \frac{2}{b-a}f^{(n-1)}\left(\frac{a+b}{2}\right)-\frac{2^2(\alpha+n-1)}{(b-a)^2}f^{(n-2)}\left(\frac{a+b}{2}\right) \\
&+\frac{2^3(\alpha+n-1)(\alpha+n-2)}{(b-a)^3}f^{(n-3)}\left(\frac{a+b}{2}\right) \\
&+\cdots+\frac{(-1)^{n-1}2^n(\alpha+n-1)(\alpha+n-2)\cdots(\alpha+1)}{(b-a)^n}f\left(\frac{a+b}{2}\right) \\
&+\frac{(-1)^n 2^n(\alpha+n-1)(\alpha+n-2)\cdots(\alpha+1)\alpha}{(b-a)^{\alpha+n}}\int_0^1 (1-t)^{\alpha-1}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \\
= \;& \sum_{j=0}^{n-1}\frac{(-1)^{n-j-1}2^{n-j}\prod_{k=j+1}^{n-1}(\alpha+k)}{(b-a)^{n-j}}f^{(j)}\left(\frac{a+b}{2}\right) \\
&+\frac{(-1)^n 2^{\alpha+n}\prod_{k=0}^{n-1}(\alpha+k)\Gamma(\alpha)}{(b-a)^{\alpha+n}}J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a).
\end{aligned}$$

Multiplying both sides of the proceeding equality by $\frac{(-1)^n(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}$, we obtain

$$\begin{aligned}
(3.4)\qquad &\frac{(-1)^n(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}\int_0^1 (1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \\
= \;& \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)-\frac{1}{2}f\left(\frac{a+b}{2}\right)-\sum_{j=1}^{n-1}\frac{(-1)^j(b-a)^j}{2^{j+1}\prod_{k=1}^{j}(\alpha+k)}f^{(j)}\left(\frac{a+b}{2}\right).
\end{aligned}$$

Similarly, using integration by parts n times again,

$$\begin{aligned}
&\int_0^1 (1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \\
= \;& -\sum_{j=0}^{n-1}\frac{2^{n-j}\prod_{k=j+1}^{n-1}(\alpha+k)}{(b-a)^{n-j}}f^{(j)}\left(\frac{a+b}{2}\right)+\frac{2^{\alpha+n}\prod_{k=0}^{n-1}(\alpha+k)\Gamma(\alpha)}{(b-a)^{\alpha+n}}J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b),
\end{aligned}$$

which means that

$$(3.5)\qquad \frac{(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}\int_0^1 (1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$

$$= \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) - \frac{1}{2} f\left(\frac{a+b}{2}\right) - \sum_{j=1}^{n-1} \frac{(b-a)^j}{2^{j+1}\prod_{k=1}^{j}(\alpha+k)} f^{(j)}\left(\frac{a+b}{2}\right).$$

Therefore we complete the proof of the lemma by (3.4) and (3.5). □

Using Lemma 1, we obtain the following fractional integral inequalities. For simplicity, we first denote

$$\mathcal{A} = \int_0^{1/2} t^{\alpha+n-1} h(t)dt, \quad \mathcal{B} = \int_{1/2}^1 (1-t)^{\alpha+n-1} h(t)dt.$$

Theorem 2. *Let $f \in C^n([a,b])$ and $f^{(n)} \in L([a,b]), n \in \mathbb{Z}^+$. Suppose that $\left|f^{(n)}\right|^q \in SX(h, \beta, [a,b]), 1 \le q < \infty$.*

(i) *If $q = 1$, then*

$$\begin{aligned} &\left|\mathfrak{L}\left(f, \frac{a+b}{2}\right)\right| \\ \le\ & \frac{2^{\alpha-1}(b-a)^n}{\prod_{k=1}^{n-1}(\alpha+k)} \left\{(\mathcal{A}+\mathcal{B})\left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right) - \frac{\beta(\alpha+n+3)(b-a)^2}{2^{n+\alpha+1}(\alpha+n+1)(\alpha+n+2)}\right\}. \end{aligned}$$

(ii) *If $1 < q < \infty$, then*

$$\begin{aligned} &\left|\mathfrak{L}\left(f, \frac{a+b}{2}\right)\right| \\ \le\ & \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^{n}(\alpha+k)} \times \\ & \left\{\left[\mathcal{B}\left|f^{(n)}(a)\right|^q + \mathcal{A}\left|f^{(n)}(b)\right|^q - \frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q}\right. \\ & \left. + \left[\mathcal{B}\left|f^{(n)}(b)\right|^q + \mathcal{A}\left|f^{(n)}(a)\right|^q - \frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q}\right\} \\ \le\ & \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^{n}(\alpha+k)} \times \\ & \left\{\left(\mathcal{A}^{1/q}+\mathcal{B}^{1/q}\right)\left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right) + 2\left[\frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q}\right\}. \end{aligned}$$

(iii) *If $1 < q < \infty$, then*

$$\begin{aligned} &\left|\mathfrak{L}\left(f, \frac{a+b}{2}\right)\right| \\ \le\ & \frac{(b-a)^n}{2^{n+1/p}\prod_{k=1}^{n-1}(\alpha+k)} \left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p} \times \\ & \left\{\left(\int_{1/2}^1 h(t)dt\left|f^{(n)}(a)\right|^q + \int_0^{1/2} h(t)dt\left|f^{(n)}(b)\right|^q - \frac{\beta(b-a)^2}{12}\right)^{1/q}\right. \end{aligned}$$

$$
\begin{aligned}
& + \left(\int_{1/2}^{1} h(t)dt \left| f^{(n)}(b) \right|^{q} + \int_{0}^{1/2} h(t)dt \left| f^{(n)}(a) \right|^{q} - \frac{\beta(b-a)^2}{12} \right)^{1/q} \Bigg\} \\
\leq \ & \frac{(b-a)^n}{2^{n+1/p} \prod_{k=1}^{n-1}(\alpha+k)} \left(\frac{1}{p(\alpha+n-1)+1} \right)^{1/p} \times \\
& \left\{ \left[\left(\int_{0}^{1/2} h(t)dt \right)^{1/q} + \left(\int_{1/2}^{1} h(t)dt \right)^{1/q} \right] \left(\left| f^{(n)}(a) \right| + \left| f^{(n)}(b) \right| \right) + 2 \left[\frac{\beta(b-a)^2}{12} \right]^{1/q} \right\},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. (1) If $q = 1$, then it follows from the fact of $|f| \in SX(h, \beta, [a,b])$ that

$$
\begin{aligned}
& \left| \int_{0}^{1} (1-t)^{\alpha+n-1} f^{(n)} \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) dt \right| \\
\leq \ & \int_{0}^{1} (1-t)^{\alpha+n-1} \left[h\left(\frac{1+t}{2}\right) \left| f^{(n)}(a) \right| + h\left(\frac{1-t}{2}\right) \left| f^{(n)}(b) \right| - \beta \frac{(1-t)(1+t)}{4} (b-a)^2 \right] dt \\
= \ & 2^{\alpha+n} \int_{1/2}^{1} (1-t)^{\alpha+n-1} h(t)dt \left| f^{(n)}(a) \right| + 2^{\alpha+n} \int_{0}^{1/2} t^{\alpha+n-1} h(t)dt \left| f^{(n)}(b) \right| \\
& - \frac{\beta(\alpha+n+3)(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)} \\
= \ & 2^{\alpha+n} \mathcal{B} \left| f^{(n)}(a) \right| + 2^{\alpha+n} \mathcal{A} \left| f^{(n)}(b) \right| - \frac{\beta(\alpha+n+3)(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \left| \int_{0}^{1} (1-t)^{\alpha+n-1} f^{(n)} \left(\frac{1-t}{2} a + \frac{1+t}{2} b \right) dt \right| \\
\leq \ & 2^{\alpha+n} \mathcal{A} \left| f^{(n)}(a) \right| + 2^{\alpha+n} \mathcal{B} \left| f^{(n)}(b) \right| - \frac{\beta(\alpha+n+3)(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)}.
\end{aligned}
$$

Then we complete the proof of (i) by the proceeding two inequalities and Lemma 1.

(2) If $1 < q < \infty$, then power-mean inequality and the fact of $|f|^q \in SX(h, \beta, [a,b])$ show that

$$
\begin{aligned}
& \left| \int_{0}^{1} (1-t)^{\alpha+n-1} f^{(n)} \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) dt \right| \\
\leq \ & \left(\int_{0}^{1} (1-t)^{\alpha+n-1} dt \right)^{1-1/q} \left(\int_{0}^{1} (1-t)^{\alpha+n-1} \left| f^{(n)} \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) \right|^{q} dt \right)^{1/q} \\
\leq \ & \left(\frac{1}{\alpha+n} \right)^{1-1/q} \left[2^{\alpha+n} \int_{1/2}^{1} (1-t)^{\alpha+n-1} h(t)dt \left| f^{(n)}(a) \right|^{q} \right. \\
& \left. + 2^{\alpha+n} \int_{0}^{1/2} t^{\alpha+n-1} h(t)dt \left| f^{(n)}(b) \right|^{q} - \frac{\beta(\alpha+n+3)(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)} \right]^{1/q}
\end{aligned}
$$

$$= \frac{2^{(\alpha+n)/q}}{(\alpha+n)^{1-1/q}}\left[\mathcal{B}\left|f^{(n)}(a)\right|^q+\mathcal{A}\left|f^{(n)}(b)\right|^q-\frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q},$$

and

$$\begin{aligned}&\left|\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\ \le\ & \frac{2^{(\alpha+n)/q}}{(\alpha+n)^{1-1/q}}\left[\mathcal{B}\left|f^{(n)}(b)\right|^q+\mathcal{A}\left|f^{(n)}(a)\right|^q-\frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q},\end{aligned}$$

which finish the proof of the first inequality in (ii) by Lemma 1 again.

For the proof of the second inequality, let

$$a_1=\int_{1/2}^1(1-t)^{\alpha+n-1}h(t)dt\left|f^{(n)}(a)\right|^q,\ b_1=2\int_0^{1/2}t^{\alpha+n-1}h(t)dt\left|f^{(n)}(b)\right|^q,$$
$$a_2=\int_{1/2}^1(1-t)^{\alpha+n-1}h(t)dt\left|f^{(n)}(b)\right|^q,\ b_2=\int_0^{1/2}t^{\alpha+n-1}h(t)dt\left|f^{(n)}(a)\right|^q,$$
$$c_1=c_2=-\tfrac{\beta(\alpha+n+3)}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}(b-a)^2.$$

According to the fact that

$$\sum_{k=1}^m(|a_k|+|b_k|+|c_k|)^s\le\sum_{k=1}^m|a_k|^s+\sum_{k=1}^m|b_k|^s+\sum_{k=1}^m|c_k|^s,\quad 0\le s<1,$$

then the desired result can be obtained easily.

(3) If $1<q<\infty$, then the Hölder inequality and the fact of $|f|^q\in SX(h,\beta,[a,b])$ tell us that

$$\begin{aligned}&\left|\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|\\ \le\ &\left(\int_0^1(1-t)^{p(\alpha+n-1)}dt\right)^{1/p}\left(\int_0^1\left|f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^q\\ \le\ &\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p}\\ &\times\left\{\int_0^1\left[h\left(\frac{1+t}{2}\right)\left|f^{(n)}(a)\right|^q+h\left(\frac{1-t}{2}\right)\left|f^{(n)}(b)\right|^q-\beta\frac{(1-t)(1+t)}{4}(b-a)^2\right]dt\right\}^{1/q}\\ =\ &\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p}\left(2\int_{1/2}^1h(t)dt\left|f^{(n)}(a)\right|^q+2\int_0^{1/2}h(t)dt\left|f^{(n)}(b)\right|^q-\frac{\beta(b-a)^2}{6}\right)^{1/q}.\end{aligned}$$

By the same way, we have

$$\begin{aligned}&\left|\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\ \le\ &\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p}\left(2\int_{1/2}^1h(t)dt\left|f^{(n)}(b)\right|^q+2\int_0^{1/2}h(t)dt\left|f^{(n)}(a)\right|^q-\frac{\beta(b-a)^2}{6}\right)^{1/q}.\end{aligned}$$

Then we complete the proof of the first inequality in (ii) by the above two inequalities and Lemma 1.

The second inequality is proved by a similar way as (2), we leave the details to readers. □

Letting $\beta \to 0$. We have the following results.

Corollary 2. *Let $f \in C^n([a,b])$ and $f^{(n)} \in L([a,b])$, $n \in \mathbb{Z}^+$. Suppose that $\left|f^{(n)}\right|^q$ is an $h-$convex function with $1 \le q < \infty$.*

(i) *If $1 \le q < \infty$, then*

$$\begin{aligned}\left|\mathfrak{L}\left(f, \frac{a+b}{2}\right)\right| &\le \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^{n}(\alpha+k)} \times \\ &\quad \left\{\left[\mathcal{B}\left|f^{(n)}(a)\right|^q + \mathcal{A}\left|f^{(n)}(b)\right|^q\right]^{1/q} + \left[\mathcal{B}\left|f^{(n)}(b)\right|^q + \mathcal{A}\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\} \\ &\le \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^{n}(\alpha+k)} \left(\mathcal{A}^{1/q} + \mathcal{B}^{1/q}\right)\left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right).\end{aligned}$$

(ii) *If $1 < q < \infty$, then*

$$\begin{aligned}\left|\mathfrak{L}\left(f, \frac{a+b}{2}\right)\right| &\le \frac{(b-a)^n}{2^{n+1/p}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p} \times \\ &\quad \left\{\left(\int_{1/2}^{1} h(t)dt\left|f^{(n)}(a)\right|^q + \int_0^{1/2} h(t)dt\left|f^{(n)}(b)\right|^q\right)^{1/q}\right. \\ &\quad \left. + \left(\int_{1/2}^{1} h(t)dt\left|f^{(n)}(b)\right|^q + \int_0^{1/2} h(t)dt\left|f^{(n)}(a)\right|^q\right)^{1/q}\right\} \\ &\le \frac{(b-a)^n}{2^{n+1/p}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p} \times \\ &\quad \left[\left(\int_0^{1/2} h(t)dt\right)^{1/q} + \left(\int_{1/2}^{1} h(t)dt\right)^{1/q}\right]\left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right),\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark. By changing of variable, it is not difficult to check that Corollary 2 extends Theorem D and Theorem E for $n = 1, h(t) = t$ and $n = 2, h(t) = t^s$, respectively.

4. Trapezoid type inequalities for n times differentiable functions

In 1998, Dragomir and Agarwal [8] established the following identity for the right hand of (1.2), and then they gained error estimates related to it. Some more studies please refer to, for examples, [1, 14, 15, 26].

Lemma D.[8] *Let $f \in C^1([a,b])$ and $f' \in L([a,b])$. Then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt.$$

In 2016, Özdemir, M. Avci-Ardinç and H. Kavurmaci-Önalan [25] (Lemma 2 for $x = (a+b)/2$) proved a trapezoid type equality for differentiable function via fractional integral.

Lemma E.[25] *Let $f \in C^1([a,b])$ and $f' \in L([a,b])$. Then*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right] - \frac{f(a)+(b)}{2}$$

$$= \frac{(b-a)}{4}\int_0^1 [1-(1-t)^\alpha]\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.$$

Thereafter, Budak [5] obtained it for generalized fractional integral in 2019 and Budak, Ertuğral and Sarikaya [6] extended it to other fractional integrals in 2020. As a consequence, the authors obtained the following results.

Theorem F.[5, 6] *Let $f \in C^1([a,b])$ and $f' \in L([a,b])$. Suppose that $|f'|^q$ is convex on $[a,b]$ for $1 \le q < \infty$.*

(i) *If $1 \le q < \infty$, then*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right| \\
\le\ & \frac{b-a}{2^{2+1/q}}\frac{\alpha}{\alpha+1}\left[\left(\frac{\alpha+1}{2(\alpha+2)}|f'(a)|^q+\frac{3\alpha+7}{2(\alpha+2)}|f'(b)|^q\right)^{1/q}\right. \\
&\left.+\left(\frac{3\alpha+7}{2(\alpha+2)}|f'(a)|^q+\frac{\alpha+1}{2(\alpha+2)}|f'(b)|^q\right)^{1/q}\right].
\end{aligned}$$

(ii) *If $1 < q < \infty$, then*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right| \\
\le\ & \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left[\left(\frac{|f'(a)|+3|f'(b)|}{4}\right)^{1/q}+\left(\frac{3|f'(a)|+4|f'(b)|}{4}\right)^{1/q}\right] \\
\le\ & \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{1/p}\left[|f'(a)|+|f'(b)|\right],
\end{aligned}$$

where $1/p + 1/q = 1$.

In this section, we will prove some similar results for strongly $h-$convex functions with n order derivatives. For simplicity, if $f \in C^n([a,b])$, we denote

$$\begin{aligned}
\Re\left(f,\frac{a+b}{2}\right) =\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2} \\
&+\sum_{j=1}^{n-1}\frac{\left[1+(-1)^j\right](b-a)^j}{2^{j+1}}\frac{\prod_{k=1}^j(\alpha+k)-j!}{j!\prod_{k=1}^j(\alpha+k)}f^{(j)}\left(\frac{a+b}{2}\right).
\end{aligned}\tag{4.1}$$

It is easy to check that if $n = 1$ or 2, $\Re\left(f,\frac{a+b}{2}\right)$ has the simplified form:

$$\Re\left(f,\frac{a+b}{2}\right)=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}.\tag{4.2}$$

Now we introduce the following key lemma.

Lemma 2. *Let $f \in C^n([a,b])$ and $f^{(n)} \in L([a,b])$, $n \in \mathbb{Z}^+$. Then*

$$\begin{aligned}
\Re\left(f,\frac{a+b}{2}\right) =\ & -\frac{(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]\times \\
&\left[(-1)^n f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt.
\end{aligned}$$

It is not difficult to check that Lemma 2 reduces to Lemma E by (4.2) for $n = 1$.

Proof. Without loss of generality, we assume that $n \geq 2$. Integrating by parts n times show that

$$\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1} - (1-t)^{\alpha+n-1} \right] f^{(n)}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt$$

$$= \frac{2}{b-a}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!} - 1\right) f^{(n-1)}\left(\frac{a+b}{2}\right)$$

$$-\frac{2(\alpha+n-1)}{b-a}\int_0^1 \left[\frac{\prod_{k=1}^{n-2}(\alpha+k)}{(n-2)!}(1-t)^{n-2} - (1-t)^{\alpha+n-2} \right] f^{(n-1)}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt$$

$$= \frac{2}{b-a}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!} - 1\right) f^{(n-1)}\left(\frac{a+b}{2}\right)$$

$$-\frac{2^2(\alpha+n-1)}{(b-a)^2}\left(\frac{\prod_{k=1}^{n-2}(\alpha+k)}{(n-2)!} - 1\right) f^{(n-2)}\left(\frac{a+b}{2}\right)$$

$$+\frac{2^2(\alpha+n-1)(\alpha+n-2)}{(b-a)^2} \times$$

$$\int_0^1 \left[\frac{\prod_{k=1}^{n-3}(\alpha+k)}{(n-3)!}(1-t)^{n-3} - (1-t)^{\alpha+n-3} \right] f^{(n-2)}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt$$

$$= \cdots$$

$$= \sum_{j=1}^{n-1} \frac{(-1)^{n-j+1}2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!} - 1\right) f^{(j)}\left(\frac{a+b}{2}\right)$$

$$+\frac{(-1)^n 2^{n-1}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n-1}}\int_0^1 [1-(1-t)^\alpha] f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt$$

$$= \sum_{j=1}^{n-1} \frac{(-1)^{n-j+1}2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!} - 1\right) f^{(j)}\left(\frac{a+b}{2}\right)$$

$$+\frac{(-1)^n 2^{n}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n}} f(a) + \frac{(-1)^{n+1}2^n\prod_{k=0}^{n-1}(\alpha+k)}{(b-a)^n}\int_0^1 (1-t)^{\alpha-1} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt$$

$$= \sum_{j=1}^{n-1} \frac{(-1)^{n-j+1}2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!} - 1\right) f^{(j)}\left(\frac{a+b}{2}\right)$$

$$+\frac{(-1)^n 2^{n}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n}} f(a) + \frac{(-1)^{n+1}2^{n+\alpha}\Gamma(\alpha)\prod_{k=0}^{n-1}(\alpha+k)}{(b-a)^{n+\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a),$$

which means that

$$\text{(4.3)}\quad \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) - \frac{f(a)}{2} + \sum_{j=1}^{n-1} \frac{(-1)^j (b-a)^j}{2^{j+1}} \frac{\prod_{k=1}^{j}(\alpha+k) - j!}{j!\prod_{k=1}^{j}(\alpha+k)} f^{(j)}\left(\frac{a+b}{2}\right)$$

$$= \frac{(-1)^{n+1}(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)} \times$$

$$\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1} - (1-t)^{\alpha+n-1} \right] f^{(n)}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt.$$

Similarly, integrating by parts n times again tell us that

$$\begin{aligned}
&\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1} - (1-t)^{\alpha+n-1} \right] f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
= \quad & -\frac{2}{b-a}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!} - 1\right) f^{(n-1)}\left(\frac{a+b}{2}\right) \\
& + \frac{2(\alpha+n-1)}{b-a}\int_0^1 \left[\frac{\prod_{k=1}^{n-2}(\alpha+k)}{(n-2)!}(1-t)^{n-2} - (1-t)^{\alpha+n-2} \right] f^{(n-1)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
= \quad & \cdots \\
= \quad & -\sum_{j=1}^{n-1} \frac{2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!} - 1\right) f^{(j)}\left(\frac{a+b}{2}\right) \\
& + \frac{2^{n-1}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n-1}}\int_0^1 \left[1-(1-t)^{\alpha}\right] f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
= \quad & -\sum_{j=1}^{n-1} \frac{2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!} - 1\right) f^{(j)}\left(\frac{a+b}{2}\right) \\
& + \frac{2^{n}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n}} f(b) - \frac{2^{n+\alpha}\Gamma(\alpha)\prod_{k=0}^{n-1}(\alpha+k)}{(b-a)^{n+\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b),
\end{aligned}$$

which implies that

$$\begin{aligned}
(4.4) \qquad & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) - \frac{f(b)}{2} + \sum_{j=1}^{n-1} \frac{(b-a)^j}{2^{j+1}} \frac{\prod_{k=1}^{j}(\alpha+k) - j!}{j!\prod_{k=1}^{j}(\alpha+k)} f^{(j)}\left(\frac{a+b}{2}\right) \\
= \quad & -\frac{(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)} \times \\
& \int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1} - (1-t)^{\alpha+n-1} \right] f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt.
\end{aligned}$$

Then we complete the proof by (4.3) and (4.4). □

Using Lemma 2, we obtain the following error estimates. For convenience, we first set

$$\begin{aligned}
\mathcal{C} &= \int_0^{1/2} \left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!} t^{n-1} - 2^{\alpha} t^{\alpha+n-1} \right) h(t)\, dt, \\
\mathcal{D} &= \int_{1/2}^{1} \left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!} (1-t)^{n-1} - 2^{\alpha} (1-t)^{\alpha+n-1} \right) h(t)\, dt.
\end{aligned}$$

Theorem 3. *Let $f \in C^n([a,b])$ and $f^{(n)} \in L([a,b]), n \in \mathbb{Z}^+$. Suppose that $\left|f^{(n)}\right|^q \in SX(h,\beta,[a,b]), 1 \leq q < \infty$.*

(i) *If $q=1$, then*

$$\left|\mathfrak{R}\left(f,\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^n}{2\prod_{k=1}^{n-1}(\alpha+k)}\left[(\mathcal{C}+\mathcal{D})\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right) -\beta\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+1}(n+2)!(\alpha+n+1)(\alpha+n+2)}(b-a)^2\right].$$

(ii) *If $1<q<\infty$, then*

$$\begin{aligned}
&\left|\mathfrak{R}\left(f,\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^n}{2^{n(1-1/q)+1}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q}\times \\
&\left\{\left[\mathcal{C}\left|f^{(n)}(a)\right|^q+\mathcal{D}\left|f^{(n)}(b)\right|^q-\beta\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}(b-a)^2\right]^{1/q}\right. \\
&\left.+\left[\mathcal{C}\left|f^{(n)}(b)\right|^q+\mathcal{D}\left|f^{(n)}(a)\right|^q-\beta\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}(b-a)^2\right]^{1/q}\right\} \\
\leq\ & \frac{(b-a)^n}{2^{n(1-1/q)+1}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q}\left\{\left(\mathcal{C}^{1/q}+\mathcal{D}^{1/q}\right)\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\right. \\
&\left.+2\left[\beta\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}(b-a)^2\right]^{1/q}\right\}.
\end{aligned}$$

(iii) *If $1<q<\infty$, then*

$$\begin{aligned}
&\left|\mathfrak{R}\left(f,\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^n}{2^{n+1-1/q}\prod_{k=1}^{n-1}(\alpha+k)}\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p}\times \\
&\left\{\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(b)\right|^q+\int_0^{1/2}h(t)dt\left|f^{(n)}(a)\right|^q-\frac{\beta(b-a)^2}{12}\right]^{1/q}\right. \\
&\left.+\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(a)\right|^q+\int_0^{1/2}h(t)dt\left|f^{(n)}(b)\right|^q-\frac{\beta(b-a)^2}{12}\right]^{1/q}\right\} \\
\leq\ & \frac{(b-a)^n}{2^{n+1-1/q}\prod_{k=1}^{n-1}(\alpha+k)}\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p}\times \\
&\left\{\left[\left(\int_0^{1/2}h(t)dt\right)^{1/q}+\left(\int_{1/2}^1 h(t)dt\right)^{1/q}\right]\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)+2\left[\frac{\beta(b-a)^2}{12}\right]^{1/q}\right\},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. (1) If $q=1$, then it follows from the fact of $|f|\in SX(h,\beta,[a,b])$ that

$$
\begin{aligned}
&\left|\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|\\
\leq\ &\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]h\left(\frac{1+t}{2}\right)dt\left|f^{(n)}(a)\right|\\
&+\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]h\left(\frac{1-t}{2}\right)dt\left|f^{(n)}(b)\right|\\
&-\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]\beta\frac{(1-t)(1+t)}{4}(b-a)^2dt\\
=\ &2^n\int_{1/2}^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-2^{\alpha}(1-t)^{\alpha+n-1}\right]h(t)\,dt\left|f^{(n)}(a)\right|\\
&+2^n\int_0^{1/2}\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-2^{\alpha}t^{\alpha+n-1}\right]h(t)\,dt\left|f^{(n)}(b)\right|\\
&-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{4(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2\\
=\ &2^n\left(\mathcal{C}\left|f^{(n)}(a)\right|+\mathcal{D}\left|f^{(n)}(b)\right|\right)-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{4(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left|\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\
\leq\ &2^n\left(\mathcal{D}\left|f^{(n)}(a)\right|+\mathcal{C}\left|f^{(n)}(b)\right|\right)-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{4(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2.
\end{aligned}
$$

Then we finish the proof of (i) by the above inequalities and Lemma 2.

(2) If $1<q<\infty$, then power-mean inequality and the fact of $|f|^q\in SX(h,\beta,[a,b])$ show that

$$
\begin{aligned}
&\left|\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|\\
\leq\ &\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]dt\right)^{1-1/q}\times\\
&\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]\left|f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q}\\
\leq\ &2^{n/q}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q}\left\{\mathcal{D}\left|f^{(n)}(a)\right|^q+\mathcal{C}\left|f^{(n)}(b)\right|^q\right.
\end{aligned}
$$

$$-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2\Bigg\}^{1/q},$$

and

$$\begin{aligned}
&\left|\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\
\le\ & 2^{n/q}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q}\Bigg\{\mathcal{D}\left|f^{(n)}(b)\right|^q+\mathcal{C}\left|f^{(n)}(a)\right|^q\\
&-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2\Bigg\}^{1/q},
\end{aligned}$$

which finish the proof of the first inequality in (ii) by Lemma 2 again.

The proof of the second inequality can be obtained by a similar method as in Theorem 2 (ii), we omit the details.

(3) If $1<q<\infty$, then Hölder's inequality and $|f|^q\in SX(h,\beta,[a,b])$ imply that

$$\begin{aligned}
&\left|\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|\\
\le\ & \left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]^p dt\right)^{1/p}\times\\
&\left(\int_0^1\left|f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q}\\
\le\ & 2^{1/q}\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p}\times\\
&\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(a)\right|^q+\int_0^{1/2}h(t)dt\left|f^{(n)}(b)\right|^q-\frac{\beta(b-a)^2}{12}\right]^{1/q},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\
\le\ & 2^{1/q}\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p}\times\\
&\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(b)\right|^q+\int_0^{1/2}h(t)dt\left|f^{(n)}(a)\right|^q-\frac{\beta(b-a)^2}{12}\right]^{1/q}.
\end{aligned}$$

Therefore, we obtained the first inequality of (iii) by the above two inequalities and Lemma 2.

The second inequality is achieved by the same way in Theorem 2 (iii), we leave it to readers. □

Letting $\beta \to 0$, we take the following conclusion.

Corollary 3. *Let $f \in C^n([a,b])$ and $f^{(n)} \in L([a,b]), n \in \mathbb{Z}^+$. Suppose that $\left|f^{(n)}\right|^q)$ is an $h-$convex function with $1 \le q < \infty$.*

(i) *If $1 \le q < \infty$, then*

$$\begin{aligned}
&\left|\mathfrak{R}\left(f, \frac{a+b}{2}\right)\right| \\
\le\ & \frac{(b-a)^n}{2^{n(1-1/q)+1}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!} - \frac{1}{\alpha+n}\right)^{1-1/q} \times \\
& \left[\left(\mathcal{C}\left|f^{(n)}(a)\right|^q + \mathcal{D}\left|f^{(n)}(b)\right|^q\right)^{1/q} + \left(\mathcal{D}\left|f^{(n)}(a)\right|^q + \mathcal{C}\left|f^{(n)}(b)\right|^q\right)^{1/q}\right] \\
\le\ & \frac{(b-a)^n}{2^{n(1-1/q)+1}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!} - \frac{1}{\alpha+n}\right)^{1-1/q} \times \\
& \left(\mathcal{C}^{1/q} + \mathcal{D}^{1/q}\right)\left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right).
\end{aligned}$$

(ii) *If $1 < q < \infty$, then*

$$\begin{aligned}
&\left|\mathfrak{R}\left(f, \frac{a+b}{2}\right)\right| \\
\le\ & \frac{(b-a)^n}{2^{n+1-1/q}\prod_{k=1}^{n-1}(\alpha+k)}\left(\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1} - t^{\alpha+n-1}\right]^p dt\right)^{1/p} \times \\
& \left\{\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(b)\right|^q + \int_0^{1/2} h(t)dt\left|f^{(n)}(a)\right|^q\right]^{1/q}\right. \\
& \left. + \left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(a)\right|^q + \int_0^{1/2} h(t)dt\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\} \\
\le\ & \frac{(b-a)^n}{2^{n+1-1/q}\prod_{k=1}^{n-1}(\alpha+k)}\left(\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1} - t^{\alpha+n-1}\right]^p dt\right)^{1/p} \times \\
& \left[\left(\int_0^{1/2} h(t)dt\right)^{1/q} + \left(\int_{1/2}^1 h(t)dt\right)^{1/q}\right]\left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark. If $n = 1$ and $h(t) = t$, then Corollary 3 reduces to Theorem F.

As a special case of Corollary 3, we have the following results.

Corollary 4. *Let $f \in C^2([a,b])$ and $f'' \in L([a,b])$. Suppose that $|f''|^q$ is a convex function with $1 \le q < \infty$.*

(i) *If $1 \le q < \infty$, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right] - \frac{f(a)+f(b)}{2}\right|$$

$$\leq \frac{(b-a)^2}{2^{3+1/q}(\alpha+1)}\left(\frac{\alpha+1}{2}-\frac{1}{\alpha+2}\right)^{1-1/q}\times$$
$$\left\{\left[\left(\frac{\alpha+1}{3}-\frac{1}{\alpha+3}\right)|f''(a)|^q+\left(\frac{2(\alpha+1)}{3}-\frac{2}{\alpha+2}+\frac{1}{\alpha+3}\right)|f''(b)|^q\right]^{1/q}\right.$$
$$\left.+\left[\left(\frac{\alpha+1}{3}-\frac{1}{\alpha+3}\right)|f''(b)|^q+\left(\frac{2(\alpha+1)}{3}-\frac{2}{\alpha+2}+\frac{1}{\alpha+3}\right)|f''(a)|^q\right]^{1/q}\right\}$$
$$\leq \frac{(b-a)^2}{2^{3+1/q}(\alpha+1)}\left(\frac{\alpha+1}{2}-\frac{1}{\alpha+2}\right)^{1-1/q}\times$$
$$\left[\left(\frac{\alpha+1}{3}-\frac{1}{\alpha+3}\right)^{1/q}+\left(\frac{2(\alpha+1)}{3}-\frac{2}{\alpha+2}+\frac{1}{\alpha+3}\right)^{1/q}\right]\left(|f''(a)|+|f''(b)|\right).$$

(ii) *If $1<q<\infty$, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{(b-a)^2}{2^{3+2/q}(\alpha+1)}\left(\int_0^1\left[(\alpha+1)t-t^{\alpha+1}\right]^p dt\right)^{1/p}\times$$
$$\left[\left(3|f''(b)|^q+|f''(a)|^q\right)^{1/q}+\left(3|f''(a)|^q+|f''(b)|^q\right)^{1/q}\right]$$
$$\leq \frac{\left(1+3^{1/q}\right)(b-a)^2}{2^{3+2/q}(\alpha+1)}\left(\int_0^1\left[(\alpha+1)t-t^{\alpha+1}\right]^p dt\right)^{1/p}\left(|f''(a)|+|f''(b)|\right)$$
$$\leq \frac{(b-a)^2}{2^{1+2/q}(\alpha+1)}\left(\int_0^1\left[(\alpha+1)t-t^{\alpha+1}\right]^p dt\right)^{1/p}\left(|f''(a)|+|f''(b)|\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark. Especially, if taking $q=1$ in Corollary 4 (i), we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{(b-a)^2}{16}\left[1-\frac{2}{(\alpha+1)(\alpha+2)}\right]\left(|f''(a)|+|f''(b)|\right).$$

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